Kinematic Metric Spaces Under Interval Uncertainty: Towards an Adequate Definition

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1. What is a kinematic metric: physical introduction

- In the physical space, we can define the distance $d(a, b)$ between two points as the length of the shortest possible path between them.
- Thus defined distance is symmetric ($d(a, b) = d(b, a)$) and satisfies the usual triangle inequality $d(a, c) \leq d(a, b) + d(b, c)$.
- The mathematical notion of a metric is a natural generalization of this physical notion.
- From the viewpoint of space-time:
  - physical space corresponds to the situation when
  - we take space-time points (“events”) $(a, t_0), (b, t_0)$, etc. corresponding to the same moment of time $t_0$.
- In relativity theory, such events cannot causally influence each other.
- Some events $a$ can causally influence events $b$.
- We will denote this strict order – i.e., irreflexive transitive – relation by $a < b$. 
2. What is a kinematic metric (cont-d)

- The causal influence is implemented by a particle or particles whose trajectories start at $a$ and end up at $b$.
- For each such trajectory, we can measure the proper time of the corresponding particle.
- In principle, particles can travel as close to the speed of light as possible.
- In this case, the proper time can be as close to 0 as possible.
- So the *smallest* proper time over all trajectories is always 0.
- Interestingly, there is the *largest* proper time $\tau(a, b)$ – which corresponds to inertial motion.
- The corresponding function $\tau(a, b)$ – defined only when $a < b$ – satisfies the “anti-triangle” inequality $\tau(a, c) \geq \tau(a, b) + \tau(b, c)$. 
3. What is a kinematic metric (cont-d)

- This inequality describes the known *twins paradox* of relativity theory:
  - when a twin brother who traveled to the stars comes back to Earth,
  - he will be younger than his twin who stayed on Earth.

- Indeed:
  - the biological age of the stay-home brother is $\tau(a, c)$, while
  - the biological age of the astronaut brother is $\tau(a, b) + \tau(b, c)$, where $b$ is the moment when the brother reached a faraway star.

- A natural generalization of this function is a notion of *kinematic metric*. 
4. Kinematic metric: definition

- Let \((X, <)\) be an ordered set.
- A function \(\tau(a, b)\) – defined for all pairs for which \(a < b\) – is called a \textit{kinematic metric} if:
  
  - all its values are non-negative and
  - it satisfies the following “anti-triangle” inequality:
  
  \[ \tau(a, c) \geq \tau(a, b) + \tau(b, c). \]
5. Need for interval uncertainty

- All information about the values of a physical quantity \( v \) – including the values of the kinematic metric – comes from measurements.
- Measurements are never absolutely accurate.
- So the measurement result \( \tilde{v} \) is, in general, different from the actual (unknown) value \( v \): there is a measurement error \( \Delta v \overset{\text{def}}{=} \tilde{v} - v \).
- Often, the only information that we have about the measurement error is an upper bound \( \Delta \) on its absolute value.
- In this case, the only information that we have about the actual value \( v \) is that this value belongs to the interval

\[
[v, \bar{v}] \overset{\text{def}}{=} [\tilde{v} - \Delta, \tilde{v} + \Delta].
\]
6. Natural question

- Suppose that we have, for all pairs \( a < b \), intervals \([\tau(a, b), \bar{\tau}(a, b)]\), with \( \tau(a, b) \geq 0 \), obtained from measurement.

- Here, \([\tau(a, b), \bar{\tau}(a, b)] = [\tilde{\tau}(a, b) - \Delta(a, b), \tilde{\tau}(a, b) + \Delta(a, b)]\).

- If all the upper bounds \( \Delta(a, b) \) are correct, then there is a kinematic metric \( \tau(a, b) \) for which \( \tau(a, b) \in [\tau(a, b), \bar{\tau}(a, b)] \) for all \( a < b \).

- However, if we – as happens – underestimated the measurement errors, we may not have such a function.

- So, a natural question is: what is the condition on the intervals \([\tau(a, b), \bar{\tau}(a, b)]\) under which such a function \( \tau(a, b) \) exists?
7. A seemingly natural idea does not work

- Anti-triangle inequality implies that
  \[ \overline{\tau}(a, c) \geq \tau(a, b) + \tau(b, c) \text{ for all } a < b < c. \]

- So, it may seem that this inequality is the right condition for the existence of the desired kinematic metric \( \tau(a, b) \).

- However, this inequality does not guarantee the existence of \( \tau(a, b) \).

- For example, for \( X = \{a_1 < a_2 < a_3 < a_4\} \) and \( [\overline{\tau}(a_i, a_j), \overline{\tau}(a_i, a_j)] = [1, 2] \) for all \( i < j \):
  - this inequality is satisfied, but
  - the desired function \( \tau(a, b) \) is not possible.

- Indeed, if \( \tau(a, b) \) existed, we would have:
  \[ 2 \geq \tau(a_1, a_4) \geq \tau(a_1, a_2) + \tau(a_2, a_3) + \tau(a_3, a_4) \geq 3, \text{ i.e., } 2 \geq 3. \]
8. Main result

- For an interval-valued function \([\underline{\tau}(a, b), \bar{\tau}(a, b)]\) defined for all \(a < b\), the following two conditions are equivalent to each other:
  
  - there exists a kinematic metric \(\tau(a, b)\) for which always
    \[
    \tau(a, b) \in [\underline{\tau}(a, b), \bar{\tau}(a, b)];
    \]
  
  - we have \(\bar{\tau}(a_1, a_n) \geq \sum_{i=1}^{n-1} \tau(a_i, a_{i+1})\) for all sequences
    \[
    a_1 < \ldots < a_n.
    \]
9. Proof

- If $\tau(a, b)$ exists, then this inequality is clearly satisfied.

- Indeed, it follows from the anti-triangle inequality

$$\tau(a_1, a_n) \geq \sum_{i=1}^{n-1} \tau(a_i, a_{i+1}) \geq \sum_{i=1}^{n-1} \tau(a_i, a_{i+1}).$$

- Vice versa, suppose that the above condition is satisfied.

- Then, we can take $\tau(a, b) = \sup \left\{ \sum_{i=1}^{n-1} \tau(a_i, a_{i+1}) \right\}.$

- Here, the supremum is taken over all the chains

$$a = a_1 < a_2 < \ldots < a_n = b$$

that connect $a$ and $b$.

- One can easily prove that thus defined function satisfies the anti-triangle inequality.

- By taking a chain $a_1 = a < a_2 = b$, we get $\tau(a, b) \geq \tau(a, b)$. 
10. Proof (cont-d)

- From the above inequality, for each chain, we get

\[ \bar{\tau}(a, b) \geq \sum_{i=1}^{n-1} \tau(a_i, a_{i+1}). \]

- Since \( \bar{\tau}(a, b) \) is greater than or equal to each sum, it is greater than or equal to their supremum:

\[ \bar{\tau}(a, b) \geq \sup \left\{ \sum_{i=1}^{n-1} \tau(a_i, a_{i+1}) \right\}. \]

- Thus, \( \underline{\tau}(a, b) \leq \tau(a, b) \leq \bar{\tau}(a, b) \), i.e.:

\[ \tau(a, b) \in [\underline{\tau}(a, b), \bar{\tau}(a, b)]. \]
11. Comment

• We need the above condition for all natural numbers \( n \):

\[
\tau(a_1, a_n) \geq \sum_{i=1}^{n-1} \tau(a_i, a_{i+1}) \text{ for all sequences } a_1 < \ldots < a_n.
\]

• If we only require it only for \( n \leq n_0 \), this does not guarantee the existence of \( \tau(a, b) \).

• Example:
  
  • \( X = \{a_1 < \ldots < a_{n_0+1}\} \) and
  
  • \([\tau(a_i, a_j), \tau(a_i, a_j)] = [1, n_0 - 1]\) for all \( i < j \).

• Indeed, if \( \tau(a, b) \) existed, we would have

\[
n_0 - 1 \geq \tau(a_1, a_{n_0+1}) \geq \tau(a_1, a_2) + \ldots + \tau(a_{n_0}, a_{n_0+1}) \geq n_0,
\]

  i.e., \( n_0 - 1 \geq n_0 \).
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13. Bibliography
