Towards the Possibility of Objective Interval Uncertainty in Physics. II

Luc Longpré and Vladik Kreinovich

Department of Computer Science
University of Texas at El Paso
500 W. University
El Paso, TX 79968, USA
longpre@utep.edu, vladik@utep.edu
1. Is Interval Uncertainty Subjective?

- Applications of interval computations usually assume that:
  - while we only know an interval \([x, \bar{x}]\) containing the actual (unknown) value of a physical quantity \(x\),
  - there is the exact value \(x\) of this quantity, and that
  - in principle, we can get more and more accurate estimates of this value.

- This assumption is in line with the usual formulations of physical theories – as
  - partial differential equations
  - relating exact values of different physical quantities, fields, etc., at different space-time locations.

- Due to uncertainty principle, there are limitations on how accurately we can measure physical quantities.
2. It Is Desirable to Take Objective Uncertainty into Account

- One of the important principles of modern physics is *operationalism*.

- According to this principle, a physical theory should only use observable quantities.

- This principle is behind most successes of the 20th-century physics, such as:
  - relativity theory (vs. un-observable aether),
  - quantum mechanics.

- Thus, it is desirable:
  - to avoid using un-measurable exact values and
  - to modify physical theories so that they explicitly take objective uncertainty into account.
3. Objective Uncertainty Is About Probabilities

- According to quantum physics, we can only predict probabilities of different events.
- Thus, uncertainty means that instead of exact values of these probabilities, we can only determine intervals.
- What is the observational meaning of probability?
- If a sequence $\omega_1 \omega_2 \ldots$ is random, it satisfies all the probability laws such as the law of large numbers.
- If a sequence satisfies all probability laws, then for all practical purposes we can consider it random.
- Thus, we can define a sequence to be random if it satisfies all probability laws.
- A probability law is a statement $S$ which is true with probability 1: $P(S) = 1$. 

- A sequence is called *random* if it satisfies all probability laws.
- A probability law is a statement $S$ which is true with probability 1: $P(S) = 1$.
- So, a sequence is random if it belongs to all definable sets of measure 1.
- A sequence belongs to a set of measure 1 iff it does not belong to its complement $C = \neg S$ with $P(C) = 0$.
- So, a sequence is random if it does not belong to any definable set of measure 0.
- There are countably many definable sets, so the union of all such sets has measure 0.
- Thus, almost all sequences are KML-random.
5. Probability Interval: Observational Meaning

- Probabilities have direct observational meaning only for repeating events.

- In mathematical terms, independent repeating events correspond to a product measure:

\[ P(A \& B) = P(A) \cdot P(B). \]

- Traditional case: we know the exact probability \( p \).

- Then, observable sequences \( \omega_1\omega_2 \ldots \) are KLM-random relative to a product of \( p \)-measures.

- It is natural to say that a sequence is \([p, \overline{p}]\)-random if it is random for some product measure with \( p_i \in [p, \overline{p}] \).

- If \( p \in [p, \overline{p}] \), then, of course, each \( p \)-random sequence is also \([p, \overline{p}]\)-random.

- In this case, the interval uncertainty is subjective.
6. Can There Be Objective Interval Uncertainty?

- We say that a sequence $\omega_1 \omega_2 \ldots$ is objectively $[p, \bar{p}]$-random if:
  - this sequence is $[p, \bar{p}]$-random, and
  - this sequence is not $[q, \bar{q}]$-random for any narrower interval $[q, \bar{q}] \subset [p, \bar{p}]$.

- Proposition. For every interval $[p, \bar{p}]$, there exist objectively $[p, \bar{p}]$-random sequences.

- Example: any sequence $\omega_1 \omega_2 \ldots$ corresponding to $p_i$ for which $\lim \inf p_i = p$ and $\lim \sup p_i = \bar{p}$.

- Proof: let us prove that this sequence $\omega_1 \omega_2 \ldots$ is not $[q, \bar{q}]$-random for any proper subinterval $[q, \bar{q}] \subset [p, \bar{p}]$.

- It is known that if two measures are mutually singular, then no sequence is random w.r.t. both measures.
7. Proof (cont-d)

- For product measures, singularity is equivalent to
  \[ \sum_{i=1}^{\infty} \left[ (\sqrt{p_i} - \sqrt{q_i})^2 + (\sqrt{1 - p_i} - \sqrt{1 - q_i})^2 \right] = +\infty. \]

- For a proper subinterval, \( p < q \) or \( q < p \).

- W.l.o.g., let us consider the case when \( p < q \).

- When \( \lim \inf p_i = p \) then, for every \( \varepsilon > 0 \), there are infinitely many \( i \) s.t. \( \sqrt{p_i} \leq \sqrt{p} + \varepsilon \).

- For these \( i \), we have \( q_i \geq q \), so \( \sqrt{q_i} \geq \sqrt{q} \).

- Thus, \( \sqrt{q_i} - \sqrt{p_i} \geq \sqrt{q} - \left( \sqrt{p} + \varepsilon \right) = \left( \sqrt{q} - \sqrt{p} \right) - \varepsilon \).

- For \( \varepsilon = (\sqrt{q} - \sqrt{p})/2 \), we have \( \sqrt{q_i} - \sqrt{p_i} > \varepsilon > 0 \) and therefore, the above sum is infinite.

- So, a \( \{p_i\} \)-random sequence \( \omega_1 \omega_2 \ldots \) cannot be \( \{q_i\} \)-random. The proposition is proven.
8. From Kolmogorov-Martin-Löf Theoretical Randomness to a More Physical One

- The above definition means that (definable) events with probability 0 cannot happen.
- In practice, physicists also assume that events with a very small probability cannot happen.
- For example, a kettle on a cold stove will not boil by itself – but the probability is non-zero.
- If a coin falls head 100 times in a row, any reasonable person will conclude that this coin is not fair.
- It is not possible to formalize this idea by simply setting a threshold \( p_0 > 0 \) below which events are not possible.
- Indeed, then, for \( N \) for which \( 2^{-N} < p_0 \), no sequence of \( N \) heads or tails would be possible at all.
9. From Kolmogorov-Martin-Löf Theoretical Randomness to a More Physical One (cont-d)

- We cannot have a universal threshold $p_0$ such that events with probability $\leq p_0$ cannot happen.

- However, we know that:
  - for each decreasing $(A_n \supseteq A_{n+1})$ sequence of properties $A_n$ with $\lim p(A_n) = 0$,
  - there exists an $N$ above which a truly random sequence cannot belong to $A_N$.

- Resulting definition: we say that $\mathcal{R}$ is a set of random elements if
  - for every definable decreasing sequence $\{A_n\}$ for which $\lim P(A_n) = 0$,
  - there exists an $N$ for which $\mathcal{R} \cap A_N = \emptyset$. 
10. Related Idea: Physical Induction

- How do we come up with physical laws?
- Someone formulates a hypothesis.
- This hypothesis is tested, and if it confirmed sufficiently many times.
- Then we conclude that this hypothesis is indeed a universal physical law.
- This conclusion is known as *physical induction*.
- Different physicists may disagree on how many experiments we need to become certain.
- However, most physicists would agree that:
  - after sufficiently many confirmations,
  - the hypothesis should be accepted as a physical law.
- Example: normal distribution :-)

10. Related Idea: Physical Induction
11. How to Describe Physical Induction in Precise Terms

• Let $s$ denote the state of the world, and let $P(s, i)$ indicate that the property $P$ holds in the $i$-th experiment.

• In these terms, physical induction means that for every property $P$, there is an integer $N$ such that:
  – if the statements $P(s, 1), \ldots, P(s, N)$ are all true,
  – then the property $P$ holds for all possible experiments – i.e., we have $\forall n P(s, n)$.

• This cannot be true for all *mathematically possible* states: we can have $P(s, 1), \ldots, P(s, N)$ and $\neg P(s, N + 1)$.

• Our understanding of the physicists’ claims is that:
  – if we restrict ourselves to *physically meaningful* states,
  – then physical induction is true.
12. Resulting Definition

• Let $S$ be a set; its elements will be called \textit{states of the world}.

• Let $T \subseteq S$ be a subset of $S$. We say that $T$ consists of \textit{physically meaningful states} if:
  
  – for every property $P$, there exists an integer $N_P$ such that
  – for each state $s \in T$ for which $P(s, i)$ holds for all $i \leq N_P$, we have $\forall n P(s, n)$.

• For this definition to be precise, we need to select a theory $\mathcal{L}$ which is:
  
  – rich enough to contain all physicists’ arguments and
  – weak enough so that we will be able to formally talk about definability in $\mathcal{L}$. 
13. Definition: Equivalent Form

- We can reformulate this definition in terms of *definable sets*, i.e.:
  - sets of the type \( \{x : P(x)\} \)
  - corresponding to definable properties \( P(x) \).

- Let \( S \) be a set; its elements will be called *states of the world*.

- Let \( T \subseteq S \) be a subset of \( S \). We say that \( T \) consists of *physically meaningful states* if:
  - for every definable sequence of sets \( \{A_n\} \), there exists an integer \( N_A \)
  - such that \( T \cap \bigcap_{n=1}^{N_A} A_n = T \cap \bigcap_{n=1}^{\infty} A_n \).
14. Existence Proof

- There are no more than countably many words, so no more than countably many definable sequences.

- For real numbers, we can enumerate all definable sequence, as \( \{A^1_n\}, \{A^2_n\}, \ldots \) Let us pick \( \varepsilon \in (0, 1) \).

- For each \( k \), for the difference sets \( D^k_n \overset{\text{def}}{=} \bigcap_{i=1}^{n} A^k_n - \bigcap_{i=1}^{\infty} A^k_n \), we have \( D^k_{n+1} \subseteq D^k_n \) and \( \bigcap_{n=1}^{\infty} D^k_n = \emptyset \), thus, \( \mu(D^k_n) \to 0 \).

- Hence, there exists \( n_k \) for which \( \mu(D^k_{n_k}) \leq 2^{-k} \cdot \varepsilon \).

- We then take \( T = S - \bigcup_{k=1}^{\infty} D^k_{n_k} \).

- Here, \( \mu \left( \bigcup_{k=1}^{\infty} D^k_{n_k} \right) \leq \sum_{k=1}^{\infty} \mu(D^k_{n_k}) \leq \sum_{k=1}^{\infty} 2^{-k} \cdot \varepsilon = \varepsilon < 1 \), and thus, the difference \( T \) is non-empty.

- For this set \( T \), we can take \( N_{A^k} = n_k \).
15. Random Sequences and Physically Meaningful Sequences

- Let $R_K$ denote the set of all elements which are random in Kolmogorov-Martin-Löf sense. Then:

- Every set of random elements consists of physically meaningful elements.

- For every set $T$ of physically meaningful elements, the intersection $T \cap R_K$ is a set of random elements.

- **Proof:** When $A_n$ is definable, for $D_n \overset{\text{def}}{=} \bigcap_{i=1}^{n} A_i - \bigcap_{i=1}^{\infty} A_i$, we have $D_n \supseteq D_{n+1}$ and $\bigcap_{n=1}^{\infty} D_n = \emptyset$, so $P(D_n) \to 0$.

- Therefore, there exists an $N$ for which the set of random elements does not contain any elements from $D_N$.

- Thus, every set of random elements indeed consists of physically meaningful elements.
16. Proof (cont-d)

- Let $T$ consist of physically meaningful elements. Let us prove that $T \cap \mathcal{R}_K$ is a set of random elements.

- If $A_n \supseteq A_{n+1}$ and $P\left(\bigcap_{n=1}^{\infty} A_n\right) = 0$, then for $B_m \overset{\text{def}}{=} A_m - \bigcap_{n=1}^{\infty} A_n$, we have $B_m \supseteq B_{m+1}$ and $\bigcap_{n=1}^{\infty} B_n = \emptyset$.

- Thus, by definition of a set consisting of physically meaningful elements, we conclude that $B_N \cap T = \emptyset$.

- Since $P\left(\bigcap_{n=1}^{\infty} A_n\right) = 0$, we also know that $\left(\bigcap_{n=1}^{\infty} A_n\right) \cap \mathcal{R}_K = \emptyset$.

- Thus, $A_N = B_N \cup \left(\bigcap_{n=1}^{\infty} A_n\right)$ has no common elements with the intersection $T \cap \mathcal{R}_K$. Q.E.D.
17. Interval Case

- **Reminder**: we want to describe the fact that events with very small probability are impossible.

- **Case of exactly known probability** $p$:
  - in addition to requiring that the sequence of observations $\omega_1\omega_2\ldots$ is $p$-random,
  - we also require that this sequence is physically meaningful.

- **Interval case** can be handled similarly:
  - in addition to requiring that the sequence of observations $\omega_1\omega_2\ldots$ is $[p, \overline{p}]$-random,
  - we also require that this sequence is physically meaningful.
18. Additional Consequence

- **Main objectives of science:**
  - *guaranteed* estimates for physical quantities;
  - *guaranteed* predictions for these quantities.
- **Problem:** estimation and prediction are ill-posed.
- **Example:**
  - measurement devices are inertial;
  - hence suppress high frequencies $\omega$;
  - so $\varphi(x)$ and $\varphi(x) + \sin(\omega \cdot t)$ are indistinguishable.
- **Existing approaches:**
  - statistical regularization (filtering);
  - Tikhonov regularization (e.g., $|\dot{x}| \leq \Delta$);
  - expert-based regularization.
- **Main problem:** no guarantee.
19. On Physically Meaningful Solutions, Problems Become Well-Posed

- **State estimation – an ill-posed problem:**
  - Measurement $f$:
    state $s \in S \to$ observation $r = f(s) \in R$.
  - *In principle*, we can reconstruct $r \to s$:
    as $s = f^{-1}(r)$.
  - **Problem**: small changes in $r$ can lead to huge changes in $s$ ($f^{-1}$ not continuous).

- **Theorem:**
  - Let $S$ be a definably separable metric space.
  - Let $\mathcal{T}$ be a set of physically meaningful elements of $S$.
  - Let $f : S \to R$ be a continuous 1-1 function.
  - Then, the inverse mapping $f^{-1} : R \to S$ is continuous for every $r \in f(\mathcal{T})$. 
20. Everything is Related – Einstein-Podolsky-Rosen (EPR) Paradox

- Due to Relativity Theory, two spatially separated simultaneous events cannot influence each other.
- Einstein, Podolsky, and Rosen intended to show that in quantum physics, such influence is possible.
- In formal terms, let $x$ and $x'$ be measured values at these two events.
- Independence means that possible values of $x$ do not depend on $x'$, i.e., $S = X \times X'$ for some $X$ and $X'$.
- Physical induction implies that the pair $(x, x')$ belongs to a set $S$ of physically meaningful pairs.
- Theorem: The set $S$ cannot be represented as $X \times X'$.
- Thus, everything is related – but we probably can’t use this relation to pass information ($S$ isn’t computable).
21. From States of the World to Specific Quantities

- Usually, we only have a partial information about a state: we have a definable function \( f : S \rightarrow X \) which maps
  - every state of the world
  - into the corresponding partial information.

- Then the range \( f(T) \) corresponding to all physically meaningful states has the same property as \( T \):

- Let a set \( T \subseteq S \) consist of physically meaningful states, and let \( f : S \rightarrow X \) be a definable function.

- Then, for every definable sequence of subsets \( B_n \subseteq X \), there exists an integer \( N_B \) such that

\[
N_B \bigcap_{n=1}^{\infty} B_n = \bigcap_{n=1}^{\infty} B_n.
\]
22. Proof

- We want to prove that for some $N_B$,
  - if an element $x \in f(T)$ belongs to the sets $B_1, \ldots, B_{N_B}$,
  - then $x \in B_n$ for all $n$.

- An arbitrary element $x \in f(T)$ has the form $x = f(s)$ for some $s \in T$.

- Let us take $A_n \overset{\text{def}}{=} f^{-1}(B_n)$.

- Since $T$ consists of physically meaningful states, there exists an appropriate integer $N_A$.

- Let us take $N_B \overset{\text{def}}{=} N_A$.

- By definition of $A_n$, the condition $x = f(s) \in B_i$ implies that $s \in A_i$; so we have $s \in A_i$ for all $i \leq N_A$.

- Thus, we have $s \in A_n$ for all $n$, which implies that $x = f(s) \in B_n$. Q.E.D.
23. Computations with Real Numbers: Reminder

- From the physical viewpoint, real numbers $x$ describe values of different quantities.
- We get values of real numbers by measurements.
- Measurements are never 100% accurate, so after a measurement, we get an approximate value $r_k$ of $x$.
- In principle, we can measure $x$ with higher and higher accuracy.
- So, from the computational viewpoint, a real number is a sequence of rational numbers $r_k$ for which, e.g.,
  $$|x - r_k| \leq 2^{-k}.$$  
- By an algorithm processing real numbers, we mean an algorithm using $r_k$ as an "oracle" (subroutine).
- This is how computations with real numbers are defined in *computable analysis*. 
24. Checking Equality of Real Numbers

- **Known:** equality of real numbers is undecidable.
- For physically meaningful real numbers, however, a deciding algorithm is possible:
  - for every set $T \subseteq \mathbb{R}^2$ which consists of physically meaningful pairs $(x, y)$ of real numbers,
  - there exists an algorithm deciding whether $x = y$.
- **Proof:** We can take $A_n = \{ (x, y) : 0 < |x - y| < 2^{-n} \}$. The intersection of all these sets is empty.
- Hence, $T$ has no elements from $\bigcap_{n=1}^{N_A} A_n = A_{N_A}$.
- Thus, for each $(x, y) \in T$, $x = y$ or $|x - y| \geq 2^{-N_A}$.
- We can detect this by taking $2^{-(N_A+3)}$-approximations $x'$ and $y'$ to $x$ and $y$. Q.E.D.
25. Finding Roots

- In general, it is not possible, given a f-n $f(x)$ attaining negative and positive values, to compute its root.
- This becomes possible if we restrict ourselves to physically meaningful functions:
  - Let $K$ be a computable compact.
  - Let $X$ be the set of all functions $f : K \to \mathbb{R}$ that attain 0 value somewhere on $K$. Then:
    - for every set $T \subseteq X$ consisting of physically meaningful functions and for every $\varepsilon > 0$,
    - there is an algorithm that, given a f-n $f \in T$, computes an $\varepsilon$-approximation to the set of roots
      \[ R \overset{\text{def}}{=} \{ x : f(x) = 0 \} . \]
- In particular, we can compute an $\varepsilon$-approximation to one of the roots.
26. Finding Roots: Proof

- To compute the set $R = \{x : f(x) = 0\}$ with accuracy $\varepsilon > 0$, let us take an $(\varepsilon/2)$-net $\{x_1, \ldots, x_n\} \subseteq K$.

- For each $i$, we can compute $\varepsilon' \in (\varepsilon/2, \varepsilon)$ for which $B_i \overset{\text{def}}{=} \{x : d(x, x_i) \leq \varepsilon'\}$ is a computable compact set.

- It is possible to algorithmically compute the minimum of a function on a computable compact set.

- Thus, we can compute $m_i \overset{\text{def}}{=} \min\{|f(x)| : x \in B_i\}$.

- Since $f \in T$, similarly to the previous proof, we can prove that $\exists N \forall f \in T \forall i (m_i = 0 \lor m_i \geq 2^{-N})$.

- Comp. $m_i$ w/acc. $2^{-(N+2)}$, we check $m_i = 0$ or $m_i > 0$.

- Let’s prove that $d_H(R, \{x_i : m_i = 0\}) \leq \varepsilon$, i.e., that $\forall i (m_i = 0 \Rightarrow \exists x (f(x) = 0 \land d(x, x_i) \leq \varepsilon))$ and $\forall x (f(x) = 0 \Rightarrow \exists i (m_i = 0 \land d(x, x_i) \leq \varepsilon))$. 
27. Finding Roots: Proof (cont-d)

- $m_i = 0$ means $\min \{|f(x)| : x \in B_i \overset{\text{def}}{=} B_{\varepsilon'}(x_i)\} = 0$.

- Since the set $K$ is compact, this value 0 is attained, i.e., there exists a value $x \in B_i$ for which $f(x) = 0$.

- From $x \in B_i$, we conclude that $d(x, x_i) \leq \varepsilon'$ and, since $\varepsilon' < \varepsilon$, that $d(x, x_i) < \varepsilon$.

- Thus, $x_i$ is $\varepsilon$-close to the root $x$.

- Vice versa, let $x$ be a root, i.e., let $f(x) = 0$.

- Since the points $x_i$ form an $(\varepsilon/2)$-net, there exists an index $i$ for which $d(x, x_i) \leq \varepsilon/2$.

- Since $\varepsilon/2 < \varepsilon'$, this means that $d(x, x_i) \leq \varepsilon'$ and thus, $x \in B_i$.

- Therefore, $m_i = \min \{|f(x)| : x \in B_i\} = 0$. So, the root $x$ is $\varepsilon$-close to a point $x_i$ for which $m_i = 0$. 
28. **Optimization**

- In general, it is not algorithmically possible to find \( x \) where \( f(x) \) attains maximum.

- Let \( K \) be a computable compact. Let \( X \) be the set of all functions \( f : K \to \mathbb{R} \). Then:
  
  - for every set \( T \subseteq X \) consisting of physically meaningful functions and for every \( \varepsilon > 0 \),
  - there is an algorithm that, given a f-n \( f \in T \), computes an \( \varepsilon \)-approx. to \( S = \{ x : f(x) = \max_y f(y) \} \).

- In particular, we can compute an approximation to an individual \( x \in S \).

- **Reduction to roots:** \( f(x) = \max_y f(y) \) iff \( g(x) = 0 \),
  
  where \( g(x) \overset{\text{def}}{=} f(x) - \max_y f(y) \).
29. Computing Fixed Points

- In general, it is not possible to compute all the fixed points of a given computable function $f(x)$.

- Let $K$ be a computable compact. Let $X$ be the set of all functions $f : K \rightarrow K$. Then:
  
  - for every set $T \subseteq X$ consisting of physically meaningful functions and for every $\varepsilon > 0$,
  - there is an algorithm that, given a function $f \in T$, computes an $\varepsilon$-approximation to the set $\{x : f(x) = x\}$.

- In particular, we can compute an approximation to an individual fixed point.

- Reduction to roots: $f(x) = x$ iff $g(x) = 0$, where $g(x) \overset{\text{def}}{=} d(f(x), x)$. 

Where $d$ denotes the distance metric.
30. Computing Limits

• *In general:* it is not algorithmically possible to find a limit \( \lim a_n \) of a convergent computable sequence.

• Let \( K \) be a computable compact. Let \( X \) be the set of all convergent sequences \( a = \{a_n\}, a_n \in K \). Then:
  
  – for every set \( T \subseteq X \) consisting of physically meaningful functions and for every \( \varepsilon > 0 \),
  
  – there exists an algorithm that, given a sequence \( a \in T \), computes its limit with accuracy \( \varepsilon \).

• *Use:* this enables us to compute limits of iterations and sums of Taylor series (frequent in physics).

• *Main idea:* for every \( \varepsilon > 0 \) there exists \( \delta > 0 \) such that when \( |a_n - a_{n-1}| \leq \delta \), then \( |a_n - \lim a_n| \leq \varepsilon \).

• *Intuitively:* we stop when two consequent iterations are close to each other.
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32. Kolmogorov Randomness


33. Randomness: From Exact to Interval Probabilities


34. A Formal Definition of Definable Sets

- Let $\mathcal{L}$ be a theory.
- Let $P(x)$ be a formula from $\mathcal{L}$ for which the set $\{x \mid P(x)\}$ exists.
- We will then call the set $\{x \mid P(x)\}$ $\mathcal{L}$-definable.
- Crudely speaking, a set is $\mathcal{L}$-definable if we can explicitly define it in $\mathcal{L}$.
- All usual sets are definable: $\mathbb{N}$, $\mathbb{R}$, etc.
- Not every set is $\mathcal{L}$-definable:
  - every $\mathcal{L}$-definable set is uniquely determined by a text $P(x)$ in the language of set theory;
  - there are only countably many texts and therefore, there are only countably many $\mathcal{L}$-definable sets;
  - so, some sets of natural numbers are not definable.
35. How to Prove Results About Definable Sets

- Our objective is to be able to make mathematical statements about $\mathcal{L}$-definable sets. Therefore:
  - in addition to the theory $\mathcal{L}$,
  - we must have a stronger theory $\mathcal{M}$ in which the class of all $\mathcal{L}$-definable sets is a countable set.

- For every formula $F$ from the theory $\mathcal{L}$, we denote its Gödel number by $\lfloor F \rfloor$.

- We say that a theory $\mathcal{M}$ is stronger than $\mathcal{L}$ if:
  - $\mathcal{M}$ contains all formulas, all axioms, and all deduction rules from $\mathcal{L}$, and
  - $\mathcal{M}$ contains a predicate $\text{def}(n, x)$ such that for every formula $P(x)$ from $\mathcal{L}$ with one free variable,
    \[
    \mathcal{M} \vdash \forall y \left( \text{def}([P(x)], y) \leftrightarrow P(y) \right).
    \]
36. Existence of a Stronger Theory

- As $\mathcal{M}$, we take $\mathcal{L}$ plus all above equivalence formulas.
- Is $\mathcal{M}$ consistent?
- Due to compactness, we prove that for any $P_1(x), \ldots, P_m(x)$, $\mathcal{L}$ is consistent with the equivalences corr. to $P_i(x)$.
- Indeed, we can take
  \[ \text{def}(n, y) \leftrightarrow (n = \lfloor P_1(x) \rfloor \& P_1(y)) \lor \ldots \lor (n = \lfloor P_m(x) \rfloor \& P_m(y)). \]
- This formula is definable in $\mathcal{L}$ and satisfies all $m$ equivalence properties.
- Thus, the existence of a stronger theory is proven.
- The notion of an $\mathcal{L}$-definable set can be expressed in $\mathcal{M}$: $S$ is $\mathcal{L}$-definable iff $\exists n \in \mathbb{N} \forall y (\text{def}(n, y) \leftrightarrow y \in S)$.
- So, all statements involving definability become statements from the $\mathcal{M}$ itself, not from metalanguage.