How the Amount of Cracks and Potholes Grows with Time: Symmetry-Based Explanation of Empirical Dependencies

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1. Cracks and Potholes

- When a road is built, it is almost perfect – it has only miniature cracks and potholes.
- However, as the road is used, cracks and potholes appear and start growing.
- The amount of cracks is gauged the overall length $C$ of longitudinal cracks outside the wheel path.
- The amount of potholes is usually gauged by the total area $P$ of potholes.
- As the road is used, the quality of the pavement deteriorates, and the values $C$ and $P$ grow.
- This growth starts at some small values corresponding to the newly built road – age $t = 0$. 
2. Cracks and Potholes (cont-d)

- It continues growing until they reach the maximum – the undesirable bad state.
- In this state, the whole road is covered by cracks and potholes.
- The empirical formulas for this growth are:

\[ C = a_C \cdot \exp(-b_C \cdot \exp(-c_C \cdot t)); \quad P = a_P \cdot \exp(-b_P \cdot \exp(-c_P \cdot t)). \]

- In this talk, we use natural symmetry ideas to provide a theoretical explanation for these empirical formulas.
3. Natural Transformations

- In science and engineering, we are interested in the values of different physical quantities.
- We describe these quantities in numerical form.
- However, the numerical values of the corresponding quantities depend on the measuring unit.
- For some quantities such as temperature or time, the values also depend on the starting point.
- If we change the measuring unit for length from meters to centimeters, then all numerical values are \( \times 100 \).
- For example, 2 m becomes \( 2 \cdot 100 = 200 \) cm.
4. Natural Transformations (cont-d)

- In general:
  - if we replace the original measuring unit with a new unit which is \( \lambda \) times smaller,
  - all numerical values are multiplied by \( \lambda \):
    \[
    x \rightarrow X = \lambda \cdot x.
    \]

- This numerical transformation is known as \emph{scaling}.

- Similarly, we can start measuring time:
  - not from our year 0,
  - but – as the French Revolution suggested – with the year 1789 when the revolution started.

- Then from all year values, we should subtract 1789.
5. Natural Transformations (cont-d)

- In general:
  - if we replace the original starting point with the one which is $x_0$ units before,
  - then we add $x_0$ to all numerical values:
    \[ x \rightarrow X = x + x_0. \]

- This numerical transformation is known as *shift*. 


6. Natural Symmetries

- For most physical quantities, there is no fixed measuring unit – and sometimes no fixed starting point.

- It is therefore reasonable to require that:
  - the dependencies $y = f(x)$ between physical quantities
  - also not depend on the choice of the measuring unit
  - (and possibly on the choice of the starting point).

- In physics, such invariance is called symmetry.
7. Natural Symmetries (cont-d)

- Of course:
  - if we just change the unit and/or starting point for $x$,
  - to keep the same formula true in the new units, we may need to appropriately change $y$.

- For example, to preserve the formula $d = v \cdot t$ – that the path is the product of speed and time:
  - when we change the unit for time,
  - we need to appropriately change the unit for speed.

- With this in mind, let us describe possible invariant dependencies.
8. Scaling-to-Scaling (sc-sc)

- Let us first consider the case when the dependence remains the same after we apply scaling to $x$ and $y$.
- In precise terms, we assume that for every $\lambda > 0$, there exists a value $\mu(\lambda)$ (depending on $\lambda$) such that:
  - if $y = f(x)$,
    - then $Y = f(X)$, where $X = \lambda \cdot x$ and $Y = \mu(\lambda) \cdot y$.
- If we plug in the expressions for $Y$ in terms of $y$ and $X$ in terms of $x$ into $Y = f(X)$, we get $f(\lambda \cdot x) = \mu(\lambda) \cdot y$.
- Here, $y = f(x)$, so $f(\lambda \cdot x) = \mu(\lambda) \cdot f(x)$.
- It is known that every measurable dependence $f(x)$ with this property has the form $f(x) = A \cdot x^a$. 

9. Comment

- The general proof is somewhat complicated.
- However, most physical dependencies are differentiable.
- For differentiable $f(x)$, this is easy to prove.
- Indeed, if $f(x)$ is differentiable, then the function $\mu(\lambda) = \frac{f(\lambda \cdot x)}{f(x)}$ is differentiable too.
- Thus, we can differentiate both sides of the equation $f(\lambda \cdot x) = \mu(\lambda) \cdot f(x)$ with respect to $\lambda$.
- As a result, we get $x \cdot f'(\lambda \cdot x) = \mu'(\lambda) \cdot f(x)$.
- In particular, for $\lambda = 1$, we get $x \cdot \frac{df}{dx} = a \cdot f$, where

$$a \overset{\text{def}}{=} \mu'(1).$$
10. Comment (cont-d)

- We can separate $x$ and $f$ if we multiply both sides of the equality by $\frac{dx}{x \cdot f} \cdot \frac{df}{f} = a \cdot \frac{dx}{x}$.

- Integrating both sides, we get $\ln(f) = a \cdot \ln(x) + C$, where $C$ is the integration constant.

- Applying the function $\exp(z)$ of both sides, we get the desired expression $f(x) = A \cdot x^a$, with $A = \exp(C')$. 
11. Shift-to-Scaling (sh-sc)

- Let us consider the case when the dependence remains the same after we apply shift to $x$ and scaling to $y$.
- In this case, for every $x_0$, there exists a value $\mu(x_0)$ (depending on $x_0$) such that:
  - if $y = f(x)$,
  - then we have $Y = f(X)$, where $X = x + x_0$ and $Y = \mu(x_0) \cdot y$.
- If we plug in the expressions for $Y$ in terms of $y$ and $X$ in terms of $x$ into $Y = f(X)$, we get $f(x + x_0) = \mu(x_0) \cdot y$.
- Here, $y = f(x)$, so $f(x + x_0) = \mu(x_0) \cdot f(x)$.
- It is known that every measurable dependence $f(x)$ with this property has the form $f(x) = A \cdot \exp(a \cdot x)$. 
12. Comment

- If \( f(x) \) is differentiable, then the function \( \mu(x_0) = \frac{f(x + x_0)}{f(x)} \) is differentiable too.

- Thus, we can differentiate both sides of the equation \( f(x + x_0) = \mu(x_0) \cdot f(x) \) with respect to \( x_0 \).

- As a result, we get \( f'(x + x_0) = \mu'(x_0) \cdot f(x) \).

- For \( x_0 = 0 \), we get \( \frac{df}{dx} = a \cdot f \), where \( a \overset{\text{def}}{=} \mu'(0) \).

- We can separate the variables \( x \) and \( f \) if we multiply both sides of the equality by \( \frac{dx}{f} : \frac{df}{f} = a \cdot dx \).

- Integrating both sides, we get \( \ln(f) = a \cdot x + C \), where \( C \) is the integration constant.

- Applying the function \( \exp(z) \) to both sides, we get \( f(x) = A \cdot \exp(a \cdot x) \), with \( A = \exp(C) \).
13. Scaling-to-Shift (sc-sh)

- Let us now consider the case when the dependence remains the same after we scale \( x \) and shift \( y \).

- In precise terms, we assume that for every \( \lambda > 0 \), there exists a value \( y_0(\lambda) \) (depending on \( \lambda \)) such that:
  
  \[- \text{ if } y = f(x), \]
  
  \[\text{ then } Y = f(X), \text{ where } X = \lambda \cdot x \text{ and } Y = y + y_0(\lambda).\]

- If we plug in the expressions for \( Y \) in terms of \( y \) and \( X \) in terms of \( x \) \( Y = f(X) \), we get \( f(\lambda \cdot x) = y + y_0(\lambda). \)

- Here, \( y = f(x) \), so \( f(\lambda \cdot x) = f(x) + y_0(\lambda). \)

- It is known that every measurable dependence \( f(x) \) with this property has the form \( f(x) = a \cdot \ln(x) + C. \)
14. Comment

- If \( f(x) \) is differentiable, then the function \( y_0(\lambda) = f(\lambda \cdot x) - f(x) \) is differentiable too.

- Thus, we can differentiate both sides of the equation \( f(\lambda \cdot x) = f(x) + y_0(\lambda) \) with respect to \( \lambda \).

- As a result, we get \( x \cdot f'(\lambda \cdot x) = y'_0(\lambda) \).

- In particular, for \( \lambda = 1 \), we get \( x \cdot \frac{df}{dx} = a \), where

\[
a \overset{\text{def}}{=} y'_0(1).
\]

- We can separate the variables \( x \) and \( f \) if we multiply both sides of the equality by \( \frac{dx}{x} : df = a \cdot \frac{dx}{x} \).

- Integrating both sides, we get \( f(x) = a \cdot \ln(x) + C' \), where \( C' \) is the integration constant.
15. Shift-to-Shift (sh-sh)

- In this case, for every \( x_0 \), there exists a value \( y_0(x_0) \) such that:
  - if \( y = f(x) \),
  - then we have \( Y = f(X) \), where \( X = x + x_0 \) and

\[
Y = y + y_0(x_0).
\]

- If we plug in the expressions for \( Y \) in terms of \( y \) and \( X \) in terms of \( x \) into \( Y = f(X) \), we get

\[
f(x + x_0) = y + y_0(x_0).
\]

- Here, \( y = f(x) \), so \( f(x + x_0) = f(x) + y_0(x_0) \).

- It is known that every measurable dependence \( f(x) \) with this property has the form \( f(x) = a \cdot x + C \).
16. Comment

• If \( f(x) \) is differentiable, then the function \( y_0(x_0) = f(x + x_0) - f(x) \) is differentiable too.

• Thus, we can differentiate both sides of the equation \( f(x + x_0) = f(x) + y_0(x_0) \) with respect to \( x_0 \).

• As a result, we get \( f'(x + x_0) = y'_0(x_0) \).

• In particular, for \( x_0 = 0 \), we get \( f'(x) = a \), where

\[
a \overset{\text{def}}{=} y'_0(0).
\]

• Integrating, we get \( f(x) = a \cdot x + C \), where \( C \) is the integration constant.
17. What We Want: A Brief Reminder

• We want to find the dependence of the quantity \( q \) (crack or pothole amount) on time \( t \); we know:
  – that the for \( t = 0 \), the value \( q(t) \) is small positive,
  – that the value \( q(t) \) increases with time, and
  – that the value \( q(t) \) tends to some large constant value when \( t \) increases.
18. What Are Possible Symmetries Here?

- For crack amount $C$ and for pothole amount $P$, there is an absolute starting point: 0.
- Then, we have no cracks and no potholes.
- However, it makes sense to use different units of length and different units of area.
- So scaling makes perfect sense.
- For time, as we have mentioned, both shift and scaling make sense.
19. First Idea

• Let us see if any of the above symmetric dependencies satisfy the desired property.

• Since for $q$, only scaling makes sense, we can only consider two possibilities: sc-sc and sh-sc.

• Let us consider them one by one.

• In the sc-sc case, we have $q(t) = A \cdot t^a$.

• Since we want a non-negative value, we have $A > 0$.

• Since we want $q(t)$ to be increasing with time, we have to take $a > 0$.

• However, in this case:
  
  – $q(0)$ is zero – while we want it to be positive, and
  
  – $q(t)$ tends to infinity as $t$ increases – while we want it to tend to some constant.
20. First Idea: sh-sc Case

- In the sh-sc case, we have $q(t) = A \cdot \exp(a \cdot t)$.
- Again, since we want a non-negative value, we have to take $A > 0$.
- Since we want $q(t)$ to be increasing with time, we have to take $a > 0$; in this case:
  - $q(0)$ is positive, which is exactly what we wanted,
  - however, $q(t)$ tends to infinity as $t$ increases – while we want it to tend to some constant.
21. So What Do We Do?

- The first idea does not work, so what should we do?
- The above arguments about possible dependencies deal with the case when $y$ directly depend on time $t$.
- However, in our case, cracks and potholes do not directly depend on time.
- What changes with time is stress, which, in its turn, causes the pavement to crack.
- In other words, instead of the direct dependence of the quantity $q$ on time:
  - we have $q$ depending on some auxiliary quantity $z$, and
  - we have $z$ depending on time $t$. 
22. So What Do We Do (cont-d)

- For both dependencies \( q(z) \) and \( z(t) \) we can have symmetry-motivated formulas.

- Let us see which combinations of these formulas provide the desired properties of \( q(t) = q(z(t)) \):
  - that this value is positive for \( t = 0 \),
  - that this value increases for \( t > 0 \), and
  - that this value tends to a finite limit when \( t \to \infty \).
23. Possible Options of the \( q(z) \) Dependence

- For \( q \), only scaling is possible.
- So, for possible dependencies \( q(z) \), we have:
  - either the sc-sc option \( q(z) = A \cdot z^a \)
  - or the sh-sc option \( q(z) = A \cdot \exp(a \cdot z) \).
- In the sc-sc option \( q(z) = A \cdot z^a \), it does not make sense to consider sh-sc or sc-sc options for \( z(t) \); indeed:
  - as one can check, this will be equivalent to sh-sc or sc-sc symmetry for \( q(t) \),
  - and we have already shown that this is not possible.
- So, to go beyond previously considered options, we need to consider two remaining options for \( z(t) \):
  - sh-sh option \( z(t) = a_1 \cdot t + C_1 \), and
  - sc-sh option \( z(t) = a_1 \cdot \ln(t) + C_1 \).
24. Possible Options (cont-d)

- In the 1st case, \( q(t) = A \cdot z^a = A \cdot (a_1 \cdot t + C_1)^a \), i.e.,
  \[ q(t) = A_1 \cdot (t + c_2)^a, \]
  where \( A_1 = A \cdot (a_1)^a \) and \( c_2 = \frac{C_1}{a_1} \).

- The need to have positive values of \( q \) implies \( A > 0 \),
  the need to have \( q(t) \) increasing leads to \( a > 0 \).

- However then, for \( t \to \infty \), the resulting expression tends to infinity – while we want it bounded.

- In the 2nd case, \( q(t) = A \cdot (a_1 \cdot \ln(t) + C_1)^a \), i.e.,
  \[ q(t) = A_1 \cdot (\ln(t) + c_2)^a, \]
  with \( A_1 = A \cdot (a_1)^a \) and \( c_2 = \frac{C_1}{a_1} \).

- The need to have positive values of \( q \) implies \( A > 0 \),
  the need to have \( q(t) \) increasing leads to \( a > 0 \).

- However then, for \( t \to \infty \), the resulting expression also tends to infinity – while we want it bounded.
25. **sh-sc Option** \( q(z) = A \cdot \exp(a \cdot z) \)

- In this option, it does not make sense to consider sh-sh or sc-sh options for \( z(t) \); indeed:
  - as one can check, this will be equivalent to sh-sc or sc-sc symmetry for \( q(t) \),
  - and we have already shown that this is not possible.

- So, to go beyond previously considered options, we need to consider two remaining options for \( z(t) \):
  - sc-sc option \( z(t) = A_1 \cdot t^{a_1} \), and
  - sh-sc option \( z(t) = A_1 \cdot \exp(a_1 \cdot t) \).

- In the 1st case, \( q(t) = A \cdot \exp(a \cdot z) = A \cdot \exp((a \cdot A_1) \cdot t^{a_1}) \).

- The need to have positive values of \( q \) implies \( A > 0 \).

- The behavior of this expression depends on the sign of the product \( a \cdot A_1 \).
26. **sh-sc Option** \( q(z) = A \cdot \exp(a \cdot z) \) (cont-d)

- If \( a \cdot A_1 > 0 \), then the need to have \( q(t) \) increasing leads to \( a_1 > 0 \).
- However then, for \( t \to \infty \), the resulting expression tends to infinity – and we want it bounded.
- If \( a \cdot A_1 < 0 \), then the need to have \( q(t) \) increasing leads to \( a_1 < 0 \).
- However then, for \( t \to 0 \), we have \( t^{-|a_1|} \to \infty \), hence \( (a \cdot A_1)t^{-|a_1|} \to -\infty \), and \( q(t) = A \cdot \exp((a \cdot A_1)t^{-|a_1|}) \to 0 \), but we want the value \( q(0) \) to be positive.
- So, the only possible case is the second case, when \( q(t) = A \cdot \exp(a \cdot z) = A \cdot ((a \cdot A_1) \cdot \exp(a_1 \cdot t)) \).
- This is exactly the desired formulas.
- Thus, we have indeed justified the empirical dependencies.
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