Why Superellipsoids: A Probability-Based Explanation

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1. Outline

- In many practical situations, possible values of the deviation vector form (approximately) a super-ellipsoid.
- In this talk, we provide a theoretical explanation for this empirical fact.
- This explanation is based on the natural notion of scale-invariance.
2. Need to Describe Uncertainty Domains

- The intent of mass production is to produce gadgets with same values \((x_1, \ldots, x_n)\) of the characteristics \(x_i\).
- In reality, different gadgets have slightly different values \(\tilde{x}_i\) of these characteristics: \(\Delta x_i \overset{\text{def}}{=} \tilde{x}_i - x_i \neq 0\).
- For each of these characteristics \(x_i\), we usually have a tolerance bound \(\Delta_i\) for which \(|\Delta x_i| \leq \Delta_i\).
- Possible values of \(\Delta x_i\) form an interval \([-\Delta_i, \Delta_i]\).
- Thus, possible values of the deviation vector \(\Delta x = (\Delta x_1, \ldots, \Delta x_n)\) are in the box
  \[[-\Delta_1, \Delta_1] \times \ldots \times [-\Delta_n, \Delta_n].\]
- In practice, not all \(\Delta x\) from this box are possible.
- It is desirable to describe the set \(S\) of all possible deviation vectors \(\Delta x\); \(S\) is called *uncertainty domain*. 
3. Shall Not We Also Determine Probabilities?

- At first glance, it seems that we should be interested:
  - not only in finding out which deviation vectors $\Delta x$ are possible and which are not,
  - but also in how frequent different possible vectors are.

- In other words, we should be interested in the probability distribution on this domain.

- In reality, however, it is not possible to find these probabilities.

- Indeed, the manufacturing process may slightly change (and often does change).

- After each such change, the tolerance intervals and the uncertainty domain remain largely unchanged.

- However, the probabilities change (often drastically).
4. Empirical Shapes of Uncertainty Domains

- In many practical cases, the uncertainty domain can be well approximated by a super-ellipsoid:
  \[ \sum_{i=1}^{n} \left( \frac{\left| \Delta x_i \right|}{\sigma_i} \right)^p \leq C. \]

- Their approximation accuracy is higher than for other families with the same number of parameters.

- Super-ellipsoids are also actively used in image processing, to describe different components of an image.

- In this talk, we provide a theoretical explanation for this empirical phenomenon.
5. First Idea: Probabilistic Approach

- In reality, there is some probability distribution $\rho_i(\Delta x_i)$ for each of the random variables $\Delta x_i$.
- We have no reason to assume that positive or negative values of $\Delta x_i$ are more probable.
- So, it makes sense to assume that they are equally probable.
- So, each distribution $\rho_i(\Delta x_i)$ is symmetric: $\rho_i(\Delta x_i) = \rho_i(|\Delta x_i|)$.
- Similarly, we have no reasons to believe that different deviations are statistically dependent.
- So, it makes sense to assume that random variables $\Delta x_i$ are independent: $\rho(\Delta x) = \prod_{i=1}^{n} \rho_i(|\Delta x_i|)$.
- Usually, we consider a deviation vector possible if its probability exceed some $t$: $S_t \overset{\text{def}}{=} \{ \Delta x : \rho(\Delta x) \geq t \}$. 
6. Second Idea: Scale Invariance

- Numerical values of the deviations $\Delta x_i$ depend on the choice of a measuring unit.
- If we replace the original unit by a $\lambda$ times smaller one, we get new numerical values $\Delta x'_i = \lambda \cdot \Delta x_i$.
- Since the physics remains the same, it makes sense to require that the uncertainty domains do not change.
- To be more precise, the pdf threshold $t$ may change, but the family of such sets should remain unchanged.
- So, we require that $\{S'_t\}_t = \{S_t\}_t$, where $S'_t$ corresponds to the re-scaled pdf $\rho'(\Delta x) = \text{const} \cdot \rho(\lambda \cdot \Delta)$.
- We will prove that under this scale-invariance, the corresponding sets $S_t$ are exactly super-ellipsoids.
- Thus, we will get the desired explanation.
7. Definitions and the Main Result

• Let $n > 1$, and let $\rho(y) = (\rho_1(y_1), \ldots, \rho_n(y_n))$ be a tuple of positive symmetric smooth functions.

• For every $t > 0$, let us denote
  
  $$S_t(\rho) \overset{\text{def}}{=} \left\{ (y_1, \ldots, y_n) : \prod_{i=1}^{n} \rho_i(y_i) \geq t \right\}.$$ 

• We say that a tuple $\rho(y)$ is bounded if the set $S_t(\rho)$ is bounded for every $t$.

• For every $\lambda > 0$, by a $\lambda$-re-scaling of the tuple $\rho(x)$, we mean a tuple $\rho_{\lambda}(y)$, for which $\rho_{\lambda,i}(y_i) \overset{\text{def}}{=} \frac{1}{\lambda} \cdot \rho_i(\lambda \cdot y_i)$.

• We say that $\rho(y)$ is scale-invariant if for every $\lambda > 0$, re-scaling does not change $\{S_t\}_t$: $\{S_t(\rho)\}_t = \{S_t(\rho_{\lambda})\}_t$.

• **Main Result.** If the tuple $\rho(y)$ is bounded and scale-invariant, then each set $S_t(\rho)$ is a super-ellipsoid.
8. Discussion

- Vice versa, it is easy to prove that:
  - each super-ellipsoid
    \[ \left\{ y : \sum_{i=1}^{n} \left( \frac{|y_i|}{\sigma_i} \right)^p \leq C \right\} \]
  - can be represented as a set \( S_t \) for some bounded and scale-invariant distributions.
- Namely, we can take \( \rho_i(y_i) = \text{const} \cdot \exp \left( -\frac{|y_i|^p}{\sigma_i^p} \right) \).
- Such probability distributions indeed occur: e.g., as probability distributions of measuring errors.
9. Proof

• For convenience, let us consider logarithms
  \[ \psi_i(y_i) \overset{\text{def}}{=} - \log(\rho_i(y_i)). \]

• Let us take the negative logarithm of both sides of the inequality
  \[ \prod_{i=1}^{n} \rho_i(y_i) \geq t \]
  that describes the set \( S_t(\rho) \).

• We then get an equivalent description
  \[ \sum_{i=1}^{n} \psi_i(y_i) \leq c, \]
  where we denoted \( c \overset{\text{def}}{=} - \log(t) \).

• In these terms, scale-invariance means that the corresponding family of sets is the same for all \( c \).

• In terms of the new functions \( \psi_i(y_i) \), scaling means
  \[ \psi_{\lambda,i}(y_i) = - \ln(\rho_{\lambda,i}(y_i)) = - \log \left( \frac{1}{\lambda} \cdot \rho_i(\lambda \cdot y_i) \right) = \log(\lambda) - \log(\rho_i(\lambda \cdot y_i)) = \psi_i(\lambda \cdot y_i) + \log(\lambda). \]
10. Proof (cont-d)

• So, scaling has the form $\psi_{\lambda,i}(y_i) = \psi_i(\lambda \cdot y_i) + \log(\lambda)$.

• In these terms, the fact that scaling does not change the family of sets $S_t$ implies that:
  
  • if two tuples $(y_1, \ldots, y_n)$ and $(z_1, \ldots, z_n)$ always belong or not belong to the same sets $S_t$,
  
  • i.e., if $\sum_{i=1}^{n} \psi_i(y_i) = \sum_{i=1}^{n} \psi_i(z_i)$,
  
  • then the re-scaled functions should also have the same value of the sum: $\sum_{i=1}^{n} \psi_{\lambda,i}(y_i) = \sum_{i=1}^{n} \psi_{\lambda,i}(z_i)$.

• Substituting $\psi_{\lambda,i}(y_i)$ into this formula, we get
  
  $\sum_{i=1}^{n} (\psi_i(\lambda \cdot y_i) + \log(\lambda)) = \sum_{i=1}^{n} (\psi_i(\lambda \cdot z_i) + \log(\lambda))$, hence

  $\sum_{i=1}^{n} \psi_i(\lambda \cdot y_i) = \sum_{i=1}^{n} \psi_i(\lambda \cdot z_i)$. 
11. Proof (cont-d)

• Thus, we have the following property:
  
  • if $\sum_{i=1}^{n} \psi_i(y_i) = \sum_{i=1}^{n} \psi_i(z_i)$,
  
  • then $\sum_{i=1}^{n} \psi_i(\lambda \cdot y_i) = \sum_{i=1}^{n} \psi_i(\lambda \cdot z_i)$.

• In particular, this property holds if we perform very small changes to only two $y_i$'s:

  $$y_a \rightarrow z_a = y_a + \delta_a, \quad y_b \rightarrow z_b = y_b + \delta_b.$$ 

• Here, $\psi_a(y_a + \delta_a) = \psi_a(y_a) + \psi'_a(y_a) \cdot \delta_a + o(\delta)$.

• Similarly, $\psi_b(y_b + \delta_b) = \psi_b(y_b) + \psi'_b(y_b) \cdot \delta_b + o(\delta)$.

• Thus, $\sum_{i=1}^{n} \psi_i(z_i) = \sum_{i=1}^{n} \psi_i(y_i) + \psi'_a(y_a) \cdot \delta_a + \psi'_b(y_b) \cdot \delta_b + o(\delta)$.

• So, the original equality $\sum_{i=1}^{n} \psi_i(y_i) = \sum_{i=1}^{n} \psi_i(z_i)$ takes the form $\psi'_a(y_a) \cdot \delta_a + \psi'_b(y_b) \cdot \delta_b + o(\delta) = 0$. 
12. Proof (cont-d)

- Similarly, the equality \( \sum_{i=1}^{n} \psi_i(\lambda \cdot y_i) = \sum_{i=1}^{n} \psi_i(\lambda \cdot z_i) \) takes the form \( \psi'_a(\lambda \cdot y_a) \cdot \delta_a + \psi'_b(\lambda \cdot y_b) \cdot \delta_b + o(\delta) = 0. \)

- So, the scale-invariance condition takes the form:
  - if \( \psi'_a(y_a) \cdot \delta_a + \psi'_b(y_b) \cdot \delta_b + o(\delta) = 0, \)
  - then \( \psi'_a(\lambda \cdot y_a) \cdot \delta_a + \psi'_b(\lambda \cdot y_b) \cdot \delta_b + o(\delta) = 0. \)

- The 1st condition \( \iff -\frac{\delta_b}{\delta_a} = \frac{\psi'_a(y_a)}{\psi'_b(y_b)} + o(\delta). \)

- The 2nd condition \( \iff -\frac{\delta_b}{\delta_a} = \frac{\psi'_a(\lambda \cdot y_a)}{\psi'_b(\lambda \cdot y_b)} + o(\delta). \)

- So, \( \frac{\psi'_a(\lambda \cdot y_a)}{\psi'_b(\lambda \cdot y_b)} = \frac{\psi'_a(y_a)}{\psi'_b(y_b)} \Rightarrow \frac{\psi'_a(\lambda \cdot y_a)}{\psi'_b(y_b)} = \frac{\psi'_a(y_a)}{\psi'_b(y_b)}. \)

- The left-hand side of this equality doesn’t depend on \( y_b; \) thus, the right-hand side doesn’t depend on \( y_b. \)
13. **Proof (cont-d)**

- Hence, this ratio depends only on $\lambda$. Let us denote this common ratio by $r(\lambda)$: $\psi'_a(\lambda \cdot y_a) = r(\lambda) \cdot \psi'_a(y_a)$.
- The derivative of a smooth function is always measurable.
- Thus, the function $r(\lambda)$ is also measurable, as a ratio of two measurable functions.
- Now, let us take arbitrary values $\lambda_1 > 0$ and $\lambda_2 > 0$.
- Then, we can re-scale first by $\lambda_1$, then by $\lambda_1$, or we can right away re-scale by $\lambda = \lambda_1 \cdot \lambda_2$.
- In the first case,
  \[ \psi' (\lambda_1 \cdot \lambda_2 \cdot y_a) = r(\lambda_1) \cdot \psi'(\lambda_2 \cdot y_a) = r(\lambda_1) \cdot r(\lambda_2) \cdot \psi'_a(y_a). \]
- In the 2nd case, $\psi'(\lambda_1 \cdot \lambda_2 \cdot y_a) = r(\lambda_1 \lambda_2) \cdot \psi'_a(y_a)$.
- So, we must have $r(\lambda_1 \cdot \lambda_2) = r(\lambda_1) \cdot r(\lambda_2)$. 

14. Proof (cont-d)

- It is known that all measurable functions satisfying this property have the form \( r(\lambda) = \lambda^\beta \) for some \( \beta \).

- So, \( \psi'_a(\lambda \cdot y_a) = r(\lambda) \cdot \psi'_a(y_a) = \lambda^\beta \cdot \psi'_a(y_a) \).

- For \( \lambda = z \) and \( y_a = 1 \), we get \( \psi'_a(z) = \psi'_a(1) \cdot z^\beta \), i.e., that \( \psi'_a(y_a) = c_a \cdot y_a^\beta \) for some \( c_a \).

- Integrating, for \( \beta \neq -1 \), for \( y_a > 0 \), we get \( \psi_a(y_a) = k_a \cdot y_a^p + C_a \) for \( p = \beta + 1 \), \( k_a \overset{\text{def}}{=} \frac{c_a}{\beta + 1} \).

- Since \( \psi_i(y_i) \) is even, we get \( \psi_i(y_i) = k_i \cdot |y_i|^p + C_i \).

- So, the condition \( \sum_{i=1}^n \psi(y_i) \leq c \) takes the super-ellipsoid form \( \sum_{i=1}^n k_i \cdot |y_i|^p \leq c_0 \overset{\text{def}}{=} c - \sum_{i=1}^n C_i \).

- For this super-ellipsoid to be bounded, we need to have \( p > 0 \).
15. Proof (final)

- To complete the proof, it is sufficient to consider the case when $\beta = -1$.

- For $\beta = -1$, integration leads to
  \[ \psi_i(y_i) = k_i \cdot \ln(|y_i|) + C_i. \]

- So the condition $\sum_{i=1}^{n} \psi_i(y_i) \leq c$ takes the form
  \[ \sum_{i=1}^{n} k_i \cdot \ln(|y_i|) \leq c_0 \defeq c - \sum_{i=1}^{n} C_i. \]

- Exponentiating both sides, we get $\prod_{i=1}^{n} |y_i|^{k_i} \leq \exp(C)$, for which the corresponding set $S_t$ is unbounded.

- So, in the bounded cases, we always have a super-ellipsoid. The result is proven.
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