From an Interval to a Natural Probability Distribution on the Interval: Weakest-Link Case, Distributions of Extremes, and Their Potential Application to Economics and to Fracture Mechanic

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1. Need to Make Decisions under Interval Uncertainty

- One of the main practical objectives is to make decisions.
- Decisions are usually made on the utility values $u(a)$ of different alternatives a.
- Under interval uncertainty, we only know the interval $\mathbf{u}(a) = [\underline{u}(a), \overline{u}(a)]$ containing $u(a)$.
- When two intervals intersect $\mathbf{u}(a) \cap \mathbf{u}(b) \neq \emptyset$, in principle, each of the two alternatives can be better.
- Intuitively, it is sometimes clear that a is "more probable" to be better than b.
- Example: if $\mathbf{u}(a) = [0, 1.1]$ and $\mathbf{u}(b) = [0.9, 2]$, the b is most probably better.

2. From the Need to Make Decisions to the Need to Assign Probabilities

- *Reminder:* in situations with interval uncertainty, we need to make decisions.
- *According to decision theory:*
  - a consistent decision making procedure under uncertainty

  is equivalent to
  - assigning "subjective" probabilities to different values within each uncertainty domain.
- *In our case:* uncertainty domain is an interval.
- *Conclusion:* we need a natural way to assign probabilities on an interval.

3. What We Do: Consider Weakest Link Case

- *General problem:* assigning probabilities on an interval.
- *What we do:* consider a practically important case of the "weakest link" arrangement.
- *What it means:* the collapse of each link is catastrophic for a system.
- *Example 1:* fracture mechanics, when a fracture in one of the areas makes the whole plane inoperable.
- *Example 2:* economics, when the collapse of one large bank or one country can have catastrophic consequences.
- *General feature:* the quality of a system is determined by the smallest $(\min_i v_i)$ of the corresponding values v_i.
- *In mathematical terms:* the distribution of $\min_i v_i$ is called the *distribution of extremes*.

4. Extreme Distributions: Standard Theory

- *We want to find:* $G(v_0) = 1 - F(v_0) = \text{Prob}(v > v_0)$, where $F(v_0)$ is a cumulative distribution function.
- *Fact:* the numerical value of a physical quantity v depends:
  - on the choice of a measuring unit $v \to a \cdot v$ (e.g., 1.7 m = 170 cm), and
  - on the choice of the starting point $v \to v + b$ (e.g.: A.D. or since the French Revolution).
- *Conclusion:* we want to find a *family* $\mathcal{G}$ of distributions $\{G(a \cdot v_0 + b)\}_{a,b}$.
- *Fact:* $v' \stackrel{\text{def}}{=} \min v_i > v_0 \Leftrightarrow v_1 > v_0 \,\&\, \ldots \,\&\, v_n > v_0$, so

$$G'(v_0) = \text{Prob}(v > v_0) = \prod_{i=1}^{n} \text{Prob}(v_i > v_0) = (G(v_0))^n.$$

5. Extreme Distributions: Standard Derivation

- *Reminder:* $G'(v_0) = (G_i(v_0))^n$.
- *Similarly:* for maximum v'' of $\alpha \cdot n$ values, we get $G''(v_0) = (G(v_0))^{\alpha \cdot n}$, hence $G''(v_0) = (G'(v_0))^{\alpha}$.
- *In the limit:* we conclude that if $G(v_0) \in \mathcal{G}$, then $G^{\alpha}(v_0) \in \mathcal{G}$ for all α.
- *Thus:* for every α, there exist $a(\alpha)$ and $b(\alpha)$ s.t.

$$G^{\alpha}(v_0) = G(a(\alpha) \cdot v_0 + b(\alpha)).$$

- *Simplification:* for $g(v_0) \stackrel{\text{def}}{=} -\ln(G(v_0))$, we get

$$\alpha \cdot g(v_0) = g(a(\alpha) \cdot v_0 + b(\alpha)).$$

- *Degenerate case:* $\alpha = 1$, $a(\alpha) = 1$, and $b(\alpha) = 0$.
- *Differentiating* both sides by α and taking $\alpha = 1$, we get $g = \dfrac{dg}{dv_0} \cdot (a \cdot v_0 + b)$, i.e., $\dfrac{dg}{g} = \dfrac{dv_0}{a \cdot v_0 + b}$ $(a \stackrel{\text{def}}{=} a'(1))$.

6. Extreme Distributions: Derivation (cont-d)

- *Reminder:* $\dfrac{dg}{g} = \dfrac{dv_0}{a \cdot v_0 + b}$.
- *When $a = 0$:* integration leads to $\ln(g) = \dfrac{v_0}{b} + c$, so $g(v_0) = \exp\left(\dfrac{v_0}{b} + c\right)$.
- *Conclusion:* $G(v_0) = \exp\left(-\exp\left(\dfrac{v_0}{b} + c\right)\right)$.
- *When $a \neq 0$:* for $v \stackrel{\text{def}}{=} v_0 + \Delta v$, with $\Delta v = b/a$, we get $\dfrac{dg}{g} = \dfrac{dv}{a \cdot v}$ hence $\ln(g) = a \cdot \ln(v) + c$.
- *Conclusion:* $g = c \cdot v^a = c \cdot (v_0 - \Delta v)^a$, hence

$$G(v_0) = \exp\left(-c \cdot (v_0 - \Delta v)^a\right).$$

- *Comment.* We get two different types of distributions depending on whether $a > 0$ or $a < 0$.

7. How to Extend This Analysis to Distributions on an Interval: Discussion

- *Symmetries:* the above derivations were based on the assumption that we have linear symmetries

$$v_0 \to a \cdot v_0 + b.$$

- *Examples:*
  - sometimes, we only have scale-invariance – 0 is fixed (height);
  - sometimes, we also have shift-invariance (temperature, time).
- *Problem:* the only linear transformation that preserves the interval is identity.
- *Our solution:* go beyond linear symmetries, to more general (non-linear) symmetries.

8. Basic Nonlinear Symmetries

- Sometimes, a system also has *nonlinear* symmetries.
- If a system is invariant under f and g, then:
  - it is invariant under their composition $f \circ g$, and
  - it is invariant under the inverse transformation f^{-1}.
- In mathematical terms, this means that symmetries form a *group*.
- In practice, at any given moment of time, we can only store and describe finitely many parameters.
- Thus, it is reasonable to restrict ourselves to *finite-dimensional* groups.
- *Question* (N. Wiener): describe all finite-dimensional groups that contain all linear transformations.
- *Answer* (for real numbers): all elements of this group are fractionally-linear $x \to (a \cdot x + b)/(c \cdot x + d)$.

9. Side Observation: Symmetries Explain the Basic Formulas of Neural Networks

- *What needs explaining:* formula for the *activation function* $f(x) = 1/(1 + e^{-x})$.
- A change in the input starting point: $x \to x + s$.
- *Reasonable requirement:* the new output $f(x+s)$ equivalent to the $f(x)$ mod. appropriate transformation.
- *Reminder:* all appropriate transformations are fractionally linear.
- *Conclusion:* $f(x+s) = \dfrac{a(s) \cdot f(x) + b(s)}{c(s) \cdot f(x) + d(s)}$.
- Differentiating both sides by s and equating s to 0, we get a differential equation for $f(x)$.
- Its known solution is the above activation function – which can thus be explained by symmetries.

10. Back to Extreme Distributions on an Interval

- *Idea:* every interval can be linearly reduced to $[0,1]$, so it is sufficient to consider $[\underline{x}, \overline{x}] = [0,1]$.
- *Reminder:* non-linear re-scalings are fractionally linear:
$$f(x) = \frac{a \cdot x + b}{c \cdot x + d}.$$
- Dividing both numerator and denominator by d, we get a simplified expression $f(x) = \dfrac{a \cdot x + b}{1 + c \cdot x}$.
- Which transformations preserve $[0,1]$:
  - we get $f(0) = 0$, so $b = 0$;
  - thus, $f(1) = 1$ implies $\dfrac{a}{1+c} = 1$, hence $c = a - 1$ and
$$f(x) = \frac{a \cdot x}{1 + (a-1) \cdot x}.$$

11. Deriving Extreme Distribution on an Interval

- *Reminder:* $f(x)=\dfrac{a\cdot x}{1+(a-1)\cdot x}.$
- *Conclusion:* $G^\alpha(v_0)=G\left(\dfrac{a(\alpha)\cdot v_0}{1+(a(\alpha)-1)\cdot v_0}\right)$, hence $\alpha\cdot g(v_0)=g\left(\dfrac{a(\alpha)\cdot v_0}{v_0+(a(\alpha)-1)}\right).$
- *Degenerate case:* $\alpha=1$ and $a(\alpha)=1$.
- *Differentiating* both sides by α and taking $\alpha=1$, we get $g=\dfrac{dg}{dv_0}\cdot(a\cdot v_0-a\cdot v_0^2)$, i.e.,

$$\frac{dg}{g}=\frac{dv_0}{a\cdot v_0-a\cdot v_0^2}=\frac{1}{a}\cdot\left(\frac{1}{v_0}+\frac{1}{1-v_0}\right).$$

12. Extreme Distribution on an Interval (cont-d)

- *Reminder:* $\dfrac{dg}{g}=\dfrac{1}{a}\cdot\left(\dfrac{1}{v_0}+\dfrac{1}{1-v_0}\right)$.
- *Integrating:* we get $\ln(g)=\dfrac{1}{a}\cdot(\ln(v_0)-\ln(1-v_0))+c=\dfrac{1}{a}\cdot\ln\left(\dfrac{v_0}{1-v_0}\right)+c$, hence $g(v_0)=\beta\cdot\left(\dfrac{1-v_0}{v_0}\right)^\alpha$.
- For a general interval $[\underline{v},\overline{v}]$, we get

$$g(v_0)=\beta\cdot\left(\frac{\overline{v}-v_0}{v_0-\underline{v}}\right)^\alpha.$$

- *Exponentiating:* we get $G(v_0)=\exp(-g(v_0))$, hence

$$G(v_0)=\exp\left(-\beta\cdot\left(\frac{\overline{v}-v_0}{v_0-\underline{v}}\right)^\alpha\right).$$

- *Fact:* these distributions were empirically found in fracture mechanics by A. Chudnovsky and B. Kunin.

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