Towards Algebraic Foundations of Algebraic Fuzzy Logic Operations: Aiming at the Minimal Number of Requirements

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1. Background. I. Why Fuzzy Logic

- In many applications, it is important to use expert knowledge.

- Experts often describe their knowledge in imprecise ("fuzzy") properties like "small".

- Example of imprecision: for a specific size, an expert may be not fully confident whether this size is small.

- To describe such properties, fuzzy logic was invented.

- In fuzzy logic, each statement is characterized by a degree of confidence.

- Usually, this degree is taken from the interval $[0, 1]$, where:
  - 0 means absolutely false and
  - 1 means absolutely true.
2. Background. II. Fuzzy Logic Operations

- **Typical situation:**
  - *we know:* the degrees $d(A)$ and $d(B)$ of expert confidence in statements $A$ and $B$;
  - *we need:* to estimate the expert’s degree of confidence in composite statements like $A \& B$, $A \lor B$:
    
    $$d(A \& B) \approx f_\&(d(A), d(B));$$
    $$d(A \lor B) \approx f_\lor(d(A), d(B));$$
    $$d(\neg A) \approx f_-(d(A)).$$

- The functions providing such estimates are called *fuzzy logic operations*:
  - and-operations (a.k.a. t-norms),
  - or-operations (a.k.a. t-conorms),
  - negation operations, etc.
3.  Background. II. Fuzzy Logic Operations (cont-d)

- Fuzzy logic operations must satisfy natural properties.
- Example 1:
  - **Fact:** $A \& B$ means the same as $B \& A$.
  - **Property:** the and-operation $f_\& (a, b)$ must be commutative:
    \[
    f_{\&}(a, b) = f_{\&}(b, a).
    \]
- Example 2:
  - **Fact:** $A \&(B \& C)$ means the same as $(A \& B) \& C$.
  - **Property:** the and-operation $f_\& (a, b)$ must be associative:
    \[
    f_{\&}(a, f_{\&}(b, c)) = f_{\&}(f_{\&}(a, b), c).
    \]
- **Known:** there exist a complete descriptions of all the operations that satisfy such properties.
4. Formulation of the Problem

- **In principle**: we can have very complex fuzzy logic operations.
- **In practice**: mostly simple algebraic operations are used:
  - linear;
  - quadratic;
  - fractional-linear; etc.
- **Foundational challenge**: how do we classify such algebraic fuzzy operations?
- **What we prove in this talk**: 
  - to classify algebraic fuzzy logic operations,
  - we do not need to use all the usual properties.
5. Motivating Result: Description of All Quadratic And-Operations

- Consider quadratic functions $f_\& : [0, 1] \times [0, 1] \rightarrow [0, 1]$: 
  $$f_\&(a, b) = c_0 + c_1 \cdot a + c_b \cdot b + c_{aa} \cdot a^2 + c_{ab} \cdot a \cdot b + c_{bb} \cdot b^2.$$ 

- **Properties:** 

  - the function $f_\&(a, b)$ is monotonic (non-decreasing) in each variable; 
  - $f_\&$ is conservative in the sense that it coincides with the usual logical operation $a \& b$ for $a, b \in \{0, 1\}$: 
    $$f_\&(0, 0) = f_\&(0, 1) = f_\&(1, 0) = 0; \quad f_\&(1, 1) = 1.$$ 

- **Result** (H.T. Nguyen, V. Kreinovich): the only quadratic and-operation with these properties is $f_\&(a, b) = a \cdot b$. 

- **Comment:** we did not use commutativity or associativity.
6. Description of All Quadratic Or-Operations

- Consider quadratic functions $f_\lor : [0, 1] \times [0, 1] \rightarrow [0, 1]$: 
  \[ f_\lor(a, b) = c_0 + c_1 \cdot a + c_b \cdot b + c_{aa} \cdot a^2 + c_{ab} \cdot a \cdot b + c_{bb} \cdot b^2. \]

- **Properties:**
  - the function $f_\lor(a, b)$ is monotonic (non-decreasing) in each variable;
  - $f_\lor$ is conservative in the sense that it coincides with the usual logical operation $a \lor b$ for $a, b \in \{0, 1\}$: 
    \[ f_\lor(0, 0) = 0, \quad f_\lor(0, 1) = f_\lor(1, 0) = f_\land(1, 1) = 1. \]

- **Result:** the only quadratic and-operation with these properties is 
  $f_\lor(a, b) = a + b - a \cdot b$.

- **Comment:** we did not use commutativity or associativity.
7. **Negation Operations: Usual Properties**

- **Main algebraic property:**
  - **Fact:** \( \neg(\neg A) \) means the same as \( A \).
  - **Property:** the negation operation \( f_{\neg}(a) \) must satisfy the property:
  \[
  f_{\neg}(f_{\neg}(a)) = a.
  \]

- **Monotonicity:** the more we believe in \( A \), the less we believe in \( \neg A \).

- **Conclusion:** the function \( f_{\neg}(a) \) must be non-increasing.

- **Conservative:** for \( a = 0 \) (“false”) and for \( a = 1 \) (“true”), \( f_{\neg}(a) \) must coincide with the truth value of “not \( a \)”:
  \[
  f_{\neg}(0) = 1, \quad f_{\neg}(1) = 0.
  \]
8. Description of All Quadratic Negation Operations

- Consider quadratic functions $f_\neg : [0, 1] \to [0, 1]$:
  \[
  f_\neg(a) = c_0 + c_1 \cdot a + c_{aa} \cdot a^2. \tag{1}
  \]

- **Properties:**
  
  - the function $f_\neg(a)$ satisfies the property
    \[
    f_\neg(f_\neg(a)) = a \text{ for all } a;
    \]
  
  - $f_\neg$ is *conservative* in the sense that it coincides with the usual logical operation $\neg a$ for $a \in \{0, 1\}$:
    \[
    f_\neg(0) = 1, \quad f_\neg(1) = 0.
    \]

- **Result:** the only quadratic negation operation with these properties is $f_\neg(a) = 1 - a$.

- **Comment:** we did not use monotonicity.
9. Description of All Fractional-Linear Negation Operations

- Consider fractional-linear functions \( f_{\neg} : [0,1] \rightarrow [0,1] \):
  \[
  f_{\neg}(a) = \frac{a + b \cdot x}{c + d \cdot x}.
  \]

- Properties:
  - the function \( f_{\neg}(a) \) satisfies the property
    \[
    f_{\neg}(f_{\neg}(a)) = a \text{ for all } a;
    \]
  - \( f_{\neg} \) is conservative in the sense that it coincides with the usual logical operation \( \neg a \) for \( a \in \{0,1\} \):
    \[
    f_{\neg}(0) = 1, \quad f_{\neg}(1) = 0.
    \]
  - Result: the only fractional-linear negation operation with these properties is \( f_{\neg}(a) = 1 - a \).
  - Comment: we did not use monotonicity.
10. Proof of the Result about Quadratic Negation Operations

- **General formula:** \( f_\neg(a) = c_0 + c_1 \cdot a + c_{aa} \cdot a^2 \).

- The condition \( f_\neg(0) = 1 \) leads to \( c_0 = 1 \).

- Now, the condition \( f_\neg(1) = 0 \) leads to \( c_{aa} = -1 - c_a \).

- Hence, \( f_\neg(a) = 1 - a^2 + c_a \cdot (a - a^2) \).

- For this expression, the condition \( f_\neg(f_\neg(a)) - a = 0 \) takes the form:

\[
(-1 - 2c_a - c_a^2) \cdot a + (2 + 3c_a - c_a^3) \cdot a^2 + (c_a + 2c_a^2 + c_a^3) \cdot a^3 + (-1 - 3c_a - 3c_a^2 - c_a^3) \cdot a^4 = 0.
\]

- **Comment:** we combined terms corresponding to different powers of \( a \).
11. Proof (cont-d)

• **Reminder:** for all $a$, we have
  
  \[
  (-1 - 2c_a - c_a^2) \cdot a + (2 + 3c_a - c_a^3) \cdot a^2 + \\
  (c_a + 2c_a^2 + c_a^3) \cdot a^3 + (-1 - 3c_a - 3c_a^2 - c_a^3) \cdot a^4 = 0.
  \]

• **Fact:** a polynomial is equal to zero only when all the coefficients are equal to zero.

• **Conclusion:** $-1 - 2c_a - c_a^2 = 0$, $2 + 3c_a - c_a^3 = 0$,
  
  \[
  c_a + 2c_a^2 + c_a^3 = 0, \quad -1 - 3c_a - 3c_a^2 - c_a^3 = 0.
  \]

• First equation means $-(1 + c_a)^2 = 0$, hence $1 + c_a = 0$ and $c_a = -1$.

• For $c_a = -1$, the formula $f_\neg(a) = 1 - a^2 + c_a \cdot (a - a^2)$ turns into $f_\neg(a) = 1 - a^2 - (a - a^2) = 1 - a$.

• So, $f_\neg(a) = 1 - a$ is the only quadratic negation operation. The result is proven.
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