Interval Computations Technology in Mathematics Research: From Help in Theoretical Breakthroughs to Practically Useful Results About Numerical Methods

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1. General Problem of Data Processing under Uncertainty

- *Indirect measurements*: way to measure $y$ that are difficult (or even impossible) to measure directly.
- *Idea*: $y = f(x_1, \ldots, x_n)$

\[
\begin{array}{cccc}
\tilde{x}_1 \\
\tilde{x}_2 \\
\vdots \\
\tilde{x}_n
\end{array} \quad \xymatrix{
\tilde{x}_1 \\
\tilde{x}_2 \\
\vdots \\
\tilde{x}_n
\ar[r]<0pt> & f \ar[r]<0pt> & \tilde{y} = f(\tilde{x}_1, \ldots, \tilde{x}_n)
}
\]

- *Problem*: measurements are never 100% accurate: $\tilde{x}_i \neq x_i$ ($\Delta x_i \neq 0$) hence

\[
\tilde{y} = f(\tilde{x}_1, \ldots, \tilde{x}_n) \neq y = f(x_1, \ldots, x_n).
\]

What are bounds on $\Delta y \overset{\text{def}}{=} \tilde{y} - y$?
2. Probabilistic and Interval Uncertainty

- **Traditional approach**: we know probability distribution for $\Delta x_i$ (usually Gaussian).
- **Where it comes from**: calibration using standard MI.
- **Problem**: calibration is not possible in:
  - fundamental science
  - manufacturing
- **Solution**: we know upper bounds $\Delta_i$ on $|\Delta x_i|$ hence
  \[ x_i \in [\tilde{x}_i - \Delta_i, \tilde{x}_i + \Delta_i]. \]
3. Interval Computations: A Problem

![Diagram of function f with intervals x_i as inputs and y as output]

- **Given:** an algorithm $y = f(x_1, \ldots, x_n)$ and $n$ intervals $x_i = [\underline{x}_i, \overline{x}_i]$.

- **Compute:** the corresponding range of $y$:
  
  $$[\underline{y}, \overline{y}] = \{ f(x_1, \ldots, x_n) | x_1 \in [\underline{x}_1, \overline{x}_1], \ldots, x_n \in [\underline{x}_n, \overline{x}_n] \}.$$  

- **Fact:** NP-hard even for quadratic $f$.

- **Challenge:** when are feasible algorithm possible?

- **Challenge:** when computing $y = [\underline{y}, \overline{y}]$ is not feasible, find a good approximation $Y \supseteq y$. 
4. **Interval Computations: A Brief History**

- **Origins**: Archimedes (Ancient Greece)
- **Modern pioneers**: Warmus (Poland), Sunaga (Japan), Moore (USA), 1956–59
- **First boom**: early 1960s.
- **First challenge**: taking interval uncertainty into account when planning spaceflights to the Moon.
- **Current applications** (sample):
  - design of elementary particle colliders: Berz, Kyoko (USA)
  - will a comet hit the Earth: Berz, Moore (USA)
  - robotics: Jaulin (France), Neumaier (Austria)
  - chemical engineering: Stadtherr (USA)
5. **Alternative Approach: Maximum Entropy**

- **Situation:** in many practical applications, it is very difficult to come up with the probabilities.

- **Traditional engineering approach:** use probabilistic techniques.

- **Problem:** many different probability distributions are consistent with the same observations.

- **Solution:** select one of these distributions – e.g., the one with the largest entropy.

- **Example – 1 variable:** if all we know is that \( x \in [x, \bar{x}] \), then MaxEnt leads to a uniform distribution on \([x, \bar{x}]\).

- **Example – multiple variables:** different variables are independently distributed.
6. Limitations of Maximum Entropy Approach

- **Example**: simplest algorithm \( y = x_1 + \ldots + x_n \).
- **Measurement errors**: \( \Delta x_i \in [-\Delta, \Delta] \).
- **Analysis**: \( \Delta y = \Delta x_1 + \ldots + \Delta x_n \).
- **Worst case situation**: \( \Delta y = n \cdot \Delta \).
- **Maximum Entropy approach**: due to Central Limit Theorem, \( \Delta y \) is \( \approx \) normal, with \( \sigma = \Delta \cdot \frac{\sqrt{n}}{\sqrt{3}} \).
- **Why this may be inadequate**: we get \( \Delta \sim \sqrt{n} \), but due to correlation, it is possible that \( \Delta = n \cdot \Delta \sim n \gg \sqrt{n} \).
- **Conclusion**: using a single distribution can be very misleading, especially if we want guaranteed results.
- **Examples**: high-risk application areas such as space exploration or nuclear engineering.
7. Interval Arithmetic: Foundations of Interval Techniques

- **Problem:** compute the range
  \[ [y, \bar{y}] = \{ f(x_1, \ldots, x_n) \mid x_1 \in [x_1, \overline{x_1}], \ldots, x_n \in [x_n, \overline{x_n}] \} . \]

- **Interval arithmetic:** for arithmetic operations \( f(x_1, x_2) \) (and for elementary functions), we have explicit formulas for the range.

- **Examples:** when \( x_1 \in \mathbf{x}_1 = [x_1, \overline{x_1}] \) and \( x_2 \in \mathbf{x}_2 = [x_2, \overline{x_2}] \), then:
  
  - The range \( \mathbf{x}_1 + \mathbf{x}_2 \) for \( x_1 + x_2 \) is \([x_1 + \overline{x_2}, \overline{x_1} + x_2]\).
  
  - The range \( \mathbf{x}_1 - \mathbf{x}_2 \) for \( x_1 - x_2 \) is \([x_1 - \overline{x_2}, \overline{x_1} - x_2]\).
  
  - The range \( \mathbf{x}_1 \cdot \mathbf{x}_2 \) for \( x_1 \cdot x_2 \) is \([y, \bar{y}]\), where
    
    \[ y = \min(x_1 \cdot \overline{x_2}, \overline{x_1} \cdot \overline{x_2}, \overline{x_1} \cdot \overline{x_2}, \overline{x_1} \cdot \overline{x_2}); \]
    \[ \bar{y} = \max(x_1 \cdot \overline{x_2}, x_1 \cdot \overline{x_2}, \overline{x_1} \cdot \overline{x_2}, \overline{x_1} \cdot \overline{x_2}); \]

- The range \( 1/\mathbf{x}_1 \) for \( 1/x_1 \) is \([1/\overline{x_1}, 1/x_1]\) (if \( 0 \notin \mathbf{x}_1 \)).
8. Straightforward Interval Computations: Example

• *Example:* \( f(x) = (x - 2) \cdot (x + 2), \ x \in [1, 2]. \)

• How will the computer compute it?
  
  • \( r_1 := x - 2; \)
  
  • \( r_2 := x + 2; \)
  
  • \( r_3 := r_1 \cdot r_2. \)

• *Main idea:* perform the same operations, but with *intervals* instead of *numbers*:
  
  • \( r_1 := [1, 2] - [2, 2] = [-1, 0]; \)
  
  • \( r_2 := [1, 2] + [2, 2] = [3, 4]; \)
  
  • \( r_3 := [-1, 0] \cdot [3, 4] = [-4, 0]. \)

• *Actual range:* \( f(x) = [-3, 0]. \)

• *Comment:* this is just a toy example, there are more efficient ways of computing an enclosure \( Y \supseteq y. \)
9. First Idea: Use of Monotonicity

- **Reminder:** for arithmetic, we had exact ranges.
- **Reason:** $+, -, \cdot$ are monotonic in each variable.
- **How monotonicity helps:** if $f(x_1, \ldots, x_n)$ is (non-strictly) increasing ($f \uparrow$) in each $x_i$, then

$$f(x_1, \ldots, x_n) = [f(x_1, \ldots, x_n), f(x_1, \ldots, x_n)].$$

- **Similarly:** if $f \uparrow$ for some $x_i$ and $f \downarrow$ for other $x_j$ ($-$).
- **Fact:** $f \uparrow$ in $x_i$ if $\frac{\partial f}{\partial x_i} \geq 0$.
- **Checking monotonicity:** check that the range $[r_i, \bar{r}_i]$ of $\frac{\partial f}{\partial x_i}$ on $x_i$ has $r_i \geq 0$.
- **Differentiation:** by Automatic Differentiation (AD) tools.
- **Estimating ranges of** $\frac{\partial f}{\partial x_i}$: straightforward interval comp.
10. Monotonicity: Example

- **Idea:** if the range $[r_i, \bar{r}_i]$ of each $\frac{\partial f}{\partial x_i}$ on $x_i$ has $r_i \geq 0$, then

$$f(x_1, \ldots, x_n) = [f(x_1, \ldots, x_n), f(\bar{x}_1, \ldots, \bar{x}_n)].$$

- **Example:** $f(x) = (x - 2) \cdot (x + 2)$, $x = [1, 2]$.

- **Case $n = 1$:** if the range $[r, \bar{r}]$ of $\frac{df}{dx}$ on $x$ has $r \geq 0$, then

$$f(x) = [f(x), f(\bar{x})].$$

- **AD:** $\frac{df}{dx} = 1 \cdot (x + 2) + (x - 2) \cdot 1 = 2x$.

- **Checking:** $[r, \bar{r}] = [2, 4]$, with $2 \geq 0$.

- **Result:** $f([1, 2]) = [f(1), f(2)] = [-3, 0]$.

- **Comparison:** this is the exact range.
11. Non-Monotonic Example

- Example: \( f(x) = x \cdot (1 - x), \ x \in [0, 1] \).
- How will the computer compute it?
  - \( r_1 := 1 - x \);
  - \( r_2 := x \cdot r_1 \).
- Straightforward interval computations:
  - \( r_1 := [1, 1] - [0, 1] = [0, 1] \);
  - \( r_2 := [0, 1] \cdot [0, 1] = [0, 1] \).
- Actual range: min, max of \( f \) at \( \underline{x}, \overline{x} \), or when \( \frac{df}{dx} = 0 \).
- Here, \( \frac{df}{dx} = 1 - 2x = 0 \) for \( x = 0.5 \), so
  - compute \( f(0) = 0, \ f(0.5) = 0.25, \) and \( f(1) = 0 \).
  - \( \underline{y} = \min(0, 0.25, 0) = 0, \overline{y} = \max(0, 0.25, 0) = 0.25 \).
- Resulting range: \( f(x) = [0, 0.25] \).
12. **Second Idea: Centered Form**

- **Main idea:** Intermediate Value Theorem

\[
f(x_1, \ldots, x_n) = f(\tilde{x}_1, \ldots, \tilde{x}_n) + \sum_{i=1}^{n} \frac{\partial f}{\partial x_i}(\chi) \cdot (x_i - \tilde{x}_i)
\]

for some \(\chi_i \in x_i\).

- **Corollary:** \(f(x_1, \ldots, x_n) \in Y\), where

\[
Y = \tilde{y} + \sum_{i=1}^{n} \frac{\partial f}{\partial x_i}(x_1, \ldots, x_n) \cdot [-\Delta_i, \Delta_i].
\]

- **Differentiation:** by Automatic Differentiation (AD) tools.

- **Estimating the ranges of derivatives:**
  - if appropriate, by monotonicity, or
  - by straightforward interval computations, or
  - by centered form (more time but more accurate).
13. Centered Form: Example

- **General formula:**

\[
Y = f(\tilde{x}_1, \ldots, \tilde{x}_n) + \sum_{i=1}^{n} \frac{\partial f}{\partial x_i}(x_1, \ldots, x_n) \cdot [-\Delta_i, \Delta_i].
\]

- **Example:** \(f(x) = x \cdot (1 - x), \quad x = [0, 1].\)

- Here, \(x = [\tilde{x} - \Delta, \tilde{x} + \Delta]\), with \(\tilde{x} = 0.5\) and \(\Delta = 0.5\).

- **Case** \(n = 1\): \(Y = f(\tilde{x}) + \frac{df}{dx}(x) \cdot [-\Delta, \Delta].\)

- **AD:** \(\frac{df}{dx} = 1 \cdot (1 - x) + x \cdot (-1) = 1 - 2x.\)

- **Estimation:** we have \(\frac{df}{dx}(x) = 1 - 2 \cdot [0, 1] = [-1, 1].\)

- **Result:** \(Y = 0.5 \cdot (1 - 0.5) + [-1, 1] \cdot [-0.5, 0.5] = 0.25 + [-0.5, 0.5] = [-0.25, 0.75].\)

- **Comparison:** actual range \([0, 0.25]\), straightforward \([0, 1]\).
14. Third Idea: Bisection

- **Known:** accuracy $O(\Delta_i^2)$ of first order formula
  $$f(x_1, \ldots, x_n) = f(\tilde{x}_1, \ldots, \tilde{x}_n) + \sum_{i=1}^{n} \frac{\partial f}{\partial x_i}(\chi) \cdot (x_i - \tilde{x}_i).$$

- **Idea:** if the intervals are too wide, we:
  - split one of them in half ($\Delta_i^2 \to \Delta_i^2/4$); and
  - take the union of the resulting ranges.

- **Example:** $f(x) = x \cdot (1 - x)$, where $x \in x = [0, 1]$.

- **Split:** take $x' = [0, 0.5]$ and $x'' = [0.5, 1]$.

- **1st range:** $1 - 2 \cdot x = 1 - 2 \cdot [0, 0.5] = [0, 1]$, so $f \uparrow$ and $f(x') = [f(0), f(0.5)] = [0, 0.25]$.

- **2nd range:** $1 - 2 \cdot x = 1 - 2 \cdot [0.5, 1] = [-1, 0]$, so $f \downarrow$ and $f(x'') = [f(1), f(0.5)] = [0, 0.25]$.

- **Result:** $f(x') \cup f(x'') = [0, 0.25]$ – exact.
15. **Use of Interval Computations in Pure Mathematics: Computer-Aided Proofs**

- **Main idea:** to prove (guarantee) that certain inequalities hold for all the values within given intervals.

- **Case study:** Lorenz equations – a simple model of atmospheric circulation.

- **Empirical fact:** solutions of Lorentz equations behave chaotically (“have a strange attractor”).

- **Theoretical confirmation:** by Warwick Tucker using interval computations (Moore prize 2002).

- **Case study:** Kepler conjecture – that the standard layer-by-layer packing of spheres in the densest.

- **Solution:** Thomas Hales (Moore prize 2004).

- **Case study:** “double bubble” iso-perimetric problem (R. Schlafly, 1995).
16. Practical Case Study: Description

- We are interested in indirect measurements:
  - we measure: $f(x)$, resulting in $[f(x), f(x)]$;
  - we know: that $L(u) = f$ for a known operator $L$;
  - we want: the range $[u(x), \overline{u}(x)]$ of possible values of $u(x)$.

- “Monotonic” case: if $f_1(x) \leq f_2(x)$ for all $x$, then $u_1(x) \leq u_2(x)$.

- In the “monotonic” case, the range is easier to compute: $u(x) \in [u, \overline{u}] = [L^{-1}(f), L^{-1}(f)]$.

- For linear operators $L$, “monotonic” means non-negativity-preserving: if $u \geq 0$, then $L(u) = f \geq 0$.

- We look for situations where $L^{-1}$ is non-negativity-preserving.

- Example: find $u$ s.t. $-u'' = f$ and $u(-1) = u(1) = 0$. 
17. Finite Element Methods (FEM) and Discrete Non-Negativity Conservation Principles

- **Problem:** find $u$ s.t. $-u'' = f$ and $u(-1) = u(1) = 0$.
- **FEM – main idea:** use (piece-wise) polynomial approximations $f_{h,p}(x)$ and $u_{h,p}(x)$ of given order $p$.
- **Problem:** we cannot have $z(x) \overset{\text{def}}{=} u''_{h,p}(x) + f_{h,p}(x) = 0$ for all $x$: e.g., for $p = 1$, we have $u''_{h,p} \equiv 0$.
- **In practice:** we never measure point values $z(x)$, only (regional) averages $\int z(x) \cdot w(x) \, dx$.
- **Solution:** require that $\int z(x) \cdot w_{h,p}(x) \, dx = 0$ for all polynomials $w_{h,p}$ which vanish at endpoints $(x = \pm 1)$.
- **So:** $\int_{-1}^{1} u'_{h,p}(x) \cdot w'_{h,p}(x) \, dx = \int_{-1}^{1} f_{h,p}(x) \cdot w_{h,p}(x) \, dx$.
- **Question:** can we guarantee $u_{h,p}(x) \geq 0$ if $f_{h,p}(x) \geq 0$?
18. Proof of Non-Negativity Preservation: Main Ideas

- **Basis:** Lobatto shape functions \( l_i(x) \) s.t. \( l'_i(x) \) are orthonormal polynomials.

- **Resulting formula:**
  \[
  u_{h,p}(x) = \int_{-1}^{1} f_{h,p}(z) \cdot \Phi_p(x, z) \, dz,
  \]
  where \( \Phi_p(x, z) \overset{\text{def}}{=} \sum_{i=1}^{p-1} l_{i+1}(x) \cdot l_{i+1}(z). \)

- **We want to prove:** if \( f_{h,p}(x) \geq 0 \), then \( u_{h,p}(x) \geq 0. \)

- **Idea of the proof:**
  - identify a subdomain \( \Omega_p^+ \) of \((-1,1)^2\) where \( \Phi_p \geq 0; \)
  - find a quadrature rule of the order of accuracy \( 2p \) with positive weights and points lying in \( \Omega_p^+ \).

- **What was known:** subdomains \( \Omega_p^+ \) and quadrature rules for \( p = 4, \ldots, 10. \)

- **Remained to be proven:** \( \Phi_p(x, z) \geq 0 \) for all \((x, z) \in \Omega_p^+. \)
19. Application of Interval Arithmetic, Case $p = 4$

- We need to show: $\Phi_4(x, z) = \sum_{i=1}^{3} l_{i+1}(x) \cdot l_{i+1}(z) \geq 0$.

- $l_{i+1}(x)$ vanishes at $\pm 1$, so $l_{i+1}(x) = (x^2 - 1) \cdot m_{i+1}(x)$.

- So, $\Phi_4(x, z) = (x^2 - 1) \cdot (z^2 - 1) \cdot \Psi_4(x, z)$, where $\Psi_4(x, z) \overset{\text{def}}{=} \sum_{i=1}^{3} m_{i+1}(x) \cdot m_{i+1}(z)$.

- Consequence: $\Phi_4(x, z) \geq 0$ for all $x, z \in (-1, 1)$ if and only if $\Psi_4(x, z) \geq 0$ for all $x, z \in (-1, 1)$.

- Here, $\Psi_4(x, z) = \frac{3}{8} + \frac{5}{8} \cdot x \cdot z + \frac{7}{128} \cdot (5x^2 - 1) \cdot (5z^2 - 1)$.

- Idea:
  - use interval arithmetic to compute the enclosure $[\Psi_4, \overline{\Psi}_4]$ for the range of the function $\Psi_4(x, z)$;
  - if $\Psi_4 \geq 0$, then $\Psi_4(x, z) \geq 0$ for all $x, z$. 
20. Application of Interval Arithmetic (cont-d)

- **Reminder:** we want to find the range of the function
  \[ \Psi_4(x, z) = \frac{3}{8} + \frac{5}{8} \cdot x \cdot z + \frac{7}{128} \cdot (5x^2 - 1) \cdot (5z^2 - 1). \]

- For \([-1, 1] \times [-1, 1]\), we get \([\Psi_4, \Psi_4] = \left[-\frac{25}{16}, \frac{95}{32}\right]\).

- Since \(\Psi_4 < 0\), we subdivide the domain and evaluate the range of each sub-domain.

- For \([0, 1] \times [0, 1]\), we get \([\Psi_4, \Psi_4] = \left[\frac{5}{32}, \frac{15}{8}\right]\), so \(\Psi_4 \geq 0\).

- For \([-1, 0] \times [-1, 0]\), we similarly get \(\Psi_4 = \frac{5}{32} \geq 0\).

- For \([0, 1] \times [-1, 0]\), we get \([\Psi_4, \Psi_4] = \left[-\frac{15}{32}, \frac{5}{4}\right]\), so \(\Psi_4 < 0\).

- Thus, we need to further subdivide this sub-domain.
21. Result for the Case $p = 4$

- We subdivide the domains until (after 6 iterations) we get a partition of $[-1, 1]^2$ where $[\Psi_4, \overline{\Psi}_4] > 0$.

- **Conclusion:** $\Psi_4(x, z) \geq 0$.

- **Conclusion:** the operator $f_{h,p} \rightarrow u_{h,p}$ is non-negativity-preserving.

- Similar results are available for $p = 5, \ldots, 10$. 
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23. **Alternative Approach: Affine Arithmetic**

- **So far:** we compute the range of $x \cdot (1 - x)$ by multiplying ranges of $x$ and $1 - x$.
- **We ignore:** that both factors depend on $x$ and are, thus, dependent.
- **Idea:** for each intermediate result $a$, keep an explicit dependence on $\Delta x_i = \tilde{x}_i - x_i$ (at least its linear terms).
- **Implementation:**
  
  $$a = a_0 + \sum_{i=1}^{n} a_i \cdot \Delta x_i + [a, \bar{a}]$$

- **We start:** with $x_i = \tilde{x}_i - \Delta x_i$, i.e.,
  
  $$\tilde{x}_i + 0 \cdot \Delta x_1 + \ldots + 0 \cdot \Delta x_{i-1} + (-1) \cdot \Delta x_i + 0 \cdot \Delta x_{i+1} + \ldots + 0 \cdot \Delta x_n + [0, 0].$$

- **Description:** $a_0 = \tilde{x}_i$, $a_i = -1$, $a_j = 0$ for $j \neq i$, and $[a, \bar{a}] = [0, 0]$. 
24. **Affine Arithmetic: Operations**

- **Representation:** \( a = a_0 + \sum_{i=1}^{n} a_i \cdot \Delta x_i + [a, \bar{a}] \).
- **Input:** \( a = a_0 + \sum_{i=1}^{n} a_i \cdot \Delta x_i + \mathbf{a} \) and \( b = b_0 + \sum_{i=1}^{n} b_i \cdot \Delta x_i + \mathbf{b} \).
- **Operations:** \( c = a \otimes b \).
- **Addition:** \( c_0 = a_0 + b_0, \ c_i = a_i + b_i, \ c = \mathbf{a} + \mathbf{b} \).
- **Subtraction:** \( c_0 = a_0 - b_0, \ c_i = a_i - b_i, \ c = \mathbf{a} - \mathbf{b} \).
- **Multiplication:** \( c_0 = a_0 \cdot b_0, \ c_i = a_0 \cdot b_i + b_0 \cdot a_i, \ c = a_0 \cdot \mathbf{b} + b_0 \cdot \mathbf{a} + \sum_{i \neq j} a_i \cdot b_j \cdot [-\Delta_i, \Delta_i] \cdot [-\Delta_j, \Delta_j] + \sum_i a_i \cdot b_i \cdot [-\Delta_i, \Delta_i]^2 + \left( \sum_i a_i \cdot [-\Delta_i, \Delta_i] \right) \cdot \mathbf{b} + \left( \sum_i b_i \cdot [-\Delta_i, \Delta_i] \right) \cdot \mathbf{a} + \mathbf{a} \cdot \mathbf{b} \).
25. **Affine Arithmetic: Example**

- **Example:** \( f(x) = x \cdot (1 - x), x \in [0, 1] \).
- Here, \( n = 1, \tilde{x} = 0.5, \) and \( \Delta = 0.5 \).
- How will the computer compute it?
  - \( r_1 := 1 - x; \)
  - \( r_2 := x \cdot r_1. \)
- **Affine arithmetic:** we start with \( x = 0.5 - \Delta x + [0, 0]; \)
  - \( r_1 := 1 - (0.5 - \Delta) = 0.5 + \Delta x; \)
  - \( r_2 := (0.5 - \Delta x) \cdot (0.5 + \Delta x), \) i.e.,
    \[
    r_2 = 0.25 + 0 \cdot \Delta x - [-\Delta, \Delta]^2 = 0.25 + [-\Delta^2, 0].
    \]
- **Resulting range:** \( y = 0.25 + [-0.25, 0] = [0, 0.25]. \)
- **Comparison:** this is the exact range.
26. **Affine Arithmetic: Towards More Accurate Estimates**

- *In our simple example:* we got the exact range.
- *In general:* range estimation is NP-hard.
- *Meaning:* a feasible (polynomial-time) algorithm will sometimes lead to excess width: $Y \supset y$.
- *Conclusion:* affine arithmetic may lead to excess width.
- *Question:* how to get more accurate estimates?
- *First idea:* bisection.
- *Second idea* (Taylor arithmetic):
  - *affine arithmetic:* $a = a_0 + \sum a_i \cdot \Delta x_i + a$;
  - *meaning:* we keep linear terms in $\Delta x_i$;
  - *idea:* keep, e.g., quadratic terms
    $$a = a_0 + \sum a_i \cdot \Delta x_i + \sum a_{ij} \cdot \Delta x_i \cdot \Delta x_j + a.$$
27. Interval Computations vs. Affine Arithmetic: Comparative Analysis

- **Objective:** we want a method that computes a reasonable estimate for the range in reasonable time.

- **Conclusion – how to compare different methods:**
  - how accurate are the estimates, and
  - how fast we can compute them.

- **Accuracy:** affine arithmetic leads to more accurate ranges.

- **Computation time:**
  - **Interval arithmetic:** for each intermediate result $a$, we compute two values: endpoints $a$ and $\bar{a}$ of $[a, \bar{a}]$.
  - **Affine arithmetic:** for each $a$, we compute $n + 3$ values:
    $$ a_0, a_1, \ldots, a_n, a, \bar{a}. $$

- **Conclusion:** affine arithmetic is $\sim n$ times slower.

- **We have:** a system of equations \( g_i(y_1, \ldots, y_n) = a_i \) with unknowns \( y_i \);
- **We know:** \( a_i \) with interval uncertainty: \( a_i \in [a_i, \bar{a}_i] \);
- **We want:** to find the corresponding ranges of \( y_j \).
- **First case:** for exactly known \( a_i \), we have an algorithm \( y_j = f_j(a_1, \ldots, a_n) \) for solving the system.
- **Example:** system of linear equations.
- **Solution:** apply interval computations techniques to find the range \( f_j([a_1, \bar{a}_1], \ldots, [a_n, \bar{a}_n]) \).
- **Better solution:** for specific equations, we often already know which ideas work best.
- **Example:** linear equations \( Ay = b \); \( y \) is monotonic in \( b \).
29. Solving Systems of Equations When No Algorithm Is Known

- **Idea:**
  - parse each equation into elementary constraints, and
  - use interval computations to improve original ranges until we get a narrow range (= solution).

- **First example:** $x - x^2 = 0.5$, $x \in [0, 1]$ (no solution).

- **Parsing:** $r_1 = x^2$, $0.5 (= r_2) = x - r_1$.

- **Rules:** from $r_1 = x^2$, we extract two rules:
  
  (1) $x \rightarrow r_1 = x^2$;  
  (2) $r_1 \rightarrow x = \sqrt{r_1}$;

  from $0.5 = x - r_1$, we extract two more rules:

  (3) $x \rightarrow r_1 = x - 0.5$;  
  (4) $r_1 \rightarrow x = r_1 + 0.5$.  

30. Solving Systems of Equations When No Algorithm Is Known: Example

- (1) \( r = x^2 \); (2) \( x = \sqrt{r} \); (3) \( r = x - 0.5 \); (4) \( x = r + 0.5 \).
- We start with: \( x = [0, 1] \), \( r = (-\infty, \infty) \).

1. \( r = [0, 1]^2 = [0, 1] \), so \( r_{\text{new}} = (-\infty, \infty) \cap [0, 1] = [0, 1] \).
2. \( x_{\text{new}} = \sqrt{[0, 1]} \cap [0, 1] = [0, 1] \) – no change.
3. \( r_{\text{new}} = ([0, 1] - 0.5) \cap [0, 1] = [-0.5, 0.5] \cap [0, 1] = [0, 0.5] \).
4. \( x_{\text{new}} = ([0, 0.5] + 0.5) \cap [0, 1] = [0.5, 1] \cap [0, 1] = [0.5, 1] \).

1. \( r_{\text{new}} = [0.5, 1]^2 \cap [0, 0.5] = [0.25, 0.5] \).
2. \( x_{\text{new}} = \sqrt{[0.25, 0.5]} \cap [0.5, 1] = [0.5, 0.71] \); round \( a \) down \( \downarrow \) and \( \bar{a} \) up \( \uparrow \), to guarantee enclosure.
3. \( r_{\text{new}} = ([0.5, 0.71] - 0.5) \cap [0.25, 5] = [0.021] \cap [0.25, 0.5] \), i.e., \( r_{\text{new}} = \emptyset \).

- Conclusion: the original equation has no solutions.
31. Solving Systems of Equations: Second Example

- **Example:** $x - x^2 = 0$, $x \in [0, 1]$.
- **Parsing:** $r_1 = x^2$, $0 (= r_2) = x - r_1$.
- **Rules:** (1) $r = x^2$; (2) $x = \sqrt{r}$; (3) $r = x$; (4) $x = r$.
- **We start with:** $x = [0, 1]$, $r = (-\infty, \infty)$.
- **Problem:** after Rule 1, we’re stuck with $x = r = [0, 1]$.
- **Solution:** bisect $x = [0, 1]$ into $[0, 0.5]$ and $[0.5, 1]$.

**For 1st subinterval:**
- Rule 1 leads to $r_{\text{new}} = [0, 0.5]^2 \cap [0, 0.5] = [0, 0.25]$;
- Rule 4 leads to $x_{\text{new}} = [0, 0.25]$;
- Rule 1 leads to $r_{\text{new}} = [0, 0.25]^2 = [0, 0.0625]$;
- Rule 4 leads to $x_{\text{new}} = [0, 0.0625]$; etc.
- we converge to $x = 0$.

**For 2nd subinterval:** we converge to $x = 1$. 
32. Optimization: Extending Known Algorithms to Situations with Interval Uncertainty

- **Problem:** find $y_1, \ldots, y_m$ for which
  \[ g(y_1, \ldots, y_m, a_1, \ldots, a_m) \to \max. \]

- **We know:** $a_i$ with interval uncertainty: $a_i \in [a_i, \bar{a}_i]$;

- **We want:** to find the corresponding ranges of $y_j$.

- **First case:** for exactly known $a_i$, we have an algorithm
  $y_j = f_j(a_1, \ldots, a_n)$ for solving the optimization problem.

- **Example:** quadratic objective function $g$.

- **Solution:** apply interval computations techniques to find the range $f_j([a_1, \bar{a}_1], \ldots, [a_n, \bar{a}_n])$.

- **Better solution:** for specific $f$, we often already know which ideas work best.
33. Optimization When No Algorithm Is Known

- **Idea:** divide the original box \( x \) into subboxes \( b \).
- If \( \max_{x \in b} g(x) < g(x') \) for a known \( x' \), dismiss \( b \).
- **Example:** \( g(x) = x \cdot (1 - x) \), \( x = [0, 1] \).
- Divide into 10 (?) subboxes \( b = [0, 0.1], [0.1, 0.2], \ldots \)
- Find \( g(\tilde{b}) \) for each \( b \); the largest is \( 0.45 \cdot 0.55 = 0.2475 \).
- Compute \( G(b) = g(\tilde{b}) + (1 - 2 \cdot b) \cdot [-\Delta, \Delta] \).
- Dismiss subboxes for which \( \overline{Y} < 0.2475 \).
- **Example:** for \([0.2, 0.3]\), we have
  \[
  0.25 \cdot (1 - 0.25) + (1 - 2 \cdot [0.2, 0.3]) \cdot [-0.05, 0.05].
  \]
- Here \( \overline{Y} = 0.2175 < 0.2475 \), so we dismiss \([0.2, 0.3]\).
- **Result:** keep only boxes \( \subseteq [0.3, 0.7] \).
- **Further subdivision:** get us closer and closer to \( x = 0.5 \).