Extending Algorithmic Randomness to the Algebraic Approach to Quantum Physics: Kolmogorov Complexity and Quantum Logics

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1. Physicists Usually Assume that Events with a Very Small Probability Cannot Occur

- Known phenomenon: Brownian motion.
- In principle: due to Brownian motion, a kettle placed on a cold stove can start boiling.
- The probability of this event is positive but very small.
- A mathematician would say that this event is possible but rare.
- A physicist would say that this event is simply not possible.
- It is desirable: to formalize this intuition of physicists.
2. Kolmogorov’s Definition of Algorithmic Randomness

- **Kolmogorov**: proposed a new definition of a random sequence, a definition that separates
  - physically random binary sequences, e.g.:
    * sequences that appear in coin flipping experiments,
    * sequences that appear in quantum measurements
  - from sequence that follow some pattern.

- **Intuitively**: if a sequence $s$ is random, it satisfies all the probability laws.

- **What is a probability law**: a statement $S$ which is true with probability 1: $P(S) = 1$.

- **Conclusion**: to prove that a sequence is not random, we must show that it does not satisfy one of these laws.
3. Kolmogorov’s Definition of Algorithmic Randomness (cont-d)

- **Reminder**: a sequence $s$ is not random if it does not satisfy one of the probability laws $S$.
- **Equivalent statement**: $s$ is not random if $s \in C$ for a (definable) set $C$ ($= -S$) with $P(C) = 0$.
- **Resulting definition** (Kolmogorov, Martin-Löf): $s$ is random if $s \notin C$ for all definable $C$ with $P(C) = 0$.
- **Consistency proof**:
  - Every definable set $C$ is defined by a finite sequence of symbols (its definition).
  - Since there are countably many sequences of symbols, there are countably many definable sets $C$.
  - So, the complement $-R$ to the class $R$ of all random sequences also has probability 0.
4. Towards a More Physically Adequate Versions of Kolmogorov Randomness

- **Problem:** the 1960s Kolmogorov’s definition only explains why events with probability 0 do not happen.

- **What we need:** formalize the physicists’ intuition that events with very small probability cannot happen.

- **Seemingly natural formalization:** there exists the “smallest possible probability” $p_0$ such that:
  - if the computed probability $p$ of some event is larger than $p_0$, then this event can occur, while
  - if the computed probability $p$ is $\leq p_0$, the event cannot occur.

- **Example:** a fair coin falls heads 100 times with prob. $2^{-100}$; it is impossible if $p_0 \geq 2^{-100}$. 
5. The Above Formalization of Randomness is Not Always Adequate

• **Problem**: every sequence of heads and tails has exactly the same probability.

• **Corollary**: if we choose $p_0 \geq 2^{-100}$, we will thus exclude all sequences of 100 heads and tails.

• However, anyone can toss a coin 100 times.

• This proves that some such sequences are physically possible.

• **Similar situation**: Kyburg’s lottery paradox:
  – in a big (e.g., state-wide) lottery, the probability of winning the Grand Prize is very small;
  – a reasonable person should not expect to win;
  – however, some people do win big prizes.
6. New Definition of Randomness

- **Example:** height:
  - if height is \( \geq 6 \text{ ft} \), it is still normal;
  - if instead of 6 ft, we consider 6 ft 1 in, 6 ft 2 in, etc.,
    then \( \exists h_0 \) s.t. everyone taller than \( h_0 \) is abnormal;
  - we are not sure what is \( h_0 \), but we are sure such \( h_0 \)
    exists.

- **General description:** on the universal set \( U \), we have
  sets \( A_1 \supseteq A_2 \supseteq \ldots \supseteq A_n \supseteq \ldots \) s.t. \( P(\cap A_n) = 0 \).

- **Example:** \( A_1 = \) people w/height \( \geq 6 \text{ ft} \), \( A_2 = \) people w/height \( \geq 6 \text{ ft 1 in} \), etc.

- A set \( R \subseteq U \) is called a set of random elements if

  \[
  \forall \text{ definable sequence of sets } A_n \text{ for which } A_n \supseteq A_{n+1} \text{ for all } n \text{ and } P(\cap A_n) = 0, \exists N \text{ for which } A_N \cap R = \emptyset.
  \]
7. Definable: Mathematical Comment

- **What is definable:**
  - let $\mathcal{L}$ be a theory,
  - let $P(x)$ be a formula from the language of the theory $\mathcal{L}$, with one free variable $x$
  - so that the set $\{x \mid P(x)\}$ is defined in $\mathcal{L}$.
  
  We will then call the set $\{x \mid P(x)\}$ $\mathcal{L}$-definable.

- **How to deal with definable sets:**
  - Our objective is to be able to make mathematical statements about $\mathcal{L}$-definable sets.
  - Thus, we must have a stronger theory $\mathcal{M}$ in which the class of all $\mathcal{L}$-definable sets is a countable set.
  - One can prove that such $\mathcal{M}$ always exists.
8. Coin Example

- Universal set $U = \{H, T\}^\mathbb{N}$

- Here, $A_n$ is the set of all the sequences that start with $n$ heads.

- The sequence $\{A_n\}$ is decreasing and definable, and its intersection has probability 0.

- Therefore, for every set $\mathcal{R}$ of random elements of $U$, there exists an integer $N$ for which $A_N \cap \mathcal{R} = \emptyset$.

- This means that if a sequence $s \in \mathcal{R}$ is random and starts with $N$ heads, it must consist of heads only.

- In physical terms: it means that

  a random sequence cannot start with $N$ heads.

- This is exactly what we wanted to formalize.
9. From Random to Typical (Not Abnormal)

- **Fact:** not all solutions to the physical equations are physically meaningful.

- **Example 1:** when a cup breaks into pieces, the corresponding trajectories of molecules make physical sense.

- **Example 2:** when we reverse all the velocities, we get pieces assembling themselves into a cup.

- **Physical fact:** this is physically impossible.

- **Mathematical fact:** the reverse process satisfies all the original (T-invariant) equations.

- **Physicist’s explanation:** the reversed process is non-physical since its initial conditions are “degenerate”.

- **Clarification:** once we modify the initial conditions even slightly, the pieces will no longer get together.
10. New Definition of Non-Abnormality

- **Example**: height:
  - if height is $\geq 6 \text{ ft}$, it is still normal;
  - if instead of 6 ft, we consider 6 ft 1 in, 6 ft 2 in, etc.,
    then $\exists h_0$ s.t. everyone taller than $h_0$ is abnormal;
  - we are not sure what is $h_0$, but we are sure such $h_0$ exists.

- **General description**: on the universal set $U$, we have
  sets $A_1 \supseteq A_2 \supseteq \ldots \supseteq A_n \supseteq \ldots$ s.t. $\cap A_n = \emptyset$.

- **Example**: $A_1 = \text{people w/height } \geq 6 \text{ ft}$, $A_2 = \text{people w/height } \geq 6 \text{ ft } 1 \text{ in}$, etc.

- A set $T \subseteq U$ is called a *set of typical elements* if
  $\forall$ definable sequence of sets $A_n$ for which $A_n \supseteq A_{n+1}$
  for all $n$ and $\cap A_n = \emptyset$, $\exists N$ for which $A_N \cap T = \emptyset$. 
11. Coin Example

- Universal set $U = \{H, T\}^\mathbb{N}$
- Here, $A_n$ is the set of all the sequences that start with $n$ heads and has a tail.
- The sequence $\{A_n\}$ is decreasing and definable, and its intersection is empty.
- Therefore, for every set $\mathcal{T}$ of typical elements of $U$, there exists an integer $N$ for which $A_N \cap \mathcal{T} = \emptyset$.
- This means that if a sequence $s \in \mathcal{T}$ is random (has both heads and tails) and starts with $N$ heads, it must consist of heads only.
- *In physical terms:* it means that
  
  a random sequence cannot start with $N$ heads.
- This is exactly what we wanted to formalize.
12. Consistency Proof

- **Statement**: \( \forall \varepsilon > 0, \) there exists a set \( \mathcal{T} \) of typical elements for which \( P(\mathcal{T}) \geq 1 - \varepsilon \).

- There are countably many definable sequences \( \{A_n\} \): \( \{A_n^{(1)}\}, \{A_n^{(2)}\}, \ldots \)

- For each \( k \), \( P\left(A_n^{(k)}\right) \to 0 \) as \( n \to \infty \).

- Hence, there exists \( N_k \) for which \( P\left(A_{N_k}^{(k)}\right) \leq \varepsilon \cdot 2^{-k} \).

- We take \( \mathcal{T} \overset{\text{def}}{=} - \bigcup_{k=1}^{\infty} A_{N_k}^{(k)} \). Since \( P\left(A_{N_k}^{(k)}\right) \leq \varepsilon \cdot 2^{-k} \), we have
  \[
  P\left(\bigcup_{k=1}^{\infty} A_{N_k}^{(k)}\right) \leq \sum_{k=1}^{\infty} P\left(A_{N_k}^{(k)}\right) \leq \sum_{k=1}^{\infty} \varepsilon \cdot 2^{-k} = \varepsilon.
  \]

- Hence, \( P(\mathcal{T}) = 1 - P\left(\bigcup_{k=1}^{\infty} A_{N_k}^{(k)}\right) \geq 1 - \varepsilon. \)
13. Ill-Posed Problems: In Brief

- **Main objectives of science:**
  - *guaranteed* estimates for physical quantities;
  - *guaranteed* predictions for these quantities.

- **Problem:** estimation and prediction are ill-posed.

- **Example:**
  - measurement devices are inertial;
  - hence suppress high frequencies $\omega$;
  - so $\varphi(x)$ and $\varphi(x) + \sin(\omega \cdot t)$ are indistinguishable.

- **Existing approaches:**
  - statistical regularization (filtering);
  - Tikhonov regularization (e.g., $|\dot{x}| \leq \Delta$);
  - expert-based regularization.

- **Main problem:** no guarantee.
14. On “Not Abnormal” Solutions, Problems Become Well-Posed

- **State estimation – an ill-posed problem:**
  - Measurement $f$:
    state $s \in S \rightarrow$ observation $r = f(s) \in R$.
  - In principle, we can reconstruct $r \rightarrow s$:
    as $s = f^{-1}(r)$.
  - Problem: small changes in $r$ can lead to huge changes in $s$ ($f^{-1}$ not continuous).

- **Theorem:**
  - Let $S$ be a definably separable metric space.
  - Let $\mathcal{T}$ be a set of all not abnormal elements of $S$.
  - Let $f : S \rightarrow R$ be a continuous 1-1 function.
  - Then, the inverse mapping $f^{-1} : R \rightarrow S$ is continuous for every $r \in f(\mathcal{T})$. 
15. Proof of Well-Posedness

- **Known:** if $f$ is continuous and 1-1 on a compact, then $f^{-1}$ is also continuous.

- **Reminder:** $X$ is compact if and only if it is closed and for every $\varepsilon$, it has a finite $\varepsilon$-net.

- **Given:** $S$ is definably separable.

- **Means:** $\exists$ def. $s_1, \ldots, s_n, \ldots$ everywhere dense in $S$.

- **Solution:** take $A_n \overset{\text{def}}{=} - \bigcup_{i=1}^{n} B_\varepsilon(s_i)$.

- Since $s_i$ are everywhere dense, we have $\cap A_n = \emptyset$.

- Hence, there exists $N$ for which $A_N \cap \mathcal{T} = \emptyset$.

- Since $A_N = - \bigcup_{i=1}^{N} B_\varepsilon(s_i)$, this means $\mathcal{T} \subseteq \bigcup_{i=1}^{N} B_\varepsilon(s_i)$.

- Hence $\{s_1, \ldots, s_N\}$ is an $\varepsilon$-net for $\mathcal{T}$. Q.E.D.
16. Other Practical Use of Algorithmic Randomness: When to Stop an Iterative Algorithm

- **Situation** in numerical mathematics:
  - we often know an iterative process whose results $x_k$ are known to converge to the desired solution $x$,
  - but we do not know when to stop to guarantee that
    $$d_X(x_k, x) \leq \varepsilon.$$ 

- **Heuristic approach**: stop when $d_X(x_k, x_{k+1}) \leq \delta$ for some $\delta > 0$.

- **Example**: in physics, if 2nd order terms are small, we use the linear expression as an approximation.
17. When to Stop an Iterative Algorithm: Result

- Let \( \{x_k\} \in S, k \) be an integer, and \( \varepsilon > 0 \) a real number.
- We say that \( x_k \) is \( \varepsilon \)-accurate if \( d_X(x_k, \lim x_p) \leq \varepsilon \).
- Let \( d \geq 1 \) be an integer.
- By a stopping criterion, we mean a function \( c : X^d \rightarrow \mathbb{R}_0^+ \) that satisfies the following two properties:
  - If \( \{x_k\} \in S \), then \( c(x_k, \ldots, x_{k+d-1}) \rightarrow 0 \).
  - If for some \( \{x_n\} \in S \) and \( k \), \( c(x_k, \ldots, x_{k+d-1}) = 0 \), then \( x_k = \ldots = x_{k+d-1} = \lim x_p \).

- Result: Let \( c \) be a stopping criterion. Then, for every \( \varepsilon > 0 \), there exists a \( \delta > 0 \) such that
  - if \( c(x_k, \ldots, x_{k+d-1}) \leq \delta \), and the sequence \( \{x_n\} \) is not abnormal,
  - then \( x_k \) is \( \varepsilon \)-accurate.
18. Need to Extend Algorithmic Randomness to Quantum Physics

• *Problem:* the original definitions assume that we have:
  – a set (of possible states) and
  – a probability measure on the set of all the states.

• *In other words:* the original definitions cover only classical (non-quantum) physics.

• *In quantum physics:*
  – for each measurable quantity, we also have a probability distribution, but
  – in general, there is no single probability distribution describing a given quantum state.

• *Instead:* for each binary (yes-no) observable $a$, we have the probability $m(a)$ of the “yes” answer.
19. Natural Extension of Randomness to Quantum Logics

• **Reminder:** A set $\mathcal{T} \subseteq U$ is called a *set of typical elements* if

$$\forall \text{definable sequence of sets } A_n \text{ for which } A_n \supseteq A_{n+1}$$

for all $n$ and $\cap A_n = \emptyset$, $\exists N$ for which $A_N \cap \mathcal{T} = \emptyset$.

• **Reminder:** a set $A$ is *possible* if $A \cap \mathcal{T} \neq \emptyset$, *impossible* if $A \cap \mathcal{T} = \emptyset$.

• **In quantum logic:** $U \Rightarrow L$, $\supseteq \Rightarrow \geq$, $\cap \Rightarrow \wedge$, $\emptyset \Rightarrow 0$.

• **Natural extension:** An element $T \in L$ is called *largest-typical* if

$$\forall \text{definable sequence } A_n \in L \text{ for which } A_n \geq A_{n+1} \text{ for all } n \text{ and } \wedge A_n = 0, \exists N \text{ for which } A_N \wedge T = 0.$$

• $A$ is *possible* if $A \wedge T \neq 0$, *impossible* if $A \wedge T = 0$. 
20. Consistency Result: Formulation

- **Desired result:** \( \forall \varepsilon > 0 \), there exists a largest-typical element \( T \) for which \( m(T) \geq 1 - \varepsilon \).

- **Requirements:** \( L \) is a complete ortholattice such that:
  - if \( A_n \geq A_{n+1} \), then \( A_n \to \bigwedge A_n \);
  - lattice operations \( \lor \) and \( \land \) are continuous;
  - the function \( m : L \to [0, 1] \) is continuous.

- **Caution:** for subspaces of \( \mathbb{R}^2 \), \( \lor \) is not continuous:
  - if \( a \) is a straight line, and
  - \( b_n \) is a line at an angle \( \alpha_n = \frac{1}{n} \to 0 \) from \( a \),
  - then \( a \lor b_n = \mathbb{R}^2 \) for all \( n \), so \( a \lor b_n \to \mathbb{R}^2 \),
  - but in the limit, \( b_n \to a \) and thus,

\[
a \lor b_n = \mathbb{R}^2 \not\to a \lor a = a.
\]
21. Consistency: Proof

• Same idea:
  – ∃ countably many definable sequences \{A_n\}:
    \{A_n^{(1)}\}, \{A_n^{(2)}\}, \ldots ;
  – we take \( T \overset{\text{def}}{=} - \bigvee_{k=1}^{\infty} A_{N_k}^{(k)} \) for some \( N_k \).

• Challenge:
  – original proof used the fact that
    \[ P(A \lor B) \leq P(A) + P(B). \]
  – in quantum logic, we may have
    \[ m(A \lor B) > m(A) + m(B). \]

• New idea: select \( N_k \) s.t.
  \[ m \left( A_{N_1}^{(1)} \lor \ldots \lor A_{N_k}^{(k)} \right) < \varepsilon. \]
22. Consistency Proof (cont-d)

• Let us assume that we have selected $N_1, \ldots, N_k$ s.t.
  \[ m \left( A_{N_1}^{(1)} \lor \ldots \lor A_{N_k}^{(k)} \right) < \varepsilon. \]

• Since $A_n^{(k+1)} \to 0$ and $\lor$ is continuous,
  \[ A_{N_1}^{(1)} \lor \ldots \lor A_{N_k}^{(k)} \lor A_n^{(k+1)} \to A_{N_1}^{(1)} \lor \ldots \lor A_{N_k}^{(k)}. \]

• Since $m$ is continuous, we have
  \[ m \left( A_{N_1}^{(1)} \lor \ldots \lor A_{N_k}^{(k)} \lor A_n^{(k+1)} \right) \to m \left( A_{N_1}^{(1)} \lor \ldots \lor A_{N_k}^{(k)} \right) < \varepsilon. \]

• So $\exists N_{k+1}$ for which
  \[ m \left( A_{N_1}^{(1)} \lor \ldots \lor A_{N_k}^{(k)} \lor A_{N_{k+1}}^{(k+1)} \right) < \varepsilon. \]

• In the limit, $m(-T) = m \left( \bigvee_{k=1}^{\infty} A_{N_k}^{(k)} \right) \leq \varepsilon$, hence
  \[ m(T) \geq 1 - \varepsilon. \]
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