

From Gauging Accuracy of Quantity Estimates to Gauging Accuracy and Resolution of Field Measurements: Geophysical Case Study

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Outline

Situations in Which...

Situations in Which...

What If a Model Is...

Situations in Which...

Example of Asymmetry

Situations with No...

Towards Optimal...

Towards Optimal...

Acknowledgments

Home Page

Title Page

⏪

⏩

◀

▶

Page 1 of 14

Go Back

Full Screen

Close

1. Traditional Applications of Interval Computations: Reminder

- *Objective:* estimate a difficult-to-measure quantity y .
- *Approach:* measure quantities x_1, \dots, x_n related to x_i by a known dependence $y = f(x_1, \dots, x_n)$.
- *Fact:* measurements are never absolutely accurate.
- *Conclusion:* the measurement results \tilde{x}_i are, in general, different from the actual (unknown) values x_i .
- *Conclusion:* the result $\tilde{y} = f(\tilde{x}_1, \dots, \tilde{x}_n)$ of data processing differs from $y = f(x_1, \dots, x_n)$.
- *Frequent situation:* we only know the upper bound Δ_i on the measurement errors $\Delta x_i \stackrel{\text{def}}{=} \tilde{x}_i - x_i$: $|\Delta x_i| \leq \Delta_i$.
- *So:* we only know that $x_i \in \mathbf{x}_i \stackrel{\text{def}}{=} [\tilde{x}_i - \Delta_i, \tilde{x}_i + \Delta_i]$.
- *Interval computations:* find the corresponding range $\mathbf{y} = \{f(x_1, \dots, x_n) : x_i \in \mathbf{x}_i\}$ of y .

Outline

Situations in Which ...

Situations in Which ...

What If a Model Is ...

Situations in Which ...

Example of Asymmetry

Situations with No ...

Towards Optimal ...

Towards Optimal ...

Acknowledgments

Home Page

Title Page



Page 2 of 14

Go Back

Full Screen

Close

2. In Practice, the Situation is Often More Complex

- *Dynamics*: we measure the values $v(t)$ of a quantity v at a certain moment of time t .
- *Spatial dependence*: we measure the value $v(x, t)$ at a certain location x .
- *Geophysical example*: we are interested in the values of the density at different locations and at different depth.
- *Traditional*: uncertainty in the measured value, $\tilde{v} \approx v$.
- *New*: uncertainty in location x , $\tilde{x} \approx x$.
- *Additional uncertainty*: the sensor picks up the “average” value of v at locations close to \tilde{x} .
- *Question*: how to describe and process the new uncertainty (*resolution*)?

Outline

Situations in Which...

Situations in Which...

What If a Model Is...

Situations in Which...

Example of Asymmetry

Situations with No...

Towards Optimal...

Towards Optimal...

Acknowledgments

Home Page

Title Page

◀

▶

◀

▶

Page 3 of 14

Go Back

Full Screen

Close

3. Outline

- *Question* (reminder): how to describe and process uncertainty both
 - in the measured value \tilde{v} and
 - in the spatial resolution \tilde{x} ?
- *Natural idea*: the answer depends on what we know about the spatial resolution.
- *Possible situations*:
 - we know exactly how the measured values \tilde{v}_i are related to $v(x)$, i.e., $\tilde{v}_i = \int w_i(x) \cdot v(x) dx + \Delta v_i$;
 - we only know the upper bound δ on the location error $\tilde{x} - x$ (this is similar to the interval case);
 - we do not even know δ .
- *What we do*: describe how to process all these types of uncertainty.

4. Situations in Which We Have Detailed Knowledge

- *Fact:* all our information about $v(x)$ is contained in the measured values \tilde{v}_i .
- *Linearity assumption:* $\tilde{v}_i = v_i + \Delta v_i$, where:
 - we have $v_i \stackrel{\text{def}}{=} \int w_i(x) \cdot v(x) dx$; and
 - Δv_i is the measurement error; e.g., $|\Delta v_i| \leq \Delta_i$.
- *Comment:* v_i can be viewed as the value of $v(x)$ at a “fuzzy” point characterized by uncertainty $w_i(x)$.
- *Description of the situation:* we know the weights $w_i(x)$.
- *Find:* range $[y, \bar{y}]$ for $y \stackrel{\text{def}}{=} \int w(x) \cdot v(x) dx$.
- *LP solution:* minimize (maximize) $\int w(x) \cdot v(x) dx$ under the constraints

$$\underline{v}_i \stackrel{\text{def}}{=} \tilde{v}_i - \Delta_i \leq \int w_i(x) \cdot v(x) dx \leq \bar{v}_i \stackrel{\text{def}}{=} \tilde{v}_i + \Delta_i.$$

Outline

Situations in Which ...

Situations in Which ...

What If a Model Is ...

Situations in Which ...

Example of Asymmetry

Situations with No ...

Towards Optimal ...

Towards Optimal ...

Acknowledgments

Home Page

Title Page

◀

▶

◀

▶

Page 5 of 14

Go Back

Full Screen

Close

5. Situations With Detailed Knowledge (cont-d)

- *Reminder*: find the range of $\int w(x) \cdot v(x) dx$ when $\underline{v}_i \leq \int w_i(x) \cdot v(x) dx \leq \bar{v}_i$.
- *General case*: when no bounds on $v(x)$, bounds are infinite – unless $w(x)$ is a linear combination of $w_i(x)$.
- *In practice* (e.g., in geophysics): $v(x) \geq 0$.
- *Similar*: imprecise probabilities (Kuznetsov, Walley).

- *Solution*: dual LP problem provides guaranteed bounds

$$\underline{v} = \sup \left\{ \sum y_i \cdot \underline{v}_i : \sum y_i \cdot w_i(x) \leq w(x) \right\};$$

$$\bar{v} = \inf \left\{ \sum y_i \cdot \bar{v}_i : w(x) \leq \sum y_i \cdot w_i(x) \right\}.$$

- *Easier* than in IP: $w_i(x)$ are localized, and we often have ≤ 2 non-zero $w_i(x)$ at each x .
- *Piece-wise linear* $w_i(x)$ and $w(x)$ – sufficient to check inequality at endpoints.

Outline

Situations in Which ...

Situations in Which ...

What If a Model Is ...

Situations in Which ...

Example of Asymmetry

Situations with No ...

Towards Optimal ...

Towards Optimal ...

Acknowledgments

Home Page

Title Page



Page 6 of 14

Go Back

Full Screen

Close

6. Situations in Which We Only Know Upper Bounds

- *Situation*: we only know;
 - the upper bound Δ on the measurement inaccuracy
 $\Delta v \stackrel{\text{def}}{=} \tilde{v} - v: |\Delta v| \leq \Delta$, and
 - the upper bound δ on the location error
 $\Delta x \stackrel{\text{def}}{=} \tilde{x} - x: |\Delta x| \leq \delta$.
- *Consequence*: the measured value \tilde{v} is Δ -close to a convex combination of values $v(x)$ for x s.t. $\|x - \tilde{x}\| \leq \Delta x$.
- *Conclusion*: $\underline{v}_\delta(\tilde{x}) - \Delta \leq \tilde{v} \leq \bar{v}_\delta(\tilde{x}) + \Delta$, where:
 - $\underline{v}_\delta(\tilde{x}) \stackrel{\text{def}}{=} \inf\{v(x) : \|x - \tilde{x}\| \leq \delta\}$, and
 - $\bar{v}_\delta(\tilde{x}) \stackrel{\text{def}}{=} \sup\{v(x) : \|x - \tilde{x}\| \leq \delta\}$.
- *Fact*: measurement errors are random.
- *So*: it makes sense to only consider *essential* ess inf and ess sup (i.e., inf and sup modulo measure 0 sets).

Outline

Situations in Which...

Situations in Which...

What If a Model Is...

Situations in Which...

Example of Asymmetry

Situations with No...

Towards Optimal...

Towards Optimal...

Acknowledgments

Home Page

Title Page

◀

▶

◀

▶

Page 7 of 14

Go Back

Full Screen

Close

7. What If a Model Is Only Known With Interval Uncertainty

- *Reminder:* we can tell when an observation (\tilde{v}, \tilde{x}) is consistent with a model $v(x)$:

$$\underline{v}_\delta(\tilde{x}) - \Delta \leq \tilde{v} \leq \overline{v}_\delta(\tilde{x}) + \Delta.$$

- *Fact:* in real life, we rarely have an *exact* model $v(x)$.
- *Usually:* we have *bounds* on $v(x)$, i.e., an interval-valued model $\mathbf{v}(x) = [v^-(x), v^+(x)]$.
- *Question:* when is an observation (\tilde{v}, \tilde{x}) consistent with an *interval-valued* model?
- *General answer:* when the observation (\tilde{v}, \tilde{x}) is consistent with *one* of the models $v(x) \in \mathbf{v}(x)$.
- *A checkable answer:* an observation (\tilde{v}, \tilde{x}) is consistent with an interval-valued model $[v^-(x), v^+(x)]$ when

$$\underline{v}_\delta^-(\tilde{x}) - \Delta \leq \tilde{v} \leq \overline{v}_\delta^+(\tilde{x}) + \Delta.$$

Outline

Situations in Which...

Situations in Which...

What If a Model Is...

Situations in Which...

Example of Asymmetry

Situations with No...

Towards Optimal...

Towards Optimal...

Acknowledgments

Home Page

Title Page

◀

▶

◀

▶

Page 8 of 14

Go Back

Full Screen

Close

8. Situations in Which We Only Know Upper Bounds (cont-d)

- *Fact:* the actual $v(x)$ is often continuous.
- *Case of continuous $v(x)$:* we can simplify the above criterion.
- *Simplification:* the set \tilde{m} of all measurement results (\tilde{x}, \tilde{x}) is consistent with the model $v(x)$ iff

$\forall(\tilde{v}, \tilde{x}) \in \tilde{m} \exists(v(x), x) \in v((\tilde{v}, \tilde{x}) \text{ is } (\Delta, \delta)\text{-close to } (v(x), x)),$
i.e., $|\tilde{v} - v| \leq \Delta$ and $\|x - \tilde{x}\| \leq \delta$.

- *Hausdorff metric:* $d_H(A, B) \leq \varepsilon$ means that:
 $\forall a \in A \exists b \in B (d(a, b) \leq \varepsilon)$ and $\forall b \in B \exists a \in A (d(a, b) \leq \varepsilon)$.
- *Conclusion:* we have an *asymmetric* version of Hausdorff metric (“quasi-metric”).

Outline

Situations in Which...

Situations in Which...

What If a Model Is...

Situations in Which...

Example of Asymmetry

Situations with No...

Towards Optimal...

Towards Optimal...

Acknowledgments

Home Page

Title Page





Page 9 of 14

Go Back

Full Screen

Close

9. Example of Asymmetry

- *Case 1:* 
 - *The actual field:* $v(0) = 1$ and $v(x) = 0$ for $x \neq 0$;
 - *Measurement results:* all zeros, i.e., $\tilde{v} = 0$ for all \tilde{x} .
 - *Conclusion:* all the measurements are *consistent* with the model.
 - *Reason:* the value $\tilde{v} = 0$ for $\tilde{x} = 0$ is consistent with $v(x) = 0$ for $x = \delta$ s.t. $|\tilde{x} - x| \leq \delta$.
- *Case 2:* 
 - *The actual field:* all zeros, i.e., $v(x) = 0$ for all x .
 - *Measurement results:* $\tilde{v} = 1$ for $\tilde{x} = 0$, and $\tilde{v} = 0$ for all $\tilde{x} \neq 0$.
 - *Conclusion* (for $\Delta < 1$): the measurement $(1, 0)$ is *inconsistent* with the model.
 - *Reason:* for all x which are δ -close to $\tilde{x} = 0$, we have $v(x) = 0$ hence we should have $|\tilde{x} - v(x)| = |\tilde{x}| \leq \Delta$.

Outline

Situations in Which...

Situations in Which...

What If a Model Is...

Situations in Which...

Example of Asymmetry

Situations with No...

Towards Optimal...

Towards Optimal...

Acknowledgments

Home Page

Title Page

◀

▶

◀

▶

Page 10 of 14

Go Back

Full Screen

Close

10. Situations with No Information about Location Accuracy

- *Example:* when we solve the seismic inverse problem to find the velocity distribution.
- *Natural heuristic idea:*
 - add a perturbation of size Δ_0 to the reconstructed field $\tilde{v}(x)$;
 - simulate the new measurement results;
 - apply the same algorithm to the simulated results, and reconstruct the new field $\tilde{v}_{\text{new}}(x)$.
- *Case 1:* perturbations are *not visible* in $\tilde{v}_{\text{new}}(x) - \tilde{v}(x)$.
- *So:* details of size Δ_0 *cannot* be reconstructed: $\delta > \Delta_0$.
- *Case 2:* perturbations are *visible* in $\tilde{v}_{\text{new}}(x) - \tilde{v}(x)$.
- *So:* details of size Δ_0 *can* be reconstructed: $\delta \leq \Delta_0$.

Outline

Situations in Which...

Situations in Which...

What If a Model Is...

Situations in Which...

Example of Asymmetry

Situations with No...

Towards Optimal...

Towards Optimal...

Acknowledgments

Home Page

Title Page

◀

▶

◀

▶

Page 11 of 14

Go Back

Full Screen

Close

11. Towards Optimal Selection of Perturbations

- *Fact:* since perturbations are small, we can safely linearize their effects.
- *Conclusion:*
 - based on the results of perturbations $e_1(x), \dots, e_k(x)$,
 - we can get the results of any linear combination

$$e(x) = c_1 \cdot e_1(x) + \dots + c_k \cdot e_k(x).$$

- *Fact:* usually, there is no preferred spatial location.
- *Conclusion:* we can choose different locations as origins ($x = 0$) of the coordinate system.
- *Natural requirement:* the results of perturbations should not change if we change the origin $x = 0$.

Outline

Situations in Which...

Situations in Which...

What If a Model Is...

Situations in Which...

Example of Asymmetry

Situations with No...

Towards Optimal...

Towards Optimal...

Acknowledgments

Home Page

Title Page

◀◀

▶▶

◀

▶

Page 12 of 14

Go Back

Full Screen

Close

12. Towards Optimal Perturbations (cont-d)

- *Reminder:* the class of perturbations should not change when we change the origin $x = 0$.
- *Fact:* in new coordinates, $x_{\text{new}} = x + x_0$.
- *Conclusion:* the set $\{c_1 \cdot e_1(x) + \dots + c_k \cdot e_k(x)\}$ must be shift-invariant:
$$e_i(x + x_0) = \sum_{j=1}^k c_{ij}(x_0) \cdot e_j(x).$$
- *When* $x_0 \rightarrow 0$, we get a system of linear differential equations with constant coefficients.
- *General solution:* linear combination of expressions $\exp(\sum a_i \cdot x_i)$ with complex a_i .
- *Fact:* perturbations must be uniformly small.
- *So:* the only *bounded* perturbations are linear combinations of sinusoids.
- *Conclusion:* use sinusoidal perturbations.

Outline

Situations in Which...

Situations in Which...

What If a Model Is...

Situations in Which...

Example of Asymmetry

Situations with No...

Towards Optimal...

Towards Optimal...

Acknowledgments

Home Page

Title Page

◀

▶

◀

▶

Page 13 of 14

Go Back

Full Screen

Close

13. Acknowledgments

This work was supported in part:

- by NSF Cyber-Share grant HRD-0734825:
 - A Center for Sharing Cyberresources to Advance Science and Education;
- by Grant 1 T36 GM078000-01 from the National Institutes of Health:
 - Enhancement of Qualitative Science;
- by the Japan Advanced Institute of Science and Technology (JAIST) Int'l Joint Research Grant 2006-08.

[Traditional . . .](#)

[In Practice, the . . .](#)

[Outline](#)

[Situations in Which . . .](#)

[Situations in Which . . .](#)

[What If a Model Is . . .](#)

[Situations in Which . . .](#)

[Example of Asymmetry](#)

[Situations with No . . .](#)

[Towards Optimal . . .](#)

[Towards Optimal . . .](#)

[Acknowledgments](#)

[Home Page](#)

[Title Page](#)



Page 14 of 14

[Go Back](#)

[Full Screen](#)

[Close](#)

[Quit](#)