Orevkov, Khalilin, and Quantum Field Theory: How Constructive Mathematics Can Help Physics

Olga Kosheleva and Vladik Kreinovich

University of Texas at El Paso
El Paso, Texas 79968, USA
olgak@utep.edu, vladik@utep.edu
http://www.cs.utep.edu/vladik/olgavita.html
http://www.cs.utep.edu/vladik
1. Orevkov’s 1972 Results

- In 1972, Vladimir Orevkov presented a talk on constructive complex analysis at LOMI.
- The main results from this talk were published in 1974.
- In that paper, he provided new more explicit constructive proofs of basic results of complex analysis:
  - that a function is differentiable iff it can be expanded in Taylor series at each point,
  - that two such (analytical) functions are equal if they coincide on a non-finite compact set, and
  - that it is possible to constructively find all the roots of such function on each bounded domain.
- These results were previously proved by Vladimir Lifschitz in a more implicit way.
2. Orevkov’s 1972 Results

- As usual, many results from classical (non-constructive) mathematics turned out to be constructively true.

- Some results from classical mathematics turned out to be constructively false, in the sense that:
  - while there is a classical existence theorem,
  - no general algorithm for constructing the corresponding object is possible.
3. Can This Result Help Physics?

- This talk attracted attention of Leonid Khalfin, Orevkov’s LOMI colleague interested in physics applications.
- Khalfin asked whether constructive mathematics can solve a problem related to physics use of complex #s.
- On macro-level, we observe many non-smooth and even discontinuous phenomena:
  - earthquakes,
  - phase transitions, etc.
- However, on the micro-level, all equations and all phenomena are smooth – and even analytical.
- Some of these phenomena are very fast – so we perceive them as discontinuous.
4. Can This Result Help Physics (cont-d)

- For complex numbers, smoothness means analyticity.
- Analyticity has been successfully used in quantum field theory.
- For example, to compute the values of some integral expressions, it is convenient to use the fact that:
  - for an analytical function,
  - a contour integral over a closed loop is 0:
    \[ \int_{\gamma} f(z) \, dz = 0 \]
  - or it is equal to an explicit expression in terms of the poles.
5. Can This Result Help Physics (cont-d)

- Thus, by using a loop $[-N, N] \cup \gamma'$, we can:
  - replace a difficult-to-compute integral over real numbers $\int_{-N}^{N} f(x) \, dx$
  - with an easier-to-compute integral over the complex values $\int_{\gamma'} f(z) \, dz$.

- This idea – mostly pioneered by Nikolai Bogolyubov – led to many successful applications.

- This “macro” analyticity has been confirmed by many experiments and makes perfect physical sense.
6. Can This Result Help Physics (cont-d)

• The problem is that in traditional mathematics:
  – such “macro” analyticity is equivalent to “micro” one,
  – that the corresponding dependencies can be expanded in Taylor series:

\[ f(z) = a_0 + a_1 \cdot (z-z_0) + a_2 \cdot (z-z_0)^2 + \ldots + a_n \cdot (z-z_0)^n + \ldots \]

• In the opinion of physicists, however:
  – this “micro” analyticity does not make direct physical sense,
  – since on the micro level, quantum uncertainty makes exact measurements impossible.
7. Can This Result Help Physics (cont-d)

• From this viewpoint, it is desirable to come up with a model in which:
  – physically meaningful macro analyticity is present, but
  – physically meaningless micro analyticity is not.

• Khalfin hoped that:
  – this “thornless rose” effect can be achieved
  – if we consider constructive mathematics instead of the traditional one.
8. Can This Result Help Physics (cont-d)

- In the early 1970s, this hope did not materialize, since:
  - as Errett Bishop has shown in his 1967 book (and as Vladimir Lifschitz pointed to Khalffin),
  - the fact that macro analyticity implies micro one can be proven in constructive mathematics as well.

- Indeed, once we know \( f(z) \), we can determine all the coefficients \( a_n \) as
  \[
  a_n = \frac{1}{2\pi i} \int_{\gamma} \frac{f(z)}{(z - z_0)^{n+1}} \, dz.
  \]

- And there are known algorithms for computing an integral of a computable function.
9. Problem Revisited

• Bishop’s derivation is based on the usual constructive mathematics.

• In this approach, existence of an object means, in effect:
  – the existence of an algorithm producing more and more accurate approximations to this object,
  – irrespective to how long this algorithm may take.

• A more realistic idea is to only allow feasible (= polynomial-time) algorithms are allowed.

• It turns out that in this case, Khalfin’s dream can be materialized.
10. Problem Revisited (cont-d)

- Indeed: while there exists an algorithm computing:
  – for each computable macro analytical function,
  – all the terms in its Taylor series expansion.
- However, the computation time of this algorithm seems to grow exponentially with the number $n$ of the term.
- Let us provide arguments for this conclusion.
11. **Explanation**

- We have a computable function $f(z)$.
- This means that we can, given $z$, compute $f(z)$.
- For simplicity, we can also assume that we know the upper bound $D$ on $|f'(z)| \leq D$.
- Computation of the $n$-th Taylor coefficient $a_n$ is based on the formula

$$a_n = \frac{1}{2\pi i} \int_{\gamma} \frac{f(z)}{(z - z_0)^{n+1}} \, dz.$$  

- Here, the simplest possible loop $\gamma$ around the point $z_0$ is a circle of some small radius $r < 1$.
- For this loop, $|z - z_0| = r$.
- We want to compute $a_n$ with a given accuracy $\varepsilon > 0$.
- This means that we need to compute the corresponding integral with accuracy $\varepsilon' = 2\pi \cdot \varepsilon$. 

12. Explanation (cont-d)

- A natural way to compute an integral \( \int g(z) \, dz \) is to consider the corresponding integral sum
  \[ \sum g(z_i) \cdot \Delta z, \text{ with } |z_{i+1} - z_i| = h \text{ for some small } h. \]

- In this approximation, we approximate \( g(z) \) with \( g(z_i) \) on each arc of length \( h \) for which \( |z - z_i| \leq h/2 \).

- The inaccuracy of this approximation is
  \[ |g(z) - g(z_i)| \leq \left( \max \limits_z |g'(z)| \cdot |z - z_i| \right) \leq \max \limits_z |g'(z)| \cdot (h/2). \]

- Here, \( g(z) = \frac{f(z)}{(z - z_0)^{n+1}} \approx \frac{f(z)}{\gamma^{n+1}}. \)

- Thus, \( \max \limits_z |g'(z)| \leq \frac{\max \limits_z |f'(z)|}{\gamma^{n+1}} = \frac{D}{\gamma^{n+1}}. \)
13. **Explanation (cont-d)**

- So, the approximation accuracy is \( \frac{D}{r^{n+1}} \cdot (h/2) \).

- To get accuracy \( \varepsilon' \), we need to take \( h \) for which
  \[
  \frac{D}{r^{n+1}} \cdot (h/2) = \varepsilon', \quad \text{i.e.,} \quad h = 2 \frac{\varepsilon'}{D} \cdot r^{n+1}.
  \]

- The whole loop \( \gamma \) of length \( 2\pi \cdot r \) should be covered by intervals of length \( h \).

- These intervals correspond to values \( z_i \) at which we compute \( f(z) \).

- Thus, we need to compute \( f(z) \) for \( N = \frac{2\pi \cdot r}{h} \) points.

- Substituting the above expression for \( h \), we conclude that we need to compute \( f(z) \) at
  \[
  N = \frac{2\pi \cdot r \cdot D}{2\varepsilon' \cdot r^{n+1}} \sim r^{-n} \text{ points}.
  \]
14. Explanation (cont-d)

- We have shown that we need to compute $f(z)$ at

$$N = \frac{2\pi \cdot r \cdot D}{2\varepsilon' \cdot r^{n+1}} \sim r^{-n} \text{ points.}$$

- Since $r < 1$, this number indeed grows exponentially with $n$.  

15. Possible Applications

- This result will probably be of interest to theoreticians like Khalfin interested:
  - in providing physical theories
  - with physically meaningful mathematical foundations.

- This result may also have practical applications if we take into account that:
  - many times when we encountered a physical process whose properties are difficult to compute,
  - it became possible to use this process to speed up computations.

- Successes of quantum computing are the latest example of this phenomenon.
16. Possible Applications (cont-d)

- From this viewpoint:
  - maybe measurement of the corresponding Taylor coefficients
  - can lead to yet another efficient quantum computing scheme?
17. Bibliography


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