Extending Algorithmic Randomness to the Algebraic Approach to Quantum Physics: Kolmogorov Complexity and Quantum Logics

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1. Physicists Usually Assume that Events with a Very Small Probability Cannot Occur

- *Known phenomenon:* Brownian motion.
- *In principle:* due to Brownian motion, a kettle placed on a cold stove can start boiling.
- *The probability* of this event is positive but very small.
- *A mathematician* would say that this event is possible but rare.
- *A physicist* would say that this event is simply not possible.
- *It is desirable:* to formalize this intuition of physicists.

2. Kolmogorov's Definition of Algorithmic Randomness

- *Kolmogorov:* proposed a new definition of a random sequence, a definition that separates
  - physically random binary sequences, e.g.:
    - sequences that appear in coin flipping experiments,
    - sequences that appear in quantum measurements
  - from sequence that follow some pattern.
- *Intuitively:* if a sequence s is random, it satisfies all the probability laws.
- *What is a probability law:* a statement S which is true with probability 1: $P(S) = 1$.
- *Conclusion:* to prove that a sequence is not random, we must show that it does not satisfy one of these laws.

3. Kolmogorov's Definition of Algorithmic Randomness (cont-d)

- *Reminder:* a sequence s is not random if it does not satisfy one of the probability laws S.
- *Equivalent statement:* s is not random if $s \in C$ for a (definable) set C $(= -S)$ with $P(C) = 0$.
- *Resulting definition* (Kolmogorov, Martin-Löf): s is random if $s \notin C$ for all definable C with $P(C) = 0$.
- *Consistency proof:*
  - Every definable set C is defined by a finite sequence of symbols (its definition).
  - Since there are countably many sequences of symbols, there are countably many definable sets C.
  - So, the complement $-\mathcal{R}$ to the class $\mathcal{R}$ of all random sequences also has probability 0.

4. Towards a More Physically Adequate Versions of Kolmogorov Randomness

- *Problem:* the 1960s Kolmogorov's definition only explains why events with probability 0 do not happen.
- *What we need:* formalize the physicists' intuition that events with very small probability cannot happen.
- *Seemingly natural formalization:* there exists the "smallest possible probability" p_0 such that:
  - if the computed probability p of some event is larger than p_0, then this event can occur, while
  - if the computed probability p is $\le p_0$, the event cannot occur.
- *Example:* a fair coin falls heads 100 times with prob. 2^{-100}; it is impossible if $p_0 \ge 2^{-100}$.

5. The Above Formalization of Randomness is Not Always Adequate

- *Problem:* every sequence of heads and tails has exactly the same probability.
- *Corollary:* if we choose $p_0 \geq 2^{-100}$, we will thus exclude all sequences of 100 heads and tails.
- However, anyone can toss a coin 100 times.
- This proves that some such sequences are physically possible.
- *Similar situation:* Kyburg's lottery paradox:
  - in a big (e.g., state-wide) lottery, the probability of winning the Grand Prize is very small;
  - a reasonable person should not expect to win;
  - however, some people do win big prizes.

6. New Definition of Randomness

- *Example:* height:
  - if height is ≥ 6 ft, it is still normal;
  - if instead of 6 ft, we consider 6 ft 1 in, 6 ft 2 in, etc., then $\exists h_0$ s.t. everyone taller than h_0 is abnormal;
  - we are not sure what is h_0, but we are sure such h_0 exists.
- *General description:* on the universal set U, we have sets $A_1 \supseteq A_2 \supseteq \ldots \supseteq A_n \supseteq \ldots$ s.t. $P(\cap A_n) = 0$.
- *Example:* $A_1 =$ people w/height ≥ 6 ft, $A_2 =$ people w/height ≥ 6 ft 1 in, etc.
- A set $\mathcal{R} \subseteq U$ is called a *set of random elements* if

  $\forall$ definable sequence of sets A_n for which $A_n \supseteq A_{n+1}$ for all n and $P(\cap A_n) = 0$, $\exists N$ for which $A_N \cap \mathcal{R} = \emptyset$.

7. Definable: Mathematical Comment

- *What is definable:*
  - let $\mathcal{L}$ be a theory,
  - let $P(x)$ be a formula from the language of the theory $\mathcal{L}$, with one free variable x
  - so that the set $\{x \mid P(x)\}$ is defined in $\mathcal{L}$.

  We will then call the set $\{x \mid P(x)\}$ $\mathcal{L}$-*definable.*

- *How to deal with definable sets:*
  - Our objective is to be able to make mathematical statements about $\mathcal{L}$-definable sets.
  - Thus, we must have a stronger theory $\mathcal{M}$ in which the class of all $\mathcal{L}$-definable sets is a countable set.
  - One can prove that such $\mathcal{M}$ always exists.

8. Coin Example

- Universal set $U = \{\mathrm{H}, \mathrm{T}\}^{\mathbb{N}}$
- Here, A_n is the set of all the sequences that start with n heads.
- The sequence $\{A_n\}$ is decreasing and definable, and its intersection has probability 0.
- Therefore, for every set $\mathcal{R}$ of random elements of U, there exists an integer N for which $A_N \cap T = \emptyset$.
- This means that if a sequence $s \in \mathcal{R}$ is random and starts with N heads, it must consist of heads only.
- *In physical terms:* it means that

  a random sequence cannot start with N heads.

- This is exactly what we wanted to formalize.

9. From Random to Typical (Not Abnormal)

- *Fact:* not all solutions to the physical equations are physically meaningful.
- *Example 1:* when a cup breaks into pieces, the corresponding trajectories of molecules make physical sense.
- *Example 2:* when we reverse all the velocities, we get pieces assembling themselves into a cup.
- *Physical fact:* this is physically impossible.
- *Mathematical fact:* the reverse process satisfies all the original (T-invariant) equations.
- *Physicist's explanation:* the reversed process is non-physical since its initial conditions are "degenerate".
- *Clarification:* once we modify the initial conditions even slightly, the pieces will no longer get together.

10. New Definition of Non-Abnormality

- *Example:* height:
  - if height is ≥ 6 ft, it is still normal;
  - if instead of 6 ft, we consider 6 ft 1 in, 6 ft 2 in, etc., then $\exists h_0$ s.t. everyone taller than h_0 is abnormal;
  - we are not sure what is h_0, but we are sure such h_0 exists.
- *General description:* on the universal set U, we have sets $A_1 \supseteq A_2 \supseteq \ldots \supseteq A_n \supseteq \ldots$ s.t. $\cap A_n = \emptyset$.
- *Example:* $A_1 =$ people w/height ≥ 6 ft, $A_2 =$ people w/height ≥ 6 ft 1 in, etc.
- A set $\mathcal{T} \subseteq U$ is called a *set of typical elements* if

$$\forall \text{ definable sequence of sets } A_n \text{ for which } A_n \supseteq A_{n+1} \text{ for all } n \text{ and } \cap A_n = \emptyset,\ \exists N \text{ for which } A_N \cap \mathcal{T} = \emptyset.$$

11. Coin Example

- Universal set $U = \{\mathrm{H}, \mathrm{T}\}^{\mathbb{N}}$
- Here, A_n is the set of all the sequences that start with n heads and has a tail.
- The sequence $\{A_n\}$ is decreasing and definable, and its intersection is empty.
- Therefore, for every set $\mathcal{T}$ of typical elements of U, there exists an integer N for which $A_N \cap T = \emptyset$.
- This means that if a sequence $s \in \mathcal{T}$ is random (has both heads and tails) and starts with N heads, it must consist of heads only.
- *In physical terms:* it means that

  a random sequence cannot start with N heads.

- This is exactly what we wanted to formalize.

12. Consistency Proof

- *Statement:* $\forall \varepsilon > 0$, there exists a set $\mathcal{T}$ of typical elements for which $\underline{P}(\mathcal{T}) \geq 1 - \varepsilon$.
- There are countably many definable sequences $\{A_n\}$: $\{A_n^{(1)}\}$, $\{A_n^{(2)}\}$, ...
- For each k, $P\left(A_n^{(k)}\right) \to 0$ as $n \to \infty$.
- Hence, there exists N_k for which $P\left(A_{N_k}^{(k)}\right) \leq \varepsilon \cdot 2^{-k}$.
- We take $\mathcal{T} \stackrel{\text{def}}{=} -\bigcup\limits_{k=1}^{\infty} A_{N_k}^{(k)}$. Since $P\left(A_{N_k}^{(k)}\right) \leq \varepsilon \cdot 2^{-k}$, we have
$$\overline{P}\left(\bigcup_{k=1}^{\infty} A_{N_k}^{(k)}\right) \leq \sum_{k=1}^{\infty} P\left(A_{N_k}^{(k)}\right) \leq \sum_{k=1}^{\infty} \varepsilon \cdot 2^{-k} = \varepsilon.$$
- Hence, $\underline{P}(\mathcal{T}) = 1 - \overline{P}\left(\bigcup\limits_{k=1}^{\infty} A_{N_k}^{(k)}\right) \geq 1 - \varepsilon$.

13. Ill-Posed Problems: In Brief

- Main *objectives* of science:
  - *guaranteed* estimates for physical quantities;
  - *guaranteed* predictions for these quantities.
- *Problem:* estimation and prediction are ill-posed.
- *Example:*
  - measurement devices are inertial;
  - hence suppress high frequencies ω;
  - so $\varphi(x)$ and $\varphi(x) + \sin(\omega \cdot t)$ are indistinguishable.
- *Existing approaches:*
  - statistical regularization (filtering);
  - Tikhonov regularization (e.g., $|\dot{x}| \le \Delta$);
  - expert-based regularization.
- *Main problem:* no guarantee.

14. On "Not Abnormal" Solutions, Problems Become Well-Posed

- *State estimation – an ill-posed problem:*
  - *Measurement f:*
    state $s \in S \to$ observation $r = f(s) \in R$.
  - *In principle,* we can reconstruct $r \to s$:
    as $s = f^{-1}(r)$.
  - *Problem:* small changes in r can lead to huge changes in s (f^{-1} *not continuous*).
- *Theorem:*
  - Let S be a definably separable metric space.
  - Let $\mathcal{T}$ be a set of all not abnormal elements of S.
  - Let $f : S \to R$ be a continuous 1-1 function.
  - Then, the inverse mapping $f^{-1} : R \to S$
    is *continuous* for every $r \in f(\mathcal{T})$.

15. Proof of Well-Posedness

- *Known:* if a f is continuous and 1-1 on a compact, then f^{-1} is also continuous.
- *Reminder:* X is compact if and only if it is closed and for every ε, it has a finite ε-net.
- *Given:* S is definably separable.
- *Means:* $\exists$ def. $s_1, \ldots, s_n, \ldots$ everywhere dense in S.
- *Solution:* take $A_n \stackrel{\text{def}}{=} -\bigcup\limits_{i=1}^{n} B_\varepsilon(s_i)$.
- Since s_i are everywhere dense, we have $\cap A_n = \emptyset$.
- Hence, there exists N for which $A_N \cap \mathcal{T} = \emptyset$.
- Since $A_N = -\bigcup\limits_{i=1}^{N} B_\varepsilon(s_i)$, this means $\mathcal{T} \subseteq \bigcup\limits_{i=1}^{N} B_\varepsilon(s_i)$.
- Hence $\{s_1, \ldots, s_N\}$ is an ε-net for $\mathcal{T}$. Q.E.D.

16. Other Practical Use of Algorithmic Randomness: When to Stop an Iterative Algorithm

- *Situation* in numerical mathematics:
  - we often know an iterative process whose results x_k are known to converge to the desired solution x,
  - but we do not know when to stop to guarantee that

$$d_X(x_k, x) \le \varepsilon.$$

- *Heuristic approach:* stop when $d_X(x_k, x_{k+1}) \le \delta$ for some $\delta > 0$.
- *Example:* in physics, if 2nd order terms are small, we use the linear expression as an approximation.

17. When to Stop an Iterative Algorithm: Result

- Let $\{x_k\} \in S$, k be an integer, and $\varepsilon > 0$ a real number.
- We say that x_k is ε-*accurate* if $d_X(x_k, \lim x_p) \le \varepsilon$.
- Let $d \ge 1$ be an integer.
- By a *stopping criterion*, we mean a function $c : X^d \to R_0^+$ that satisfies the following two properties:
  - If $\{x_k\} \in S$, then $c(x_k, \ldots, x_{k+d-1}) \to 0$.
  - If for some $\{x_n\} \in S$ and k, $c(x_k, \ldots, x_{k+d-1}) = 0$, then $x_k = \ldots = x_{k+d-1} = \lim x_p$.
- *Result:* Let c be a stopping criterion. Then, for every $\varepsilon > 0$, there exists a $\delta > 0$ such that
  - if $c(x_k, \ldots, x_{k+d-1}) \le \delta$, and the sequence $\{x_n\}$ is not abnormal,
  - then x_k is ε-accurate.

18. Need to Extend Algorithmic Randomness to Quantum Physics

- *Problem:* the original definitions assume that we have:
  - a set (of possible states) and
  - a probability measure on the set of all the states.
- *In other words:* the original definitions cover only classical (non-quantum) physics.
- *In quantum physics:*
  - for each measurable quantity, we also have a probability distribution, but
  - in general, there is no single probability distribution describing a given quantum state.
- *Instead:* for each binary (yes-no) observable a, we have the probability $m(a)$ of the "yes" answer.

19. Natural Extension of Randomness to Quantum Logics

- *Reminder:* A set $\mathcal{T} \subseteq U$ is called a *set of typical elements* if

$$\forall \text{ definable sequence of sets } A_n \text{ for which } A_n \supseteq A_{n+1} \text{ for all } n \text{ and } \cap A_n = \emptyset, \ \exists N \text{ for which } A_N \cap \mathcal{T} = \emptyset.$$

- *Reminder:* a set A is *possible* if $A \cap \mathcal{T} \neq \emptyset$, *impossible* if $A \cap \mathcal{T} = \emptyset$.
- *In quantum logic:* $U \Rightarrow L$, $\supseteq \Rightarrow \geq$, $\cap \Rightarrow \wedge$, $\emptyset \Rightarrow 0$.
- *Natural extension:* An element $T \in L$ is called *largest-typical* if

$$\forall \text{ definable sequence } A_n \in L \text{ for which } A_n \geq A_{n+1} \text{ for all } n \text{ and } \wedge A_n = 0, \ \exists N \text{ for which } A_N \wedge T = 0.$$

- A is *possible* if $A \wedge T \neq 0$, *impossible* if $A \wedge T = 0$.

20. Consistency Result: Formulation

- *Desired result:* $\forall \varepsilon > 0$, there exists a largest-typical element T for which $m(T) \geq 1 - \varepsilon$.
- *Requirements:* L is a complete ortholattice such that:
  - if $A_n \geq A_{n+1}$, then $A_n \to \wedge A_n$;
  - lattice operations $\vee$ and $\wedge$ are continuous;
  - the function $m : L \to [0, 1]$ is continuous.
- *Caution:* for subspaces of R^2, $\vee$ is not continuous:
  - if a is a straight line, and
  - b_n is a line at an angle $\alpha_n = \dfrac{1}{n} \to 0$ from a,
  - then $a \vee b_n = \mathbb{R}^2$ for all n, so $a \vee b_n \to \mathbb{R}^2$,
  - but in the limit, $b_n \to a$ and thus,

$$a \vee b_n = \mathbb{R}^2 \not\to a \vee a = a.$$

21. Consistency: Proof

- *Same idea:*
  - $\exists$ countably many definable sequences $\{A_n\}$:
$$\{A_n^{(1)}\}, \quad \{A_n^{(2)}\}, \quad \ldots;$$
  - we take $T \stackrel{\text{def}}{=} -\bigvee_{k=1}^{\infty} A_{N_k}^{(k)}$ for some N_k.
- *Challenge:*
  - original proof used the fact that
$$P(A \vee B) \leq P(A) + P(B).$$
  - in quantum logic, we may have
$$m(A \vee B) > m(A) + m(B).$$
- *New idea:* select N_k s.t.
$$m\left(A_{N_1}^{(1)} \vee \ldots \vee A_{N_k}^{(k)}\right) < \varepsilon.$$

22. Consistency Proof (cont-d)

- Let us assume that we have selected $N_1, \ldots, N_k$ s.t.

$$m\left(A^{(1)}_{N_1} \vee \ldots \vee A^{(k)}_{N_k}\right) < \varepsilon.$$

- Since $A^{(k+1)}_n \to 0$ and $\vee$ is continuous,

$$A^{(1)}_{N_1} \vee \ldots \vee A^{(k)}_{N_k} \vee A^{(k+1)}_n \to A^{(1)}_{N_1} \vee \ldots \vee A^{(k)}_{N_k}.$$

- Since m is continuous, we have

$$m\left(A^{(1)}_{N_1} \vee \ldots \vee A^{(k)}_{N_k} \vee A^{(k+1)}_n\right) \to m\left(A^{(1)}_{N_1} \vee \ldots \vee A^{(k)}_{N_k}\right) < \varepsilon.$$

- So $\exists N_{k+1}$ for which

$$m\left(A^{(1)}_{N_1} \vee \ldots \vee A^{(k)}_{N_k} \vee A^{(k+1)}_{N_{k+1}}\right) < \varepsilon.$$

- In the limit, $m(-T) = m\left(\bigvee\limits_{k=1}^{\infty} A^{(k)}_{N_k}\right) \le \varepsilon$, hence

$$m(T) \ge 1 - \varepsilon.$$

23. Acknowledgments

This work was supported in part:

- by National Science Foundation grants HRD-0734825, EAR-0225670, and DMS-0532645 and
- by Grant 1 T36 GM078000-01 from the National Institutes of Health.