

Extending Algorithmic Randomness to the Algebraic Approach to Quantum Physics: Kolmogorov Complexity and Quantum Logics

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1. Physicists Usually Assume that Events with a Very Small Probability Cannot Occur

- *Known phenomenon*: Brownian motion.
- *In principle*: due to Brownian motion, a kettle placed on a cold stove can start boiling.
- *The probability* of this event is positive but very small.
- *A mathematician* would say that this event is possible but rare.
- *A physicist* would say that this event is simply not possible.
- *It is desirable*: to formalize this intuition of physicists.

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2. Kolmogorov's Definition of Algorithmic Randomness

- *Kolmogorov*: proposed a new definition of a random sequence, a definition that separates
 - physically random binary sequences, e.g.:
 - * sequences that appear in coin flipping experiments,
 - * sequences that appear in quantum measurements
 - from sequence that follow some pattern.
- *Intuitively*: if a sequence s is random, it satisfies all the probability laws.
- *What is a probability law*: a statement S which is true with probability 1: $P(S) = 1$.
- *Conclusion*: to prove that a sequence is not random, we must show that it does not satisfy one of these laws.

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3. Kolmogorov's Definition of Algorithmic Randomness (cont-d)

- *Reminder:* a sequence s is not random if it does not satisfy one of the probability laws S .
- *Equivalent statement:* s is not random if $s \in C$ for a (definable) set C ($= -S$) with $P(C) = 0$.
- *Resulting definition* (Kolmogorov, Martin-Löf): s is random if $s \notin C$ for all definable C with $P(C) = 0$.
- *Consistency proof:*
 - Every definable set C is defined by a finite sequence of symbols (its definition).
 - Since there are countably many sequences of symbols, there are countably many definable sets C .
 - So, the complement $-\mathcal{R}$ to the class \mathcal{R} of all random sequences also has probability 0.

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4. Towards a More Physically Adequate Versions of Kolmogorov Randomness

- *Problem:* the 1960s Kolmogorov's definition only explains why events with probability 0 do not happen.
- *What we need:* formalize the physicists' intuition that events with very small probability cannot happen.
- *Seemingly natural formalization:* there exists the “smallest possible probability” p_0 such that:
 - if the computed probability p of some event is larger than p_0 , then this event can occur, while
 - if the computed probability p is $\leq p_0$, the event cannot occur.
- *Example:* a fair coin falls heads 100 times with prob. 2^{-100} ; it is impossible if $p_0 \geq 2^{-100}$.

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5. The Above Formalization of Randomness is Not Always Adequate

- *Problem:* every sequence of heads and tails has exactly the same probability.
- *Corollary:* if we choose $p_0 \geq 2^{-100}$, we will thus exclude all sequences of 100 heads and tails.
- However, anyone can toss a coin 100 times.
- This proves that some such sequences are physically possible.
- *Similar situation:* Kyburg's lottery paradox:
 - in a big (e.g., state-wide) lottery, the probability of winning the Grand Prize is very small;
 - a reasonable person should not expect to win;
 - however, some people do win big prizes.

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6. New Definition of Randomness

- *Example:* height:
 - if height is ≥ 6 ft, it is still normal;
 - if instead of 6 ft, we consider 6 ft 1 in, 6 ft 2 in, etc., then $\exists h_0$ s.t. everyone taller than h_0 is abnormal;
 - we are not sure what is h_0 , but we are sure such h_0 exists.
- *General description:* on the universal set U , we have sets $A_1 \supseteq A_2 \supseteq \dots \supseteq A_n \supseteq \dots$ s.t. $P(\cap A_n) = 0$.
- *Example:* $A_1 =$ people w/height ≥ 6 ft, $A_2 =$ people w/height ≥ 6 ft 1 in, etc.
- A set $\mathcal{R} \subseteq U$ is called a *set of random elements* if
 \forall definable sequence of sets A_n for which $A_n \supseteq A_{n+1}$ for all n and $P(\cap A_n) = 0$, $\exists N$ for which $A_N \cap \mathcal{R} = \emptyset$.

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7. Definable: Mathematical Comment

- *What is definable:*
 - let \mathcal{L} be a theory,
 - let $P(x)$ be a formula from the language of the theory \mathcal{L} , with one free variable x
 - so that the set $\{x \mid P(x)\}$ is defined in \mathcal{L} .

We will then call the set $\{x \mid P(x)\}$ \mathcal{L} -definable.

- *How to deal with definable sets:*
 - Our objective is to be able to make mathematical statements about \mathcal{L} -definable sets.
 - Thus, we must have a stronger theory \mathcal{M} in which the class of all \mathcal{L} -definable sets is a countable set.
 - One can prove that such \mathcal{M} always exists.

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8. Coin Example

- Universal set $U = \{H, T\}^{\mathbb{N}}$
- Here, A_n is the set of all the sequences that start with n heads.
- The sequence $\{A_n\}$ is decreasing and definable, and its intersection has probability 0.
- Therefore, for every set \mathcal{R} of random elements of U , there exists an integer N for which $A_N \cap \mathcal{R} = \emptyset$.
- This means that if a sequence $s \in \mathcal{R}$ is random and starts with N heads, it must consist of heads only.
- *In physical terms:* it means that
 a random sequence cannot start with N heads.
- This is exactly what we wanted to formalize.

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9. From Random to Typical (Not Abnormal)

- *Fact:* not all solutions to the physical equations are physically meaningful.
- *Example 1:* when a cup breaks into pieces, the corresponding trajectories of molecules make physical sense.
- *Example 2:* when we reverse all the velocities, we get pieces assembling themselves into a cup.
- *Physical fact:* this is physically impossible.
- *Mathematical fact:* the reverse process satisfies all the original (T-invariant) equations.
- *Physicist's explanation:* the reversed process is non-physical since its initial conditions are “degenerate”.
- *Clarification:* once we modify the initial conditions even slightly, the pieces will no longer get together.

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10. New Definition of Non-Abnormality

- *Example:* height:
 - if height is ≥ 6 ft, it is still normal;
 - if instead of 6 ft, we consider 6 ft 1 in, 6 ft 2 in, etc., then $\exists h_0$ s.t. everyone taller than h_0 is abnormal;
 - we are not sure what is h_0 , but we are sure such h_0 exists.
- *General description:* on the universal set U , we have sets $A_1 \supseteq A_2 \supseteq \dots \supseteq A_n \supseteq \dots$ s.t. $\bigcap A_n = \emptyset$.
- *Example:* $A_1 =$ people w/height ≥ 6 ft, $A_2 =$ people w/height ≥ 6 ft 1 in, etc.
- A set $\mathcal{T} \subseteq U$ is called a *set of typical elements* if
 \forall definable sequence of sets A_n for which $A_n \supseteq A_{n+1}$
for all n and $\bigcap A_n = \emptyset$, $\exists N$ for which $A_N \cap \mathcal{T} = \emptyset$.

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11. Coin Example

- Universal set $U = \{\text{H}, \text{T}\}^{\mathbb{N}}$
- Here, A_n is the set of all the sequences that start with n heads and has a tail.
- The sequence $\{A_n\}$ is decreasing and definable, and its intersection is empty.
- Therefore, for every set \mathcal{T} of typical elements of U , there exists an integer N for which $A_N \cap \mathcal{T} = \emptyset$.
- This means that if a sequence $s \in \mathcal{T}$ is random (has both heads and tails) and starts with N heads, it must consist of heads only.
- *In physical terms:* it means that
 a random sequence cannot start with N heads.
- This is exactly what we wanted to formalize.

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12. Consistency Proof

- *Statement:* $\forall \varepsilon > 0$, there exists a set \mathcal{T} of typical elements for which $\underline{P}(\mathcal{T}) \geq 1 - \varepsilon$.
- There are countably many definable sequences $\{A_n\}$: $\{A_n^{(1)}\}, \{A_n^{(2)}\}, \dots$
- For each k , $P\left(A_n^{(k)}\right) \rightarrow 0$ as $n \rightarrow \infty$.
- Hence, there exists N_k for which $P\left(A_{N_k}^{(k)}\right) \leq \varepsilon \cdot 2^{-k}$.
- We take $\mathcal{T} \stackrel{\text{def}}{=} \bigcup_{k=1}^{\infty} A_{N_k}^{(k)}$. Since $P\left(A_{N_k}^{(k)}\right) \leq \varepsilon \cdot 2^{-k}$, we have
$$\overline{P}\left(\bigcup_{k=1}^{\infty} A_{N_k}^{(k)}\right) \leq \sum_{k=1}^{\infty} P\left(A_{N_k}^{(k)}\right) \leq \sum_{k=1}^{\infty} \varepsilon \cdot 2^{-k} = \varepsilon.$$
- Hence, $\underline{P}(\mathcal{T}) = 1 - \overline{P}\left(\bigcup_{k=1}^{\infty} A_{N_k}^{(k)}\right) \geq 1 - \varepsilon$.

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13. Ill-Posed Problems: In Brief

- Main *objectives* of science:
 - *guaranteed* estimates for physical quantities;
 - *guaranteed* predictions for these quantities.
- *Problem*: estimation and prediction are ill-posed.
- *Example*:
 - measurement devices are inertial;
 - hence suppress high frequencies ω ;
 - so $\varphi(x)$ and $\varphi(x) + \sin(\omega \cdot t)$ are indistinguishable.
- *Existing approaches*:
 - statistical regularization (filtering);
 - Tikhonov regularization (e.g., $|\dot{x}| \leq \Delta$);
 - expert-based regularization.
- *Main problem*: no guarantee.

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14. On “Not Abnormal” Solutions, Problems Become Well-Posed

- *State estimation – an ill-posed problem:*
 - *Measurement f :*
state $s \in S \rightarrow$ observation $r = f(s) \in R$.
 - *In principle, we can reconstruct $r \rightarrow s$:*
as $s = f^{-1}(r)$.
 - *Problem:* small changes in r can lead to huge changes in s (f^{-1} not continuous).
- *Theorem:*
 - Let S be a definably separable metric space.
 - Let T be a set of all not abnormal elements of S .
 - Let $f : S \rightarrow R$ be a continuous 1-1 function.
 - Then, the inverse mapping $f^{-1} : R \rightarrow S$ is *continuous* for every $r \in f(T)$.

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15. Proof of Well-Posedness

- *Known:* if a f is continuous and 1-1 on a compact, then f^{-1} is also continuous.
- *Reminder:* S is compact if and only if it is closed and for every ε , it has a finite ε -net.
- *Given:* S is definably separable.
- *Means:* \exists def. s_1, \dots, s_n, \dots everywhere dense in S .
- *Solution:* take $A_n \stackrel{\text{def}}{=} \bigcup_{i=1}^n B_\varepsilon(s_i)$.
- Since s_i are everywhere dense, we have $\bigcap A_n = \emptyset$.
- Hence, there exists N for which $A_N \cap T = \emptyset$.
- Since $A_N = \bigcup_{i=1}^N B_\varepsilon(s_i)$, this means $T \subseteq \bigcup_{i=1}^N B_\varepsilon(s_i)$.
- Hence $X \stackrel{\text{def}}{=} \{s_1, \dots, s_N\}$ is an ε -net for S . Q.E.D.

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16. Other Practical Use of Algorithmic Randomness: When to Stop an Iterative Algorithm

- *Situation* in numerical mathematics:
 - we often know an iterative process whose results x_k are known to converge to the desired solution x ,
 - but we do not know when to stop to guarantee that

$$d_X(x_k, x) \leq \varepsilon.$$

- *Heuristic approach*: stop when $d_X(x_k, x_{k+1}) \leq \delta$ for some $\delta > 0$.
- *Example*: in physics, if 2nd order terms are small, we use the linear expression as an approximation.

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17. When to Stop an Iterative Algorithm: Result

- Let $\{x_k\} \in S$, k be an integer, and $\varepsilon > 0$ a real number.
- We say that x_k is ε -accurate if $d_X(x_k, \lim x_p) \leq \varepsilon$.
- Let $d \geq 1$ be an integer.
- By a *stopping criterion*, we mean a function $c : X^d \rightarrow R_0^+$ that satisfies the following two properties:
 - If $\{x_k\} \in S$, then $c(x_k, \dots, x_{k+d-1}) \rightarrow 0$.
 - If for some $\{x_n\} \in S$ and k , $c(x_k, \dots, x_{k+d-1}) = 0$, then $x_k = \dots = x_{k+d-1} = \lim x_p$.
- *Result:* Let c be a stopping criterion. Then, for every $\varepsilon > 0$, there exists a $\delta > 0$ such that
 - if $c(x_k, \dots, x_{k+d-1}) \leq \delta$, and the sequence $\{x_n\}$ is not abnormal,
 - then x_k is ε -accurate.

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18. Need to Extend Algorithmic Randomness to Quantum Physics

- *Problem:* the original definitions assume that we have:
 - a set (of possible states) and
 - a probability measure on the set of all the states.
- *In other words:* the original definitions cover only classical (non-quantum) physics.
- *In quantum physics:*
 - for each measurable quantity, we also have a probability distribution, but
 - in general, there is no single probability distribution describing a given quantum state.
- *Instead:* for each binary (yes-no) observable a , we have the probability $m(a)$ of the “yes” answer.

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19. Natural Extension of Randomness to Quantum Logics

- *Reminder:* A set $\mathcal{T} \subseteq U$ is called a *set of typical elements* if

\forall definable sequence of sets A_n for which $A_n \supseteq A_{n+1}$ for all n and $\bigcap A_n = \emptyset$, $\exists N$ for which $A_N \cap \mathcal{T} = \emptyset$.

- *Reminder:* a set A is *possible* if $A \cap \mathcal{T} \neq \emptyset$, *impossible* if $A \cap \mathcal{T} = \emptyset$.

- *In quantum logic:* $U \Rightarrow L$, $\supseteq \Rightarrow \geq$, $\cap \Rightarrow \wedge$, $\emptyset \Rightarrow 0$.

- *Natural extension:* An element $T \in L$ is called *largest-typical* if

\forall definable sequence $A_n \in L$ for which $A_n \geq A_{n+1}$ for all n and $\bigwedge A_n = 0$, $\exists N$ for which $A_N \wedge T = 0$.

- A is *possible* if $A \wedge T \neq 0$, *impossible* if $A \wedge T = 0$.

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20. Consistency Result: Formulation

- *Desired result:* $\forall \varepsilon > 0$, there exists a largest-typical element T for which $m(T) \geq 1 - \varepsilon$.
- *Requirements:* L is a complete ortholattice such that:
 - if $A_n \geq A_{n+1}$, then $A_n \rightarrow \bigwedge A_n$;
 - lattice operations \vee and \wedge are continuous;
 - the function $m : L \rightarrow [0, 1]$ is continuous.
- *Caution:* for subspaces of R^2 , \vee is not continuous:
 - if a is a straight line, and
 - b_n is a line at an angle $\alpha_n = \frac{1}{n} \rightarrow 0$ from a ,
 - then $a \vee b_n = R^2$ for all n , so $a \vee b_n \rightarrow R^2$,
 - but in the limit, $b_n \rightarrow a$ and thus,

$$a \vee b_n = R^2 \not\rightarrow a \vee a = a.$$

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21. Consistency: Proof

- *Same idea:*

- \exists countably many definable sequences $\{A_n\}$:

$$\{A_n^{(1)}\}, \{A_n^{(2)}\}, \dots;$$

- we take $T \stackrel{\text{def}}{=} \bigvee_{k=1}^{\infty} A_{N_k}^{(k)}$ for some N_k .

- *Challenge:*

- original proof used the fact that

$$P(A \vee B) \leq P(A) + P(B).$$

- in quantum logic, we may have

$$m(A \vee B) > m(A) + m(B).$$

- *New idea:* select N_k s.t.

$$m\left(A_{N_1}^{(1)} \vee \dots \vee A_{N_k}^{(k)}\right) < \varepsilon.$$

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22. Consistency Proof (cont-d)

- Let us assume that we have selected N_1, \dots, N_k s.t.

$$m \left(A_{N_1}^{(1)} \vee \dots \vee A_{N_k}^{(k)} \right) < \varepsilon.$$

- Since $A_n^{(k+1)} \rightarrow 0$ and \vee is continuous,

$$A_{N_1}^{(1)} \vee \dots \vee A_{N_k}^{(k)} \vee A_n^{(k+1)} \rightarrow A_{N_1}^{(1)} \vee \dots \vee A_{N_k}^{(k)}.$$

- Since m is continuous, we have

$$m \left(A_{N_1}^{(1)} \vee \dots \vee A_{N_k}^{(k)} \vee A_n^{(k+1)} \right) \rightarrow m \left(A_{N_1}^{(1)} \vee \dots \vee A_{N_k}^{(k)} \right) < \varepsilon.$$

- So $\exists N_{k+1}$ for which

$$m \left(A_{N_1}^{(1)} \vee \dots \vee A_{N_k}^{(k)} \vee A_{N_{k+1}}^{(k+1)} \right) < \varepsilon.$$

- In the limit, $m(-T) = m \left(\bigvee_{k=1}^{\infty} A_{N_k}^{(k)} \right) \leq \varepsilon$, hence

$$m(T) \geq 1 - \varepsilon.$$

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