Computing the Cube of an Interval Matrix Is NP-Hard

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1. **Why Intervals**

- In many real-life situations, we do not know the exact value of a physical quantity $x$.
- We only know the interval $x$ of possible values of $x$.
- This happens, e.g.:
  - if our information about $x$ comes from measurement, and
  - the only information that we have about the possible error of the measuring instrument is that this error is $\leq$ a certain bound $\Delta$.
- In this case, let the measurement result is $\tilde{x}$.
- We know that $|\tilde{x} - x| \leq \Delta$, where $x$ is the (unknown) actual value of the measured quantity.
- We can conclude that $x$ belongs to the interval $x \overset{\text{def}}{=} [\tilde{x} - \Delta, \tilde{x} + \Delta]$.
2. Why Interval Matrices

- In some physical situations, quantities form a matrix

\[
A = \begin{pmatrix}
  a_{11} & \cdots & a_{1j} & \cdots & a_{1n} \\
  \vdots & \ddots & \vdots & \ddots & \vdots \\
  a_{i1} & \cdots & a_{ij} & \cdots & a_{in} \\
  \vdots & \ddots & \vdots & \ddots & \vdots \\
  a_{m1} & \cdots & a_{mj} & \cdots & a_{mn}
\end{pmatrix}.
\]

- *Example:* system’s dynamics

\[
s_i(t + 1) = f_i(s_1(t), \ldots, s_n(t)).
\]

- Often, we are interested in small deviations \(\Delta s_i(t) \overset{\text{def}}{=} s_i(t) - s_i^{(0)}\) from the stable state \(s^{(0)}\).

- Linearization leads to \(\Delta s_i(t + 1) = \sum_{j=1}^{n} a_{ij} \cdot \Delta s_i(t)\) or \(\Delta s(t + 1) = A \Delta s(t)\).

- Often, for each \(i\) and \(j\), we only know the interval \(a_{ij}\) of possible values of \(a_{ij}\) – an interval matrix.
3. Why Products of Interval Matrices

- **Why product:**
  - if transition $t \rightarrow t + 1$ is described by a matrix $A$,
  - transition $t + 1 \rightarrow t + 2$ is described by $B$,
  - then transition $t \rightarrow t + 2$ is described by the product $C = BA$, with entries
    \[ c_{ij} = \sum_{k=1}^{n} a_{ik} \cdot b_{kj}. \]

- In case of interval uncertainty, we know $A$ and $B$, and we want to know
  \[ (AB)_{ij} \stackrel{\text{def}}{=} \{ (AB)_{ij}; A \in A, B \in B \}. \]

- Similar, for a transition $t \rightarrow t + 3$, we must know:
  \[ (ABC)_{ij} \stackrel{\text{def}}{=} \{ (ABC)_{ij}; A \in A, B \in B, C \in C \}. \]

- **Problem:** How can we compute these products?
4. The Problem of Multiplying Interval Matrices is a Particular Case of a General Problem

- **General problem:**
  - we have a function \( f(x_1, \ldots, x_n) \) of \( n \) variables,
  - we know the interval \( x_i \) of possible values of each of these variables, and
  - we must find the range
    \[
    f(x_1, \ldots, x_n) \overset{\text{def}}{=} \{ f(x_1, \ldots, x_n); x_1 \in x_1, \ldots, x_n \in x_n \}
    \]
    of this function when \( x_i \in x_i \).

- This general problem is called the problem of *interval computations*.

- **Known:** in general, NP-hard.
5. Interval Computations

- **Interval arithmetic**: explicit formulas when \( f = +, -, \cdot, \text{etc.} \):

\[
[x_1, x_1] + [x_2, x_2] = [x_1 + x_2, x_1 + x_2];
\]

\[
[x_1, x_1] \cdot [x_2, x_2] = \left[ \min(x_1 \cdot x_2, x_1 \cdot x_2, x_1 \cdot x_2, x_1 \cdot x_2), \right.
\]
\[
\left. \max(x_1 \cdot x_2, x_1 \cdot x_2, x_1 \cdot x_2, x_1 \cdot x_2) \right].
\]

- **Straightforward interval computations**:
  - replace each operation forming the algorithm \( f \)
  - with the corresponding operation from interval arithmetic.

- **Case of single-use expressions (SUE)**: exact result.

- **Conclusion**: we get the exact product of two interval matrices:

\[
c_{ij} = \sum_{k=1}^{n} a_{ik} \cdot b_{kj}.
\]
6. Computing the Product of Three Interval Matrices is NP-Hard

- **Problem**: computing the product \( D = ABC \) of three interval matrices.

- **Situation**: the expression \( d_{ij} = \sum_{k=1}^{n} \sum_{l=1}^{n} a_{ik} \cdot b_{kl} \cdot c_{lj} \) is not SUE.

- **Conclusion**: we can only guarantee that the straightforward interval computation leads to an enclosure – i.e., the result may not be always exact.

- **Our first (simple) result**: The problem of computing the exact product of three interval matrices is NP-hard.

- **Idea of the proof**: it is NP-hard, given a square matrix \( B = (b_{ij})_{i,j} \), to compute the range of \( x^T By \), where \( x_i = y_j = [-1, 1] \).
7. Why Power of a Matrix

- **Situation**: in many practical situations, we know that the system is stationary.

- This means that the transition from each moment of time to the next is described by the same matrix $A$.

- Then:
  - transition $t \rightarrow t + 2$ is described by $A^2$,
  - transition $t \rightarrow t + 3$ is described by $A^3$, etc.

- In case of interval uncertainty, we only know that $A \in A$ for a given interval matrix $A$.

- **Problem**: compute, for every $i$ and $j$, the set (interval) of possible values of $A^2$ and/or $A^3$:

$$ (A^2)_{ij} \overset{\text{def}}{=} \{ (A^2)_{ij} ; A \in A \}; \quad (A^3)_{ij} \overset{\text{def}}{=} \{ (A^3)_{ij} ; A \in A \}. $$
8. Feasible Algorithm for Computing the Square of an Interval Matrix

- **Situation:** for \( B = A^2 \), the expression

\[
b_{ij} = \sum_{k=1}^{n} a_{ik} \cdot a_{kj}
\]

is not SUE.

- **Example:** for \( i \neq j \), we have two occurrences of \( a_{ij} \): \( a_{ij} \cdot a_{jj} \) (when \( k = j \)) and \( a_{ii} \cdot a_{ij} \) (when \( k = j \)).

- **Idea:** reformulate into SUE:

\[
b_{ij} = \sum_{k \neq i, k \neq j} a_{ik} \cdot a_{kj} + a_{ij} \cdot (a_{ii} + a_{jj}) \quad (i \neq j);
\]

\[
b_{ii} = \sum_{k \neq i} a_{ik} \cdot a_{ki} + a_{ii}^2.
\]

- **Solution:** a feasible algorithm for computing \( A^2 \):

\[
b_{ij} = \sum_{k \neq i, k \neq j} a_{ik} \cdot a_{kj} + a_{ij} \cdot (a_{ii} + a_{jj}) \quad (i \neq j);
\]

\[
b_{ii} = \sum_{k \neq i} a_{ik} \cdot a_{ki} + a_{ii}^2.
\]
9. Interval Matrix Product Is Not Associative: Example

\[
A = \begin{pmatrix} 1 & [0,1] \\ 1 & -1 \end{pmatrix}; \text{ then } A \ast_s A = \begin{pmatrix} [1,2] & [-1,1] \\ 0 & [1,2] \end{pmatrix};
\]

\[
A \ast_s (A \ast_s A) = \begin{pmatrix} [1,2] & [-1,3] \\ [1,2] & [-3,0] \end{pmatrix};
\]

\[
(A \ast_s A) \ast_s A = \begin{pmatrix} [0,3] & [-1,3] \\ [1,2] & [-2,-1] \end{pmatrix} \neq A \ast_s (A \ast_s A).
\]

Here, \( A = \begin{pmatrix} 1 & a_{12} \\ 1 & -1 \end{pmatrix} \); so \( A^2 = \begin{pmatrix} 1 + a_{12} & 0 \\ 0 & 1 + a_{12} \end{pmatrix} \);

\[
A^3 = \begin{pmatrix} 1 + a_{12} & a_{12} + a_{12}^2 \\ 1 + a_{12} & -(1 + a_{12}) \end{pmatrix}; \text{ hence}
\]

\[
A^2 = \begin{pmatrix} [1,2] & 0 \\ 0 & [1,2] \end{pmatrix}; \quad A^3 = \begin{pmatrix} [1,2] & [0,2] \\ [1,2] & [-2,-1] \end{pmatrix}.
\]
10. Computing the Cube of an Interval Matrix Is NP-Hard

• **Result:** in general, computing $A^3$ is NP-hard.

• **Conclusions:**
  
  – Computing the product of interval matrices is important in many applications.
  
  – For two matrices, the corresponding problems are computationally feasible:
    
    * computing the exact range for the product of two interval matrices;
    * computing the square of an interval matrix.
  
  – The following 3-matrix problems are NP-hard:
    
    * computing the exact range for the product of three matrices;
    * computing the third power of a matrix.
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12. Proof

- **Idea:** use the same known NP-hard problem:
  - given a square matrix $B = (b_{ij})$,
  - compute the range of $x^T By$, where
    
    $x_i = y_j = [-1, 1]$.

- Specifically, for each $n \times n$ matrix $B$, we will consider the following $(2n + 2) \times (2n + 2)$ interval matrix:

  $$A = \begin{pmatrix}
  0 & U \\
  L & 0
  \end{pmatrix},$$

  where

  $$L = \begin{pmatrix}
  0 & \cdots & 0 & \cdots \\
  \vdots & \ddots & \vdots & \ddots \\
  0 & \cdots & B & \cdots \\
  \cdots & \cdots & \cdots & \cdots
  \end{pmatrix};
  U = \begin{pmatrix}
  0 & x_1 & \cdots & x_n \\
  y_1 & \cdots & \cdots & \cdots \\
  \vdots & \cdots & \cdots & \cdots \\
  y_n & \cdots & \cdots & \cdots
  \end{pmatrix}.$$
13. **Proof (cont-d)**

- For every matrix

\[ A = \begin{pmatrix} 0 & U \\ L & 0 \end{pmatrix} \in A = \begin{pmatrix} 0 & U \\ L & 0 \end{pmatrix}, \]

we have

\[ A^2 = \begin{pmatrix} 0 & U \\ L & 0 \end{pmatrix} \begin{pmatrix} 0 & U \\ L & 0 \end{pmatrix} = \begin{pmatrix} UL & 0 \\ 0 & LU \end{pmatrix}, \]

- Hence

\[ A^3 = A^2 A = \begin{pmatrix} UL & 0 \\ 0 & LU \end{pmatrix} \begin{pmatrix} 0 & U \\ L & 0 \end{pmatrix} = \begin{pmatrix} 0 & ULU \\ LUL & 0 \end{pmatrix}. \]

- Here,

\[ UL = \begin{pmatrix} 0 & x^T \\ y & 0 \end{pmatrix} \begin{pmatrix} 0 & 0^T \\ 0 & B \end{pmatrix} = \begin{pmatrix} 0 & x^T B \\ 0 & 0 \end{pmatrix}. \]
14. Proof (final part)

- Hence

\[
ULU = \begin{pmatrix}
0 & x^T B \\
0 & 0
\end{pmatrix} \begin{pmatrix}
0 & x^T \\
y & 0
\end{pmatrix} = \begin{pmatrix}
z & 0 \\
0 & 0
\end{pmatrix},
\]

where \( z = x^T By \).

- We have shown that

\[
A^3 = \begin{pmatrix}
0 & ULU \\
LUL & 0
\end{pmatrix}.
\]

- So, \((ULU)_{11} = (A^3)_{1,n+2} = x^T By\).

- We know that computing the range of \( x^T By \) is NP-hard.

- We conclude that computing the range \( A^3 \) is also an NP-hard problem.