Dynamic Fuzzy Logic Leads to More Adequate “And” and “Or” Operations

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1. Outline

- In the traditional (static) fuzzy logic, we select an “and”-operation (t-norm) and an “or”-operation (t-conorm).
- The result of applying these operations may differ from the expert’s degrees of belief in $A \& B$ and $A \lor B$.
- Reason: the degrees $d(A \& B)$ and $d(A \lor B)$ depend:
  - not only on the expert’s degrees of belief in statements $A$ and $B$,
  - but also in the extent to which the statements $A$ and $B$ are dependent.
- We show that dynamic fuzzy logic enables us to automatically take this dependence into account.
- Thus, dynamic fuzzy logic leads to more adequate “and”- and “or”-operations.
2. **Fuzzy Logic: Brief Reminder**

- Expert rules are often formulated by using imprecise ("fuzzy") words, like "small", "medium size", or "large".
- For example, a medical recommendation depends on whether the tumor is small, medium size, or large.
- How to avoid collision with a car depends on whether the distance to the car is small, medium, or large.
- To describe such words, L. Zadeh proposed *fuzzy logic*.
- In fuzzy logic, we assign, to each value $x$, the degree $\mu_P(x) \in [0, 1]$ to which $P$ is satisfied; e.g.:
  - as a proportion of the experts who believe that $x$ satisfies the given property,
  - or as a subjective probability.
3. “And” and “Or” Operations in Fuzzy Logic

- Often, an expert rule contains several conditions, e.g.:
  - If an obstacle is close and the car is going fast, then we need to break fast.
  - If a skin tumor is large or bleeding or has irregular shape, then we need to operate on it.

- Thus, we need to:
  - combine the degrees of confidence \( a = d(A) \) and \( b = d(B) \) in the corresponding component statements
  - into a single degree \( d(S) \) to which the rule \( S \) is applicable.

- An algorithm \( f_\&(a, b) \) that transforms \( a \) and \( b \) into \( d(A \& B) \) is called an “and”-operation or a t-norm.

- An algorithm \( f_\lor(a, b) \) that transforms \( a \) and \( b \) into \( d(A \lor B) \) is called an “or”-operation or a t-conorm.
4. Variety of t-Norms and t-Conorms

- In fuzzy logic, there are numerous t-norms and t-conorms.
- Which one to apply depends on the relation between the statements $A$ and $B$.
- This dependence can be illustrated in the probabilistic approaches, when $a = \text{Prob}(A)$.
- If $A$ and $B$ are independent, then the probability $f_\& (a, b)$ of $A \& B$ is equal to the product $a \cdot b = P(A) \cdot P(B)$.
- In this case, the most adequate t-norm is a product $f_\& (a, b) = a \cdot b$.
- If $A$ and $B$ are strongly correlated, then we should have $P(A \& B) = P(A) = P(B)$ when $A = B$.
- In this case, a t-norm $f_\& (a, b) = \min(a, b)$ is more adequate.
5. Formulation of the Problem

- The problem is that in many cases, we do not know whether $A$ and $B$ are correlated or not.

- In such cases, we select some t-norm.

- The selected t-norm may not necessarily coincide with the ideal one.

- Hence, the resulting recommendations may not be always adequate.

- The problem is with “truth-functionality”: 
  - the degree of confidence in $A \& B$ depends only on the degrees of confidence in $A$ and $B$
  - without fully adequately taking into account the possibility of different correlations.

- This is often cited as one of the main limitations of fuzzy techniques.
6. Dynamic Fuzzy Logic

- The traditional fuzzy logic assumes that the expert's degrees of confidence do not change.

- In reality, the expert’s opinions often change with time; thus:
  - to get a more adequate description of the expert opinions and rules,
  - it is necessary to take these changes into account.

- In other words,
  - to describe the expert’s opinion about a statement $A$, instead of a single value $a \in [0, 1]$,
  - we need to use a function $a(t)$ that describes how this degree changes with time $t$.

- Such dynamic fuzzy logic was proposed by Leonid Perlovsky and others.
7. What We Do in This Talk

- In this paper, we show that:
  - if we take this dynamics into consideration,
  - then we can get a more adequate description of “and” and “or” operations.

- Specifically, we get a description in which it is possible to distinguish between:
  - the cases when the statements are independent and
  - the cases when the statements are strongly dependent.

- This possibility will be illustrated on the example when the fuzzy degrees have a probabilistic meaning.
8. Correlation: Reminder

- In statistics:
  - the most frequent way to describe correlation between two random variables $x$ and $y$ is
  - to use the correlation coefficient.

- Usually:
  - the mean (expected value) of the variable $x$ is denoted by $E[x]$, and
  - the variance $V[x]$ is defined as
  \[ V[x] \overset{\text{def}}{=} E[(x - E[x])^2] = E[x^2] - (E[x])^2. \]

- The correlation coefficient is then defined as
  \[ \rho = \frac{E[x \cdot y] - E[x] \cdot E[y]}{\sqrt{V[x] \cdot V[y]}}. \]
9. Relation between Correlation and the Probability $P(A \& B)$

- We consider a statement $A$ which is true with probability $a$ and false with the remaining probability $1 - a$.

- $A$ can be viewed as a random variable that is equal to 1 ("true") w/prob. $a$ and to 0 ("false") w/prob. $1 - a$.

- For this variable, $E[A] = 1 \cdot a + 0 \cdot (1 - a) = a$ and similarly, $E[B] = b$ and $E[A \& B] = P(A \& B)$.


- Similarly, we can conclude that $V[B] = b \cdot (1 - b)$.

- For true and false statements, "and" is simply a product, so $A \& B = A \cdot B$ and thus,

$$E[A \& B] = P(A \& B) = E[A \cdot B].$$
10. Relation between Correlation and the Probability $P(A \& B)$ (cont-d)

• In general, $\rho = \frac{E[A \cdot B] - E[A] \cdot E[B]}{\sqrt{V[A] \cdot V[B]}}$.

• Here, $E[A \cdot B] = P(A \& B)$, $E[A] = a$, $E[B] = b$, $V[A] = a \cdot (1 - a)$, and $V[B] = b \cdot (1 - b)$; thus:

$$\rho = \frac{P(A \& B) - a \cdot b}{\sqrt{a \cdot (1 - a) \cdot b \cdot (1 - b)}}.$$  

• Thus, once we know $P(A) = a$, $P(B) = b$, and $\rho$, we can uniquely reconstruct $P(A \& B)$ as

$$P(A \& B) = a \cdot b + \rho \cdot \sqrt{a \cdot (1 - a) \cdot b \cdot (1 - b)}.$$  

• From $P(A \& B) + P(A \lor B) = P(A) + P(B)$, we conclude that $P(A \lor B) = P(A) + P(B) - P(A \& B)$, so:

$$P(A \lor B) = a + b - a \cdot b - \rho \cdot \sqrt{a \cdot (1 - a) \cdot b \cdot (1 - b)}.$$
11. How Do We Find the Correlation Coefficient: Idea

- In the dynamic case:
  - we not only know the current expert’s degrees of confidence \( a \) and \( b \) in statements \( A \) and \( B \),
  - we also know the past degrees \( a(t) \) and \( b(t) \) which were, in general, different from \( a \) and \( b \).
- When \( A \) and \( B \) are strongly correlated, it is reasonable to expect that \( a(t) \) and \( b(t) \) are also correlated.
- If \( A \) and \( B \) are independent, then it is reasonable to expect that \( a(t) \) and \( b(t) \) are also independent.
- In general:
  - to find the correlation coefficient between \( A \) and \( B \),
  - we can use, as random variables, the values \( a(t) \) and \( b(t) \) corresponding to \( T \) known moments of time.
12. How Do We Find the Correlation Coefficient: Resulting Formulas

- Under this idea,

\[
E[A] = \frac{1}{T} \cdot \sum_{t} a(t), \quad E[B] = \frac{1}{T} \cdot \sum_{t} b(t),
\]

\[
V[A] = \frac{1}{T} \cdot \sum_{t} a^2(t) - \left( \frac{1}{T} \cdot \sum_{t} a(t) \right)^2,
\]

\[
V[B] = \frac{1}{T} \cdot \sum_{t} b^2(t) - \left( \frac{1}{T} \cdot \sum_{t} b(t) \right)^2,
\]

\[
E[A \cdot B] = \frac{1}{T} \cdot \sum_{t} a(t) \cdot b(t), \quad \text{so} \quad \rho = \frac{E[A \cdot B] - E[A] \cdot E[B]}{\sqrt{V[A] \cdot V[B]}}.
\]

- Using this value \(\rho\), we get the desired estimates for

\[
P(A \& B) = a \cdot b + \rho \cdot \sqrt{a \cdot (1 - a) \cdot b \cdot (1 - b)} \quad \text{and}
\]

\[
P(A \lor B) = a + b - a \cdot b - \rho \cdot \sqrt{a \cdot (1 - a) \cdot b \cdot (1 - b)}.
\]
13. Mathematical Comment: Ergodicity

- In producing these estimates, we implicitly assumed that:
  - averaging over time leads to the same result as
  - averaging over a sample.
- This property is called *ergodicity*.
- This property is often assumed and/or proved:
  - in statistical physics and
  - in statistical data analysis.
14. Need for Weighted Averages

- In the above formulas, we implicitly assumed that the correlation does not change in time.
- In reality, just like the expert degrees change with time, the correlation between these degrees may also change.
- It is therefore necessary to take this change into account when estimating correlation.
- One way to do that is to consider the recent values with higher weights than past values.
- In other words, we take $E[A] = \sum_t w(t) \cdot a(t)$ for some weights $w(t) \geq 0$ for which $\sum_t w(t) = 1$.
- A usual selection of “discount” weights is a geometric progression $w(t) = C \cdot q^t$ for some $q < 1$.
- In this case, $\sum_{t=1}^{T} w_t = 1$ implies that $C = \frac{1 - q}{1 - q^{T+1}}$. 
15. Weighted Averages: Resulting Formulas

- First, we compute the values

\[ E[A] = \sum_{t} w(t) \cdot a(t), \quad E[B] = \sum_{t} w(t) \cdot b(t), \]

\[ V[A] = \sum_{t} w(t) \cdot a^2(t) - \left( \sum_{t} w(t) \cdot a(t) \right)^2, \]

\[ V[B] = \sum_{t} w(t) \cdot b^2(t) - \left( \sum_{t} w(t) \cdot b(t) \right)^2, \]

\[ E[A \cdot B] = \sum_{t} w(t) \cdot a(t) \cdot b(t); \quad \rho = \frac{E[A \cdot B] - E[A] \cdot E[B]}{\sqrt{V[A] \cdot V[B]}}. \]

- Using this value \( \rho \), we then compute

\[ P(A \& B) = a \cdot b + \rho \cdot \sqrt{a \cdot (1 - a) \cdot b \cdot (1 - b)} \] and

\[ P(A \lor B) = a + b - a \cdot b - \rho \cdot \sqrt{a \cdot (1 - a) \cdot b \cdot (1 - b)}. \]
16. First Limitation of This Approach: Computational Complexity

- In the *static* fuzzy logic:
  - to find the degree of confidence in $A \& B$ or in $A \lor B$,
  - we simply applying a t-norm or a t-conorm to two numbers.

- In the *dynamic* case, we need to perform a large number of computations instead.

- This is unavoidable in the dynamic fuzzy logic:
  - we have *more values* for representing the expert’s degree of confidence in each statement,
  - so processing these degrees takes *more computation time*.
17. Another Limitation: Non-Associativity

- Another limitation is that:
  - in contrast to the usual (static) fuzzy logic,
  - dynamic logic operations are not necessarily associative.

- In other words, the estimates for \((A \lor B) \lor C\) and for \(A \lor (B \lor C)\) are, in general, different.

- We will show that this non-associativity is also a limitation
  - not of a specific method of extending “and”- and “or”-operations to dynamic fuzzy logic, but
  - of the very dynamic character of these logics.

- We will show that non-associativity occurs even if we restrict ourselves to linear operations.
18. Non-Associativity: Linear Restriction

• We plan to show that non-associativity occurs even if we restrict ourselves to linear operations.

• Why is such a restriction reasonable?

• One of the most frequently used probability-related fuzzy “or”-operation \( f_\lor(a, b) = a + b - a \cdot b \) is:
  
  – approximately linear for small \( a \) and \( b \);
  
  – isomorphic to \( a + b \) if we appropriately re-scale the values from the interval \([0, 1]\) to \( \mathbb{IR}_0^+ \).
19. Definitions

- For every integer $t$, by a *dynamical fuzzy* $t$-*value*, we mean a sequence $a = \{a_s\}_{s \leq t}$, $a_s \geq 0$.

- For every $t_0$ and $a$, by a *shift* $S_{t_0}(a)$, we mean a sequence $a' = \{a'_s\}_{s \leq t+t_0}$ with $a'_s = a_{s-t_0}$.

- By a *aggregation operation*, we mean an operation $f$ that transforms $t$-sequences $a$ and $b$ into a value $c_t \geq 0$.

- An operation $f$ is called *shift-invariant* if:
  - whenever it transforms $a$ and $b$ into a value $c_t$,
  - it transforms shifted values $S_{t_0}(a)$ and $S_{t_0}(b)$ into the same value $c_{t+t_0}$.

- We say that an aggregation operation $f$ is *linear* if $c_t = Z_t + \sum_{s \leq t} A_{t,s} \cdot a_s + \sum_{s \leq t} B_{t,s} \cdot b_s$.

- By the *result* $c = f(a, b)$ of applying $f$ to sequences $a$ and $b$, we mean a sequence $c_s = f(\{a_u\}_{u \leq s}, \{b_u\}_{u \leq s})$. 
20. Main Result about Non-Associativity

- **Proposition.** If \( c = f(a, b) \) is a shift-invariant linear commutative and associative operation, then:
  - the value \( c_t \) depends only on \( a_t \) and \( b_t \) and
  - does not depend on the values \( a_s \) and \( b_s \) for \( s < t \).

- So, any commutative linear operation that takes into account previous fuzzy estimates is *not* associative.

- Similar results are known in other application areas:
  - if we formulate natural requirements for a reasonable next step in a bargaining process,
  - then every function satisfying these requirements does not depend on the bargaining pre-history.
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22. Proof: Meaning of Shift-Invariance

- Shift-invariance: for $a' = S_{t_0}(a)$ and $b' = S_{t_0}(b)$,
  \[ c_t = Z_t + \sum_{s \leq t} A_{t,s} \cdot a_s + \sum_{s \leq t} B_{t,s} \cdot b_s \text{ implies} \]
  \[ c_t = Z_{t+t_0} + \sum_{s \leq t+t_0} A_{t+t_0,s} \cdot a'_s + \sum_{s \leq t} B_{t+t_0,s} \cdot b'_s. \]

- Substituting $a'_s = a_{s-t_0}$ and $b'_s = b_{s-t_0}$, we get:
  \[ c_t = Z_{t+t_0} + \sum_{s \leq t+t_0} A_{t+t_0,s} \cdot a_{s-t_0} + \sum_{s \leq t} B_{t+t_0,s} \cdot b_{s-t_0}. \]

- Introducing a new variable $s' \overset{\text{def}}{=} s - t_0$, we get:
  \[ c_t = Z_{t+t_0} + \sum_{s' \leq t} A_{t+t_0,s'+t_0} \cdot a_{s'} + \sum_{s \leq t} B_{t+t_0,s'+t_0} \cdot b_{s'}. \]

- Two linear functions coincide if and only if all their coefficients coincide, so:
  \[ Z_t = Z_{t+t_0}, \quad A_{t,s} = A_{t+t_0,s+t_0}, \quad \text{and} \quad B_{t,s} = B_{t+t_0,s+t_0}. \]
23. **Meaning of Shift-Invariance (cont-d)**

- **Reminder:**
  \[ Z_t = Z_{t+t_0}, \quad A_{t,s} = A_{t+t_0,s+t_0}, \quad \text{and} \quad B_{t,s} = B_{t+t_0,s+t_0}. \]

- For every two values \( t \) and \( t' \), we can take \( t_0 = t' - t \), then \( t + t_0 = t' \) hence \( Z_t = Z_{t'} \).

- Thus, \( Z_t \) does not depend on \( t \): \( Z_t = Z \).

- From \( A_{t,s} = A_{t+t_0,s+t_0} \), by taking \( t_0 = -s \), we conclude that \( A_{t,s} = A_{t-s,0} \).

- Thus, \( A_{t,s} = A_{t-s} \), where \( A_t \overset{\text{def}}{=} A_{t,0} \).

- Similarly, we conclude that \( B_{t,s} = B_{t-s} \), for \( B_t \overset{\text{def}}{=} B_{t,0} \).

- Thus, a shift-invariant linear operation has the form
  \[
  c_t = Z + \sum_{s \leq t} A_{t-s} \cdot a_s + \sum_{s \leq t} B_{t-s} \cdot b_s.
  \]
24. Meaning of Commutativity

- Reminder: $c_t = Z + \sum_{s \leq t} A_{t-s} \cdot a_s + \sum_{s \leq t} B_{t-s} \cdot b_s$.

- Commutativity means that

$$
Z + \sum_{s \leq t} A_{t-s} \cdot a_s + \sum_{s \leq t} B_{t-s} \cdot b_s = 
$$

$$
Z + \sum_{s \leq t} A_{t-s} \cdot b_s + \sum_{s \leq t} B_{t-s} \cdot a_s.
$$

- Here again, the fact that the two linear functions coincide means that all their coefficients must coincide.

- So, we conclude that $A_t = B_t$ for all $t$.

- Thus, the above formula for $c_t$ takes the form

$$
c_t = Z + \sum_{s \leq t} A_{t-s} \cdot (a_s + b_s).
$$
25. Meaning of Associativity

- **Reminder:** \( c_t = \sum_{s \leq t} A_{t-s} \cdot (a_s + b_s) \).

- **Associativity** means that \( f(f(a, b), c) = f(a, f(b, c)) \).

- In \( f(f(a, b), c) \), we first combine \( a \) and \( b \) into \( d = f(a, b) \), and then combine \( d \) and \( c \) into \( e = f(d, c) \).

- If we keep track only of the dependence on \( a_t, b_t, \) and \( c_t \), we get \( d_t = A_0 \cdot (a_t + b_t) + \ldots \) and thus:
  \[
  e_t = A_0 \cdot (d_t + c_t) + \ldots = A_0^2 \cdot (a_t + b_t) + A_0 \cdot c_t + \ldots
  \]

- **Associativity** implies that
  \[
  A_0^2 \cdot (a_t + b_t) + A_0 \cdot c_t = A_0^2 \cdot (b_t + c_t) + A_0 \cdot a_t.
  \]

- Since the two linear functions coincide, their coefficients must coincide, i.e., we must have \( A_0 = A_0^2 \).

- Thus, we have \( A_0 = 0 \) or \( A_0 = 1 \).
26. How We Will Prove Non-Associativity

- *Reminder:* \( c_t = \sum_{s \leq t} A_{t-s} \cdot (a_s + b_s) \).

- We have proven that in the associativity case, \( A_0 = 0 \) or \( A_0 = 1 \).

- We will show that in both cases \( A_0 = 0 \) and \( A_0 = 1 \), we have \( A_1 = A_2 = \ldots = 0 \).

- This will prove that \( c_t \) depends only on \( a_t \) and \( b_t \) and does not depend on the previous values \( a_s \) and \( b_s \).

- In both cases, we will prove it by contradiction.

- We will assume that \( A_j \neq 0 \) for some \( j \geq 1 \).

- In this proof, \( k \) will denote the smallest index \( k \geq 0 \) for which \( A_k \neq 0 \).
27. Case $A_0 = 0$

- **Reminder:** $c_t = \sum_{s \leq t} A_{t-s} \cdot (a_s + b_s)$, with $A_0 = A_1 = \ldots = A_{k-1} = 0$, $A_k \neq 0$.

- For $d = f(a, b)$ and $e = f(d, c)$, we get
  
  $d_t = Z + A_k \cdot (a_{t-k} + b_{t-k}) + \ldots$; $e_t = Z + A_k \cdot (d_{t-k} + c_{t-k}) + \ldots$

- Here, $d_{t-k} = Z + A_k \cdot (a_{t-2k} + b_{t-2k}) + \ldots$; thus, we have
  
  $e_t = Z + A_k \cdot Z + A_k^2 \cdot (a_{t-2k} + b_{t-2k}) + A_k \cdot c_{t-k} + \ldots$

- Similarly, $f(a, f(b, c))$ leads to
  
  $e_t = Z + A_k \cdot Z + A_k^2 \cdot (b_{t-2k} + c_{t-2k}) + A_k \cdot a_{t-k} + \ldots$

- So, $Z + A_k \cdot Z + A_k^2 \cdot (a_{t-2k} + b_{t-2k}) + A_k \cdot c_{t-k} + \ldots = Z + A_k \cdot Z + A_k^2 \cdot (b_{t-2k} + c_{t-2k}) + A_k \cdot a_{t-k} + \ldots$

- The left-hand side of this equality does not depend on $a_{t-k}$, while the right-hand side does ($A_k \neq 0$).

- Thus, the equality is indeed impossible.
28. Case $A_0 = 1$

- We find $d = f(a, b)$ and $e = f(d, c)$, w/ $A_0 = 1$, $A_1 = \ldots = A_{k-1} = 0$, $A_k \neq 0$, $c_t = \sum_{s \leq t} A_{t-s} \cdot (a_s + b_s)$.

- Here, $d_t = Z + a_t + b_t + A_k \cdot (a_{t-k} + b_{t-k}) + \ldots$ so $e_t = Z + d_t + c_k + A_k \cdot (d_{t-k} + c_{t-k}) + \ldots$

- Here, $d_{t-k} = Z + a_{t-k} + b_{t-k} + \ldots$; thus, we have

\[
e_t = Z + (Z + a_t + b_t + A_k \cdot (a_{t-k} + b_{t-k}) + \ldots) + c_t + A_k \cdot ((Z + a_{t-k} + b_{t-k} + \ldots) + c_{t-k}) + \ldots =
\]

\[2Z + a_t + b_t + c_t + A_k \cdot (2a_{t-k} + 2b_{t-k} + c_{t-k}) + \ldots\]

- Similarly, the expression $f(a, f(b, c))$ leads to

\[e_t = 2Z + a_t + b_t + c_t + A_k \cdot (2b_{t-k} + 2c_{t-k} + a_{t-k}) + \ldots\]

- Thus \[2Z + a_t + b_t + c_t + A_k \cdot (2a_{t-k} + 2b_{t-k} + c_{t-k}) + \ldots =
\]

\[2Z + a_t + b_t + c_t + A_k \cdot (2b_{t-k} + 2c_{t-k} + a_{t-k}) + \ldots\]
29. Case $A_0 = 1$ (cont-d)

- We have concluded that:

$$2Z + a_t + b_t + c_t + A_k \cdot (2a_{t-k} + 2b_{t-k} + c_{t-k}) + \ldots =$$

$$2Z + a_t + b_t + c_t + A_k \cdot (2b_{t-k} + 2c_{t-k} + a_{t-k}) + \ldots$$

- Reminder: $A_k \neq 0$.

- Here:
  - The left-hand contains $a_{t-k}$ with a coefficient $2A_k$,
    while
  - the right-hand side has this variable with a different coefficient $A_k \neq 2A_k$.

- Thus, the equality is impossible in this case as well.

- The proposition is proven.