How to Make a Solution to a Territorial Dispute More Realistic: Taking into Account Uncertainty, Emotions, and Step-by-Step Approach

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1. Territorial Division: Formulation of the Problem

- In many real-life situations, there is a dispute over a territory:
  - from conflicts between neighboring farms
  - to conflict between states.

- As a result of a conflict, none of the sides can use this territory efficiently.

- In such situations, it is desirable to come up with a mutually beneficial agreement.

- The current solution is based on the work by the Nobelist J. Nash.

- Nash showed that the best mutually beneficial solution maximizes the product of all the utilities.
2. Nash’s Solution: From a Theoretical Formulation to Practical Recommendations

- Let \( u_i(x) \) be the utility (per area) of the \( i \)-th participant at location \( x \).

- We should select a partition for which the product \( \prod_{i=1}^{n} U_i \) is the largest, where:
  - \( U_i \overset{\text{def}}{=} \int_{S_i} u_i(x) \, dx \) and
  - \( S_i \) is the set allocated to the \( i \)-th participant.

- **Solution:** for some \( t_i \), to assign each location \( x \) to the participant \( i \) with the largest ratio \( u_i(x)/t_i \).

- The parameters \( t_i \) must be determined from the requirement that the \( \prod_{i=1}^{n} U_i \rightarrow \text{max} \).

- For two participants, \( x \in S_1 \) if \( \frac{u_1(x)}{u_2(x)} \geq t \overset{\text{def}}{=} \frac{t_1}{t_2} \).
3. Nash’s Solution: Advantages and Limitations

- Nash’s solution is in perfect agreement with common sense description as formalized by fuzzy logic:
  - we want the first participant to be happy and the second participant to be happy, etc.
  - the degree of happiness of each participant can be described by his or her utility;
  - to represent “and”, it’s reasonable to use one of the most frequently used fuzzy “and”-operations \( a \cdot b \).

- Nash’s solution assumes that we know the exact values \( u_i(x) \).

- In reality, we know the values \( u_i(x) \) only approximately.

- For example, we only know the interval \([u_i(x), \overline{u}_i(x)]\) containing \( u_i(x) \).

- How can we take this uncertainty into account?
4. Nash’s Solution: Limitations (cont-d)

- The above solution assumes that all the sides are making their decisions on a purely rational basis.
- In reality, emotions are often involved.
- How can we take these emotions into account?
- Finally, the above formula proposes an immediate solution.
- But participants often follow step-by-step approach:
  – they first divide a small part,
  – then another part, etc.
- This also needs to be taken into account.
- In this talk, we show how to take all this into account.
5. How to Take Uncertainty into Account

- **Reminder**: we often only know the bounds on $u_i(x)$: 
  $u_i(x) \leq u_i(x) \leq \bar{u}_i(x)$.

- In this case, for each allocation $S_i$, we only know the *interval* $[\underline{U}_i, \overline{U}_i]$ of possible values of utility:

  $\underline{U}_i \overset{\text{def}}{=} \int_{S_i} u_i(x) \, dx; \quad \overline{U}_i \overset{\text{def}}{=} \int_{S_i} \bar{u}_i(x) \, dx$

- In situations with interval uncertainty, decision theory recommends using $\tilde{U}_i = \alpha_i \cdot \overline{U}_i + (1 - \alpha_i) \cdot \underline{U}_i$

  - $\alpha_i \in [0, 1]$ be $i$-th participant’s degree of optimism
  - Similarly, we can use $\tilde{U}_i = \int_{S_i} \tilde{u}_i \, dx$, where  
    $\tilde{u}_i(x) \overset{\text{def}}{=} \alpha_i \cdot \bar{u}_i(x) + (1 - \alpha_i) \cdot u_i(x)$

- We acquire the same formulation, so, we assign each location $x$ to a participant with the largest ratio $\tilde{u}_i(x)/t_i$. 
6. Example

- Let us assume that different participants assign the same utility to all the locations:

\[ u_i(x) = u_j(x) \text{ and } \bar{u}_i(x) = \bar{u}_j(x) \text{ for all } i \text{ and } j. \]

- The only difference between the participants is that they have different optimism degrees \( \alpha_i \neq \alpha_j \).

- Without losing generality, let \( \alpha_i > \alpha_j \).

- Then, the above optimization implies that a point is allocated to \( i \)-th zone if

\[ \frac{\bar{u}(x) - u(x)}{u(x)} \geq t; \text{ so:} \]

  - a more optimistic participant gets the locations with higher uncertainty, while
  - a more pessimistic one get locations with lower uncertainty.
7. How to Take Emotions Into Account

- Emotions mean that instead of maximizing $U_i$, participants maximize $U_{i}^{\text{emo}} = U_i + \sum_j \alpha_{ij} \cdot U_j$.

- Here, $\alpha_{ij}$ describes the feelings of the $i$-th participant towards the $j$-th one:
  - $\alpha_{ij} > 0$ indicate positive feelings;
  - $\alpha_{ij} < 0$ indicate negative feelings;
  - $\alpha_{ij} = 0$ indicate indifference.

- Nash’s solution is to maximize the product $\prod_i U_{i}^{\text{emo}}$.

- Result: for some $t_i$ we assign each location $x$ to a participant with the largest ratio $\tilde{u}_i(x)/t_i$.

- Main difference: the thresholds $t_i$ change.

- A participant with $\alpha_{ij} > 0$ gets fewer locations $x$, since his utility is improved via happiness of others.
8. What If Emotions Are Negative?

- When emotions are negative, i.e., when $\alpha_{ij} < 0$, then, somewhat surprisingly, we get a positive effect.
- Specifically, negative emotions stimulate equality.
- Indeed, all the sides agree to a division only if their utilities $U_i^{\text{emo}}$ are non-negative.
- For example, when $\alpha_{12} = \alpha_{21} = -1$, then:
  - the only way to guarantee that both values $U_1^{\text{emo}} = U_1 - U_2$ and $U_2^{\text{emo}} = U_2 - U_1$ are non-negative is
  - when the values $U_1$ and $U_2$ are equal to each other.
- For other values $\alpha_{ij}$:
  - we do not get $U_i = U_j$, but
  - we get bounds limiting how much $U_i$ and $U_j$ can differ from each other: $0 < c \leq \frac{U_i}{U_j} \leq C$. 
9. Immediate Solution vs. Step-by-Step Approach

- It is desirable to arrive at an immediate solution, but in international affairs, this is not common.
- So, we approach the problem in a location-by-location basis.
- It turns out that the resulting arrangement is not optimal.
- In small vicinities of each location $x$, utility functions $u_i(x)$ do not change much.
- So, we can safely assume that in the vicinity, each utility function is a constant $u_i(x) = u_i$.
- Thus, the utility $U_i = \int_{S_i} u_i(x) \, dx$ is proportional to the area $A_i$ of the set $S_i$: $U_i = u_i \cdot A_i$.
- Then, the optimal division means selecting $A_i$ for which $\sum_{i=1}^{n} A_i = A$ and $\prod_{i=1}^{n} U_i = \prod_{i=1}^{n} (u_i \cdot A_i) \rightarrow \text{max.}$
10. Immediate Solution vs. Step-by-Step (cont-d)

• The optimal division means selecting $A_i$ that maximize

$$\prod_{i=1}^{n} U_i = \prod_{i=1}^{n} (u_i \cdot A_i) \rightarrow \text{max under the constraint}$$

$$\sum_{i=1}^{n} A_i = A.$$ 

• Solution is $A_i = \frac{A}{n}$: each vicinity is divided equally.

• Let us show that this is not optimal.
11. **Step-by-Step Approach: An Example**

- An area $S$ consists of two equal parts:
  - the first part is useless for the 1st participant, but valuable to the second one
  - the second part is valuable for the first participant, but useless for the second one
- A clear optimal solution is to allocate:
  - the first part to the second participant and
  - the second part to the first participant.
- In a step-by-step solution, we divide each part equally.
- As a result, each participant gets only half of the area which is useful to this participant (non-optimal)
- Recommendation: try to solve the problem as a whole, and avoid step-by-step solutions.
12. Auxiliary Question: Should we Divide in the First Place?

- At first glance, it may seem that:
  - instead of dividing a disputed territory,
  - it is desirable to show a brotherly/sisterly spirit and control it jointly.
- This may work at times.
- However, as we show, in general, this strategy will lead to a suboptimal solution: in almost all cases,
  - the product of utilities is the largest when we divide
  - and not when we share the control.
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14. How to Take Emotions Into Account: Proof

- Let us assume that in the optimal division, location $x_0$ is allocated to the $i_0$-th participant.
- This means that:
  - if re-allocate a small neighborhood of $x_0$ (of area $\delta$) to participant $j_0$,
  - then the product $\prod_i U_{i_0}^{\text{emo}}$ will decrease;
  - so its logarithm $L = \ln \left( \prod_i U_{i_0}^{\text{emo}} \right)$ also decreases.
- Here, $U_{i_0} = \int_{S_{i_0}} u_{i_0}(x) \, dx$ decreases by
  $$\Delta U_{i_0} = -u_{i_0}(x_0) \cdot \delta.$$
- $U_{j_0} = \int_{S_{j_0}} u_{j_0}(x) \, dx$ increases by $\Delta U_{j_0} = u_{j_0}(x_0) \cdot \delta$.
- All other $U_i$ remain unchanged: $\Delta U_i = 0$ for $i \neq i_0, j_0$. 
15. Emotions: Proof (cont-d)

- Thus, for the changes $\Delta U_{i}^{emo}$:
  - for $i = i_0$, we have $\Delta U_{i_0}^{emo} = \Delta U_{i_0} + \alpha_{i_0 j_0} \cdot \Delta U_{j_0}$;
  - for $i = j_0$, we have $\Delta U_{j_0}^{emo} = \Delta U_{i_0} + \alpha_{j_0 i_0} \cdot \Delta U_{i_0}$;
  - for all other $i$, $\Delta U_{i}^{emo} = \alpha_{ii_0} \cdot \Delta U_{i_0} + \alpha_{ij_0} \cdot \Delta U_{j_0}$.

- For $L = \sum_i \ln(U_{i}^{emo})$, we have $\Delta L = \sum_i \frac{\Delta U_{i}^{emo}}{U_{i}^{emo}}$.

- So $\Delta L \leq 0$ takes the form $a \cdot u_{i_0}(x_0) + b \cdot u_{j_0}(x_0) \leq 0$, or, equivalently, $\frac{u_{i_0}(x_0)}{u_{j_0}(x_0)} \geq c$.

- If $x_0$ was originally allocated to $j_0$, we get same inequality with $-\delta$ instead of $\delta$, so $\frac{u_{i_0}(x_0)}{u_{j_0}(x_0)} \leq c$.

- Thus, in the optimal partition, each participant $i$ indeed gets all locations for which $u_i(t)/t_i$ is the largest.