Products of Partially Ordered Sets (Posets) and Intervals in Such Products, with Potential Applications to Uncertainty Logic and Space-Time Geometry

Francisco Zapata[1], Olga Kosheleva[1], and Karen Villaverde[2]

[1]University of Texas at El Paso
El Paso, TX 79968, USA
olgak@utep.edu

[2]Department of Computer Science
New Mexico State University
Las Cruces, NM 88003, USA
kvillave@cs.nmsu.edu

1. Posets in Space-Time Geometry

- Starting from general relativity, space-time models are usually formulated in terms of physical fields.
- Typical example: a metric field $g_{ij}(x)$.
- These fields assume that the space-time is *smooth*.
- However, there are important situations of *non-smoothness*:
  - *singularities* like the Big Bang or a black hole, and
  - *quantum fluctuations*.
- According to modern physics, a proper description of the corresponding non-smooth space-time models means:
  - that we no longer have a metric field,
  - that we only have a *causality* relation $\preceq$ between events – a partial order.

2. Intervals in Space-Time Posets

- Due to measurement inaccuracy, we rarely know the *exact* space-time location of an event e.
- Often, we only know:
  - an event $\underline{e}$ that precedes e:
  $$\underline{e} \preceq e,$$
  and
  - an event $\overline{e}$ that follows e:
  $$e \preceq \overline{e}.$$
- In this case, we only know that e belongs to the *interval*
$$[\underline{e}, \overline{e}] \stackrel{\text{def}}{=} \{e : \underline{e} \preceq e \preceq \overline{e}\}.$$
- *Comment:* In the 1-D case, we get standard intervals on the real line.

3. Products of Space-Time Posets

- Sometimes, we need to consider *pairs* of events.
- *Example:* situations like quantum entanglement, situations of importance to quantum computing.
- *Question:* how to extend partial orders on posets A_1 and A_2 to a partial order on the set $A_1 \times A_2$ of all pairs?
- *Reasonable assumption:* the validity of $(a_1, a_2) \preceq (a'_1, a'_2)$ depends only on:
  - whether $a_1 \preceq_1 a'_1$,
  - whether $a'_1 \preceq_1 a_1$,
  - whether $a_2 \preceq_2 a'_2$, and/or
  - whether $a'_2 \preceq_2 a_2$.
- It is also reasonable to assume that:

$$\text{if } a_1 \preceq_1 a'_1 \text{ and } a_2 \preceq_2 a'_2 \text{ then } (a_1, a_2) \preceq (a'_1, a'_2).$$

4. Posets in Uncertainty Logic: Need for Intervals and Products

- A similar partial order $\preceq$ is useful in describing degrees of expert's certainty, where

$$a \preceq a' \Leftrightarrow a \text{ corresponds to less certainty than } a'.$$

- Often, we cannot determine the exact value a of the expert's degree of certainty.
- In many cases, we can only determine the *interval* $[\underline{a}, \overline{a}]$ of possible values of a.
- Sometimes, two (or more) experts evaluate a statement S.
- Then, our certainty in S is described by a *pair* (a_1, a_2), where $a_i \in A_i$ is the i-th expert's degree of certainty.

5. Products of Ordered Sets: What Is Known

- At present, two product operations are known:
  - *Cartesian* product

$$(a_1, a_2) \preceq (a'_1, a'_2) \Leftrightarrow (a_1 \preceq_1 a'_1 \,\&\, a_2 \preceq_2 a'_2);$$

  and

  - *lexicographic* product

$$(a_1, a_2) \preceq (a'_1, a'_2) \Leftrightarrow$$

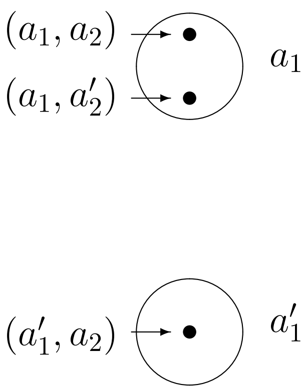
$$((a_1 \preceq_1 a'_1 \,\&\, a_1 \neq a'_1) \vee (a_1 = a'_1 \,\&\, a_2 \preceq_2 a'_2).$$

- *Question:* what other operations are possible?

6. Possible Physical Meaning of Lexicographic Order

Idea:

- A_1 is *macroscopic* space-time,
- A_2 is *microscopic* space-time:



Posets in Space-Time . . .
Intervals in Space- . . .
Products of Space- . . .
Posets in Uncertainty . . .
Main Theorem
Auxiliary Results: . . .
Second Example: . . .
Intersection Property . . .
Space-Time . . .
Title Page
Page 7 of 23
Go Back
Full Screen
Close
Quit

7. Possible Logical Meaning of Different Orders

- *Reminder:* our certainty in S is described by a *pair* $(a_1, a_2) \in A_1 \times A_2$.
- We must therefore define a *partial order* on the set $A_1 \times A_2$ of all pairs.
- *Cartesian product:* our confidence in S is higher than in S' if and only if it is higher for both experts.
- *Meaning:* a *maximally cautious* approach.
- *Lexicographic product:* means that we have *absolute confidence* in the first expert.
- We only use the opinion of the 2nd expert when, to the 1st expert, the degrees of certainty are equivalent.

8. Main Theorem

- By a *product operation*, we mean a Boolean function

$$P : \{T, F\}^4 \to \{T, F\}.$$

- For every two partially ordered sets A_1 and A_2, we define the following relation on $A_1 \times A_2$:

$$(a_1, a_2) \preceq (a'_1, a'_2) \stackrel{\text{def}}{=}$$

$$P(a_1 \preceq_1 a'_1, a'_1 \preceq_1 a_1, a_2 \preceq_2 a'_2, a'_2 \preceq_2 a_2).$$

- We say that a product operation is *consistent* if $\preceq$ is always a partial order, and

$$(a_1 \preceq_1 a'_1 \,\&\, a_2 \preceq_2 a'_2) \Rightarrow (a_1, a_2) \preceq (a'_1, a'_2).$$

- **Theorem:** *Every consistent product operation is the Cartesian or the lexicographic product.*

9. Auxiliary Results: General Idea and First Example

- For each property of intervals in an ordered set A, we analyze:
  - which properties need to be satisfied for A_1 and A_2
  - so that the corresponding property is satisfies for intervals in $A_1 \times A_2$.
- *Connectedness property (CP):* for every two points $a, a' \in A$, there exists an interval that contains a and a':

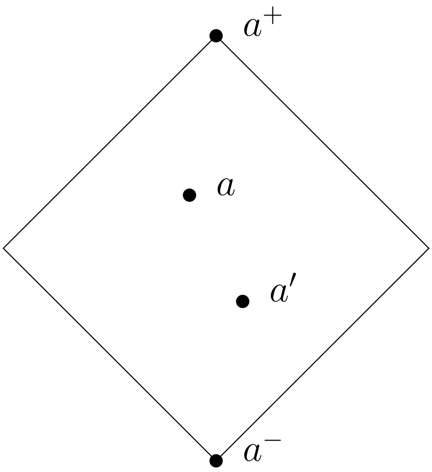
$$\forall a\, \forall a'\, \exists a^-\, \exists a^+\, (a^- \preceq a, a' \preceq a^+).$$

- This property is equivalent to two properties:
  - A is *upward-directed*: $\forall a\, \forall a'\, \exists a^+\, (a, a' \preceq a^+)$;
  - A is *downward-directed*: $\forall a\, \forall a'\, \exists a^-\, (a^- \preceq a, a')$.
- *Cartesian product*: A is upward-(downward-) directed $\Leftrightarrow$ both A_1 and A_2 are upward-(downward-) directed.

10. Connectedness Property Illustrated

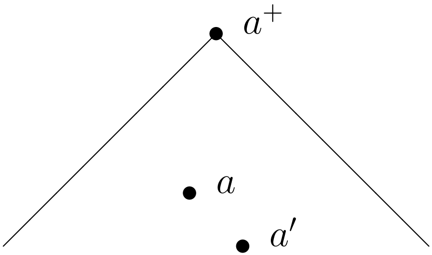
Connectedness property (CP): for every two points $a, a' \in A$, there exists an interval that contains a and a':

$$\forall a\, \forall a'\, \exists a^-\, \exists a^+\, (a^- \preceq a, a' \preceq a^+).$$

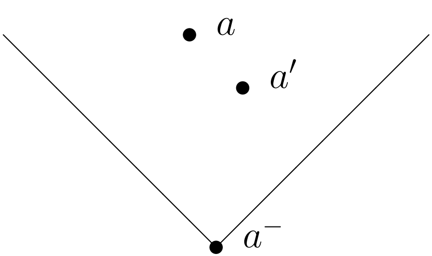


11. Upward and Downward Directed: Illustrated

Upward-directed: $\forall a\, \forall a'\, \exists a^+\, (a, a' \preceq a^+)$;



Downward-directed: $\forall a\, \forall a'\, \exists a^-\, (a^- \preceq a, a')$.



12. First Example, Case of Cartesian Product: Proof

- *Part 1:*
  - Let us assume that $A_1 \times A_2$ is upward-directed.
  - We want to prove that A_1 is upward-directed.
  - For any $a_1, a_1' \in A_1$, take any $a_2 \in A_2$, then
  $$\exists a^+ = (a_1^+, a_2^+) \text{ such that } (a_1, a_2), (a_1', a_2) \preceq a^+.$$
  - Hence $a_1, a_1' \preceq_1 a_1^+$, i.e., A_1 is upward-directed.
- *Part 2:*
  - Assume that both A_i are upward-directed.
  - We want to prove that $A_1 \times A_2$ is upward-directed.
  - For any $a = (a_1, a_2)$ and $a' = (a_1', a_2')$, for $i = 1, 2$,
  $$\exists\, a_i^+ \,(a_i, a_i' \preceq_i a_i^+).$$
  - Hence $(a_1, a_2), (a_1', a_2') \preceq (a_1^+, a_2^+)$, i.e., $A_1 \times A_2$ is upward-directed.

13. First Example: Case of Lexicographic Product

- $A_1 \times A_2$ is upward-directed $\Leftrightarrow$ the following two conditions hold:
  - the set A_1 is upward-directed, and
  - if A_1 has a maximal element $\overline{a}_1$ (= for which there are no a_1 with $\overline{a}_1 \prec_1 a_1$), then A_2 is upward-directed.
- $A_1 \times A_2$ is downward-directed $\Leftrightarrow$ the following two conditions hold:
  - the set A_1 is downward-directed, and
  - if A_1 has a minimal element $\underline{a}_1$ (= for which there are no a_1 for which $a_1 \prec_1 \underline{a}_1$), then A_2 is downward-directed.

14. Case of Lexicographic Product: Proof

- Let us assume that $A_1 \times A_2$ is upward-directed.
- *Part 1:*
  - We want to prove that A_1 is upward-directed.
  - For any $a_1, a_1' \in A_1$, take any $a_2 \in A_2$, then

    $$\exists a^+ = (a_1^+, a_2^+) \text{ for which } (a, a_2), (a', a_2) \preceq a^+.$$
  - Hence $a_1, a_1' \preceq_1 a_1^+$, i.e., A_1 is upward-directed.
- *Part 2:*
  - Let $\overline{a}_1$ be a maximal element of A_1.
  - For any $a_2, a_2' \in A_2$, we have

    $$\exists a^+ = (a_1^+, a_2^+) \text{ for which } (\overline{a}_1, a_2), (\overline{a}_1, a_2') \preceq a^+.$$
  - Here, $\overline{a}_1 \preceq_1 a_1^+$ and since $\overline{a}_1$ is maximal, $a_1^+ = \overline{a}_1$.
  - Hence $a_2, a_2' \preceq_2 a_2^+$, i.e., A_2 is upward-directed.

15. Case of Lexicographic Product: Proof (cont-d)

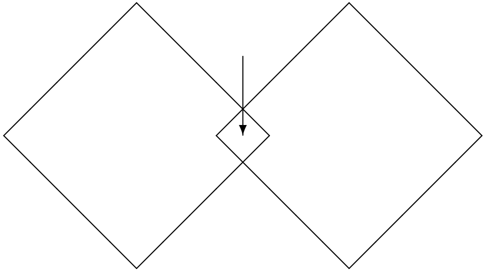
- Let us assume that A_1 is upward-directed.
- Let us assume that if A_1 has a maximal element, then A_2 is upward-directed.
- We want to prove that $A_1 \times A_2$ is upward-directed.
- Take any $a = (a_1, a_2)$ and $a' = (a'_1, a'_2)$ from $A_1 \times A_2$.
- Since A_1 is upward-directed, $\exists a_1^+ \, (a_1, a'_1 \preceq_1 a_1^+)$.
- If $a_1 \prec_1 a_1^+$, then $(a_1, a_2), (a'_1, a'_2) \preceq (a_1^+, a'_2)$.
- If $a'_1 \prec_1 a_1^+$, then $(a_1, a_2), (a'_1, a'_2) \preceq (a_1^+, a_2)$.
- If $a_1 = a_1^+ = a'_1$, and a_1 is not a maximal element, then $\exists a''_1 \, (a_1 \prec_1 a''_1)$, hence $(a_1, a_2), (a'_1, a'_2) \preceq (a''_1, s_2)$.
- If $a_1 = a_1^+ = a'_1$, and a_1 is a maximal element, then A_2 is upward-directed, hence $\exists a_2^+ \, (a_2, a'_2 \preceq_2 a_2^+)$ and
$$(a_1, a_2), (a_1, a'_2) \preceq (a_1, a_2^+).$$

16. Second Example: Intersection Property

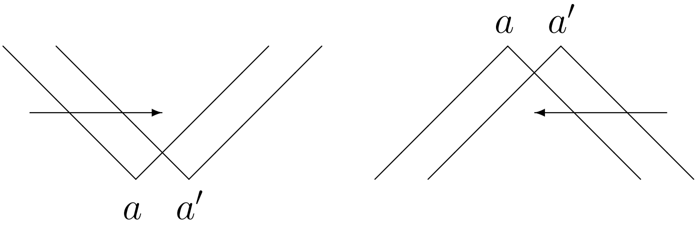
- The intersection of every two intervals is an interval.
- *Comment:* this is true for intervals on the real line.
- This can be similarly reduced to two properties:
  - the intersection of every two future cones $Q^+_a \stackrel{\text{def}}{=} \{b : a \preceq b\}$ is a future cone;
  - the intersection of every two past cones $Q^-_a \stackrel{\text{def}}{=} \{b : b \preceq a\}$ is a past cone.
- If both properties hold, then the intersection of every two intervals $[a, b] = Q^+_a \cap Q^-_b$ is an interval.
- Ordered sets with Q^+ and Q^- properties are called *upper* and *lower* semi-lattices.
- *For Cartesian product:* $A_1 \times A_2$ is an upper (lower) semi-lattice $\Leftrightarrow$ both A_i are upper (lower) semi-lattices.

17. Intersection Property Illustrated

Intersection property for intervals:



Upper and lower semi-lattice properties:



18. Intersection Property: Lexicographic Product

- $A_1 \times A_2$ is an upper semi-lattice $\Leftrightarrow$ A_1 is an upper semi-lattice and one of the following conditions holds:
  - A_1 is linearly (lin.) ordered and A_2 is an upper semi-lattice;
  - A_2 is an upper semi-lattice that has the smallest element;
  - A_1 is sequential up, A_2 is a conditional upper semi-lattice, and A_2 has the smallest element.
- *Proof:* Let us assume that $A_1 \times A_2$ is an upper semi-lattice.
- *Notation:* the element a'' s.t. $Q^+_a \cap Q^+_{a'} = Q^+_{a''}$ is called a *join* and denoted $a \vee a'$.
- Let us prove that A_1 is an upper semi-lattice.

19. Proof for Lexicographic Product (idea)

- For any $a_1, a'_1 \in A_1$, for any $a_2 \in A_1$, take

$$(a_1^+, a_2^+) \stackrel{\text{def}}{=} (a_1, a_2) \vee (a'_1, a_2).$$

- One can prove that $a_1^+ = a_1 \vee a'_1$, so A_1 is an upper semi-lattice.
- If A_1 is not lin. ordered, $\exists a_1, a'_1\, (a_1 \not\preceq_1 a'_1 \,\&\, a'_1 \not\preceq_1 a_1)$.
- For $(a_1^+, a_2^+) = (a_1, a_2) \vee (a'_1, a_2)$, we have $a_1 \preceq_1 a_1^+$ and $a'_1 \preceq_1 a_1^+$, hence $a_1^+ \neq a_1$ and $a_1^+ \neq a'_1$, i.e.,

$$a_1 \prec_1 a_1^+ \text{ and } a'_1 \prec_1 a_1^+.$$

- For every $a'_2 \in A_2$, we have $(a_1^+, a'_2) \in Q^+_{(a_1,a_2)}$ and $(a_1^+, a'_2) \in Q^+_{(a'_1,a_2)}$, hence $(a_1^+, a_2^+) \preceq (a_1^+, a'_2)$ and

$$a_2^+ \preceq_2 a'_2.$$

- Thus, a_2^+ is the smallest element of A_2.

20. Space-Time Geometry: Physical References

- H. Busemann, *Timelike spaces*, PWN: Warszawa, 1967.
- E. H. Kronheimer and R. Penrose, "On the structure of causal spaces", *Proc. Cambr. Phil. Soc.*, Vol. 63, No. 2, pp. 481–501, 1967.
- C. W. Misner, K. S. Thorne, and J. A. Wheeler, *Gravitation*, New York: W. H. Freeman, 1973.
- R. I. Pimenov, *Kinematic spaces: Mathematical Theory of Space-Time*, N.Y.: Consultants Bureau, 1970.

21. Space-Time Geometry: Mathematical and Computational References

- V. Kreinovich and O. Kosheleva, "Computational complexity of determining which statements about causality hold in different space-time models", *Theoretical Computer Science*, 2008, Vol. 405, No. 1–2, pp. 50–63.
- A. Levichev and O. Kosheleva, "Intervals in space-time", *Reliable Computing*, 1998, Vol. 4, No. 1, pp. 109–112.
- P. G. Vroegindeweij, V. Kreinovich, and O. M. Kosheleva. "From a connected, partially ordered set of events to a field of time intervals", *Foundations of Physics*, 1980, Vol. 10, No. 5/6, pp. 469–484.

22. References: Uncertainty Logic

- G. Klir and B. Yuan, *Fuzzy Sets and Fuzzy Logic: Theory and Applications*, Upper Saddle River, New Jersey: Prentice Hall, 1995.
- J. M. Mendel, *Uncertain Rule-Based Fuzzy Logic Systems: Introduction and New Directions*, Prentice-Hall, 2001.
- H. T. Nguyen, V. Kreinovich, and Q. Zuo, "Interval-valued degrees of belief: applications of interval computations to expert systems and intelligent control", *International Journal of Uncertainty, Fuzziness, and Knowledge-Based Systems (IJUFKS)*, 1997, Vol. 5, No. 3, pp. 317–358.
- H. T. Nguyen and E. A. Walker, *A First Course in Fuzzy Logic*, Chapman & Hall/CRC, Boca Raton, Florida, 2006.