Why Min-Based Conditioning

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1. Need for Ordinal-Scale Possibility Degrees

- It is often useful to describe,
  - for each theoretically possible alternative $\omega$ from the set of all *theoretically* possible alternatives $\Omega$,
  - to what extent this alternative is, in the expert’s opinion, *actually* possible.

- Often, the only information that we can extract from experts is the qualitative one:
  - which alternatives have a higher degree of possibility and
  - which have lower degree.

- In some cases, we have a linear order.

- We could use this order to process this information.

- However, computers have been designed to process numbers; they are still best in processing numbers.
2. From Ordinal Scale to Numbers

• So, degrees of possibility are usually described by numbers $\pi(\omega) \in [0, 1]$:
  
  – the higher the degree,
  – the larger the value $\pi(\omega)$.

• These numbers by themselves do not have an exact meaning, the only meaning is in the order.

• So, the same meaning can be described if we apply any strictly increasing transformation to $[0, 1]$.

• Usually, some of this freedom is eliminated by the convention that the largest degree is set to 1.

• We can always achieve this with an appropriate transformation (normalization).

• Definition. Let $\Omega$ be a finite set. A possibility distribution is a function $\pi : \Omega \to [0, 1]$ s.t. $\max_{\omega \in \Omega} \pi(\omega) = 1$. 
3. Need for Conditioning and Normalization

- Often, we acquire an additional information:
  - some of the alternatives that we originally thought to be possible
  - are actually not possible: $\Psi \subset \Omega$, $\Psi \neq \Omega$.
- Example: some original suspects have alibis.
- We have $\pi'(\omega) = 0$ for all $\omega \notin \Psi$; but we may have $\max_{\omega \in \Psi} \pi'(\omega) < 1$, so we need normalization.
- Definition. By a conditioning operator, we mean a mapping $(\pi \mid \Psi)$ that:
  - inputs a possibility distribution $\pi$ on a set $\Omega$ and a non-empty set $\Psi \subseteq \Omega$, and
  - returns a new possibility distribution for which $(\pi \mid \Psi)(\omega) = 0$ for all $\omega \notin \Psi$.
- What are the reasonable conditioning operators?
4. Reasonable Properties

• A first reasonable requirement is that:
  – since alternatives $\omega \notin \Psi$ are excluded,
  – their original possibility degrees should not affect the resulting degrees.

• C1. If $\pi|_\Psi = \pi'|_\Psi$, i.e., if $\pi(\omega) = \pi'(\omega)$ for all $\omega \in \Psi$, then $(\pi|_\Psi) = (\pi'|_\Psi)$.

• Another reasonable condition is that:
  – while the numerical values of possibility degrees may change,
  – the order between these degrees should not change.

• C2. If $\pi(\omega) < \pi(\omega')$ for some $\omega, \omega' \in \Psi$, then $(\pi|_\Psi)(\omega) < (\pi|_\Psi)(\omega')$.

• C3. If $\pi(\omega) = \pi(\omega')$ for some $\omega, \omega' \in \Psi$, then $(\pi|_\Psi)(\omega) = (\pi|_\Psi)(\omega')$. 
5. Reasonable Properties (cont-d)

- Often, after learning $\Psi \subset \Omega$, we learn additional information $\Psi' \subset \Psi$. In this case:
  - first compute $\pi' = (\pi | \Psi)$, and then
  - compute $\pi'' = (\pi' | \Psi') = ((\pi | \Psi) | \Psi')$.

- Alternative, we could learn both pieces of the information at the same time, and get $(\pi | \Psi')$.

- In both cases, we gain the exact same new information.

- So, the resulting changes in possibility degrees should be the same:

- C4. If $\Psi' \subset \Psi$, then $((\pi | \Psi) | \Psi') = (\pi | \Psi')$. 
6. Reasonable Properties (cont-d)

- Another condition is that if had an alternative \( \omega_0 \) which we originally believed to be impossible, then:
  - this alternative should remain impossible, and
  - the possibility degrees of all other alternatives \( \omega \neq \omega_0 \) should remain the same.

- **C5.** If \( \pi(\omega_0) = 0 \) for some \( \omega_0 \in \Psi \), then \( (\pi | _\Psi)(\omega_0) = 0 \) and \( (\pi | _{\Psi-\{\omega_0\}} | _\Psi) = (\pi | _\Psi) | _{\Psi-\{\omega_0\}} \).
7. Final Property: Invariance

- What matters is the order between the degrees, not the numerical values of the degrees.
- So, the situations should not change if we apply a re-scaling $T$ that doesn’t change the order (e.g., $x \rightarrow x^2$).
- The result of applying the conditioning operator not change if we apply such a re-scaling.
- We should get the exact same result:
  - if we apply conditioning $\pi \rightarrow (\pi \mid \Psi)$ in the original scale, and then re-scale to $T(\pi \mid \Psi)$;
  - or we first apply the re-scaling, resulting in $T\pi$, and then apply the conditioning, resulting in $(T\pi \mid \Psi)$.
- **C6.** For every increasing one-to-one function $T : [0, 1] \rightarrow [0, 1]$, we have $(T\pi \mid \Psi) = T(\pi \mid \Psi)$. 
8. Main Result

- **Proposition.** The only conditioning operator satisfying C1–C6 is the min-based operator for which:
  
  - $(\pi \mid \Psi)(\omega) = 1$ when $\omega \in \Omega$ and $\pi(\omega) = \max_{\omega' \in \Psi} \pi(\omega')$;
  - $(\pi \mid \Psi)(\omega) = \pi(\omega)$ when $\omega \in \Omega$ and $\pi(\omega) < \max_{\omega' \in \Psi} \pi(\omega')$; and
  - $(\pi \mid \Psi)(\omega) = 0$ when $\omega \not\in \Psi$.

- The usual derivation selects $(A \mid B)$ as the maximal value s.t. $d(A \& B) = d((A \mid B) \& B)$, with
  \[d(A \& B) \overset{\text{def}}{=} \min(d(A), d(B)).\]

- We show that **maximality** can be replaced with **invariance** – reflecting ordinal character of degrees.
9. **Proof**

- It is easy to show that the min-based operator satisfies the properties $C_1$–$C_6$.
- To complete the proof, we need to prove that, vice versa,
  - every conditioning operator that satisfies these five properties
  - is indeed the min-based operator.
- To prove this statement, we will consider two possible cases:
  - the case when the set $\Psi$ contains some alternative $\omega$ for which $\pi(\omega) = 1$, and
  - the case when the set $\Psi$ does not contain any alternative $\omega$ for which $\pi(\omega) = 1$. 
10. **Proof: First Case**

- Let us first consider the case when the set $\Psi$ contains some alternative $\omega$ for which $\pi(\omega) = 1$.
- In this case, the min-based formula leads to $(\pi \mid \Psi)(\omega) = \pi(\omega)$ for all $\omega \in \Psi$.
- Let us show that this equality holds for all conditioning operators that satisfy the properties $C1$–$C6$.
- If there is no $\omega_0 \in \Psi$ for which $\pi(\omega_0) = 0$, let us add such an element to our set $\Omega$.
- According to Property $C5$, this will not change the result.
- Thus, without losing generality, we can safely assume that there is an element $\omega_0 \in \Psi$ for which $\pi(\omega_0) = 0$.
- As for the values $\pi(\omega)$ for $\omega \not\in \Psi$, we can use the property $C1$ to replace them with zeros.
11. First Case (cont-d)

- Let us sort values $\psi(\omega)$ corresponding to different alternatives $\omega \in \Psi$ in increasing order.

- We know that the resulting list of values includes 0 and 1, so this list has the form

$$v_1 = 0 < v_2 < \ldots < v_{k-1} < v_k = 1.$$ 

- Let us use property $C6$ to prove that the values $(\pi | \Psi)$ should also be from this list.

- Indeed, let us consider the following strictly increasing function $T(v)$: for $v_i \leq v \leq v_{i+1}$, we take

$$T(v) = v_i + \left( \frac{v - v_i}{v_{i+1} - v_i} \right)^2 \cdot (v_{i+1} - v_i).$$ 

- One can easily check that for this function, $T(v_i) = v_i$ for all $i$, so $T(\pi) = \pi$. 
12. First Case (cont-d)

• Thus, the property $C6$ implies that $T(\pi \mid \Psi) = (\pi \mid \Psi)$.
• So, for each value $v = (\pi \mid \Psi)(\omega)$, we should have $T(v) = v$.
• But for the above function $T(v)$, the only such values are $v_1, \ldots, v_k$.
• So, indeed, the values $v_1 < \ldots < v_k$ are mapped to the same $k$ values.
• By properties $C2$ and $C3$:
  – equal values of $\pi(\omega)$ are mapped into equal values of $(\pi \mid \Psi)(\omega)$, and
  – smaller values of $\pi(\omega)$ are mapped into smaller values of $(\pi \mid \Psi)(\omega)$.
• Thus, the values $v'_i$ corresponding to $v_i$ are also sorted in increasing order: $v'_1 < \ldots < v'_k$. 
13. First Case (final)

- Each new value $v'_i$ must coincide with one of the original values $v_j$.

- So, in the increasing list $v_1 < \ldots < v_k$ of $k$ values, we have $k$ new values $v'_i$ which have the same order.

- This implies:
  - that $v'_1$ must be the smallest of $v_i$, i.e., $v'_1 = v_1$,
  - that $v'_2$ be the second smallest, i.e., $v'_2 = v_2$, and,
  - in general, $v'_i = v_i$.

- So, indeed, $(\pi | \Psi)(\omega) = \pi(\omega)$ for all $\omega \in \Psi$. 
14. Proof: Second Case

- Let us now consider the case when the set $\Psi$ does not contain some alternative $\omega$ for which $\pi(\omega) = 1$.

- In this case, we can also:
  - add (if needed) an element $\omega_0$ for which $\pi(\omega_0) = 0$, and
  - sort the values $\pi(\omega)$ corresponding to $\omega \in \Psi$ into an increasing sequence $v_1 = 0 < v_2 < \ldots < v_k < 1$.

- The only difference is that in this case, the largest value $v_k$ in this increasing sequence is smaller than 1.

- One of the new values should be equal to 1.

- So, due to Properties C2 and C3, only the largest degree $v_k$ should be mapped into 1.
15. Second Case (cont-d)

- Similarly to the first case, we can prove:
  - that each of the the values \(v_1, \ldots, v_{k-1}\) maps into one of the values \(v_1, \ldots, v_k\), and
  - that if \(v_i < v_j\), then \(v'_i < v'_j\).
- By induction, we can prove that \(v'_i \geq v_i\).
- Since we have only one additional value to move to, for each \(i\), we have either \(c'_i = v_i\) or \(v'_i = v_{i+1}\).
- Let use the Property C4 to prove, by contradiction, that \(v_i < v_k\) cannot be transformed into \(v_{i+1}\).
- Let us assume that, vice versa, there is an element \(\omega_i \in \Psi\) for which \(\pi(\omega_i) = v_i\) and \((\pi | \Omega)(\omega_i) = v_{i+1}\).
16. Second Case (cont-d)

• To get a contradiction, let us consider:
  – the new set $\Omega^* = \Omega \cup \{\omega^*\}$, with a new element $\omega^*$, and
  – a new possibility distribution $\pi^*$ for which we have $v_i < \pi^*(\omega^*) < v_{i+1}$ and $\pi^*(\omega) = \pi(\omega)$ for all $\omega \neq \omega_i$.

• Let us consider two conditioning paths from $\Omega^*$ to $\Psi$:
  – in the first path, we go from $\Omega^*$ to $\Omega$ and then from $\Omega$ to $\Psi$;
  – in the second path, we go from $\Omega^*$ to $\Psi^* = \Psi \cup \{\omega^*\}$ and then from $\Psi^*$ to $\Psi$.

• According to the Property C4, the resulting value $(\pi^* | \Psi)(\omega_i)$ should be the same for both paths.

• In the first path, first, we go from $\Omega^*$ to $\Omega$. 
17. Second Case (cont-d)

• The transition from $\Omega^*$ to $\Omega$ eliminates a single element $\omega^*$ for which $\pi^*(\omega^*) < 1$.

• Thus, according to the first case, possibility degrees of remaining elements remain unchanged: $(\pi^* | \Omega) = \pi$.

• We already know that $(\pi | \Psi)(\omega_i) = v_{i+1}$.

• Thus, due to Property C4, we have

$$(\pi^* | \Psi)(\omega_i) = ((\pi^* | \Omega) | \Psi)(\omega_i) = v_{i+1}.$$ 

• On the other hand, in the second path, we first move from $\Omega^*$ to $\Psi^*$.

• In this transition, we have $v_k$ transformed into 1, and the original value $\pi^*(\omega_i) = v_i$:

  – can either remain the same,
  – or it can be transformed to the next value which is now $\pi^*(\omega^*) < v_{i+1}$.
18. Second Case (cont-d)

- In both cases, the new possibility degree is smaller than $v_{i+1}$: $\pi(\omega_i) < v_{i+1}$.
- When we then reduce $\Psi^*$ to $\Psi$, then:
  - all alternatives for which originally $\pi^*(\omega) = \pi(\omega) = v_k$ and now $\pi'(\omega) = 1$
  - remain in the set.
- Thus, all other alternatives – including the alternative $\omega_i$ – according to first case, retain their values.
- For $\omega_i$, this implies that $(\pi' \mid \Psi)(\omega_i) = \pi'(\omega_i) < v_{i+1}$.
- Thus, we have $(\pi^* \mid \Psi)(\omega_i) = \pi'(\omega_i) < v_{i+1}$.
- This contradicts to $(\pi^* \mid \Psi)(\omega_i) = v_{i+1}$.
- This contradiction shows that the transformation from $v_i$ to $v_{i+1}$ is indeed impossible. So, $v'_i = v_i$. Q.E.D.
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