Why Feynman Path Integration?

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1. Why Use Quantum Effects in Computing?

- **Fact:** computers are fast.
- **Challenge:** computer are not yet fast enough:
  - to predict weather more accurately and earlier,
  - to run industrial robots more efficiently,
  - to power complex simulations of forest fires,
  and for many other applications, we need higher computing speed.
- **How:** to increase the computing speed, we must make signals travel faster between computer components.
- **Limitation:** signals cannot travel faster than the speed of light, and they already travel at about that speed.
- **Conclusion:** the size of computer components must be reduced.
2. Why Quantum Effects in Computing? (cont-d)

- *Reminder*: the size of computer components must be reduced.

- *Fact*: the sizes of these components have almost reached the sizes of atoms and molecules.

- *Fact*: these molecules follow the rules of quantum mechanics.

- *Conclusion*: quantum computing is necessary.

- *Resulting question*: how to describe these quantum effects?
3. Need for Quantization

- Since the early 1900s, we know that we need to take into account quantum effects. Thus:
  - for every non-quantum physical theory describing a certain phenomenon, be it
    * mechanics
    * or electrodynamics
    * or gravitation theory,
  - we must come up with an appropriate quantum theory.

- Traditional quantization methods: replace the scalar physical quantities with the corresponding operators.

- Problem: operators are non-commutative, $px \neq xp$.

- Conclusion: several different quantum versions of each classical theory.
4. Towards Feynman’s Approach: Least Action Principle

- *Laws of physics* have been traditionally described in terms of differential equations.
- For *fundamental* physical phenomena, not all differential equations make sense.
- *Example:* we need conservation of fundamental physical quantities (energy, momentum, etc.)
- It turns out that
  - all known fundamental physical equations can be described in terms of minimization, and
  - in general, equations following from a minimization principle lead to conservation laws.
5. Towards Feynman’s Approach: Least Action Principle (cont-d)

- **Idea:** we can assign, to each trajectory $\gamma(t)$, we can assign a value $S(\gamma)$ such that
  - among all possible trajectories,
  - the actual one is the one for which the value $S(\gamma)$ is the smallest possible.

- This value $S(\gamma)$ is called *action*.

- The principle that action is minimized along the actual trajectory is called the *minimal action principle*.

- **Feynman’s idea:** the probability to get from the state $\gamma$ to the state $\bar{\gamma}$ is proportional to $|\psi(\gamma \rightarrow \bar{\gamma})|^2$, where

$$
\psi = \sum_{\gamma: \gamma \rightarrow \bar{\gamma}} \exp \left(i \cdot \frac{S(\gamma)}{\hbar} \right).
$$
6. Feynman’s Approach: Successes and Challenges

- **Successes:** Feynman’s approach is an efficient computing tool:
  - we can expand the corresponding expression, and
  - represent the resulting probability as a sum of an infinite series.

- Each term of this series can be described by an appropriate graph called *Feynman diagram*.

- **Foundational challenge:** why the above formula?

- **What we do in this talk:** we provide a natural explanation for Feynman’s path integration formula.
7. First Idea: An Alternative Representation of the Original Theory

- *Reminder:* a physical theory is a functional $S$ that assigns, to every path $\gamma$, the value of the action $S(\gamma)$.

- From this viewpoint, a priori, all the paths are equivalent, they only differ by the corresponding values $S(\gamma)$.

- In other words, what is important is the frequency with which we encounter different values $S(\gamma)$:
  
  - if among $N$ paths, only one has this value of the action, this frequency is $1/N$,
  
  - if two, the frequency is $2/N$, etc.

- In mathematical terms, this means that we consider the action $S(\gamma)$ as a *random variable*.

- One possible way to describe a random variable $\alpha$ is by its characteristic function $\chi_\alpha(\omega) \overset{\text{def}}{=} E[\exp(i \cdot \omega \cdot \alpha)]$. 
8. An Alternative Representation of the Original Theory (cont-d)

• **Reminder:** we consider $\alpha = S(\gamma)$ as a random variable.

• **Reminder:** we describe $\alpha$ via its characteristics function $\chi_\alpha(\omega) \overset{\text{def}}{=} E[\exp(i \cdot \omega \cdot \alpha)]$.

• **Conclusion:**

$$\chi(\omega) = \frac{1}{N} \cdot \sum_{\gamma} \exp(i \cdot S(\gamma) \cdot \omega).$$

• **Reminder:** Feynman’s formula

$$\psi = \sum_{\gamma: \gamma \rightarrow \bar{\gamma}} \exp \left( i \cdot \frac{S(\gamma)}{\hbar} \right).$$

• **Observation:** Feynman’s formula is $\chi(1/\hbar)$.

• **Comment:** this is not yet a derivation, since there are many ways to represent a random variable.

- **Objective:** derive a formula that transforms a functional $S(\gamma)$ into transition probabilities.

- **Typical situation:** the physical system consists of two subsystems.

- In this case, each state $\gamma$ of the composite system is a pair $\gamma = (\gamma_1, \gamma_2)$ consisting of
  
  - a state $\gamma_1$ of the first subsystem and
  - the state $\gamma_2$ of the second subsystem.

- Often, these subsystems are independent.

- Due to this independence,
  
  $$P((\gamma_1, \gamma_2) \rightarrow (\gamma'_1, \gamma'_2)) = P_1(\gamma_1 \rightarrow \gamma'_1) \cdot P_2(\gamma_2 \rightarrow \gamma'_2).$$
10. Independent Physical Systems (cont-d)

- **Reminder:**
  
  \[ P((\gamma_1, \gamma_2) \rightarrow (\gamma'_1, \gamma'_2)) = P_1(\gamma_1 \rightarrow \gamma'_1) \cdot P_2(\gamma_2 \rightarrow \gamma'_2). \]

- In physics, independence is usually described as
  
  \[ S((\gamma_1, \gamma_2)) = S_1(\gamma_1) + S_2(\gamma_2). \]

- In probabilistic terms, this means that we have the sum of two independent random variables. So:
  
  - the probability corresponding to the sum of independent random variables
  - is equal to the product of corresponding probabilities.

- **Fact:** for the sum, characteristic functions multiply:
  
  \[ \chi_{\alpha_1+\alpha_2}(\omega) = \chi_{\alpha_1}(\omega) \cdot \chi_{\alpha_2}(\omega). \]
11. Independent Physical Systems (cont-d)

- **Reminder:** we have \( p(\chi_1 \cdot \chi_2) = p(\chi_1) \cdot p(\chi_2) \), i.e., to
  \[
p(\chi_1(\omega_1) \cdot \chi_2(\omega_1), \ldots, \chi_1(\omega_n) \cdot \chi_2(\omega_n), \ldots) = \]
  \[
p(\chi_1(\omega_1), \ldots, \chi_1(\omega_n), \ldots) \cdot p(\chi_2(\omega_1), \ldots, \chi_2(\omega_n), \ldots).
  \]

- **To simplify:** use log-log scale:
  - \( P \overset{\text{def}}{=} \ln(p) \) as the new dependent variable, and
  - the values \( Z_i = X_i + i \cdot Y_i \overset{\text{def}}{=} \ln(\chi(\omega_i)) \) as the new independent variables:
    \[
P(Z_1, \ldots, Z_n, \ldots) = \ln p(\exp(Z_1), \ldots, \exp(Z_n), \ldots)
    \]

- Then, \( P(Z + Z') = P(Z) + P(Z') \), so \( P \) is linear:
  \[
P(Z) = \sum_i (a(\omega_i) \cdot X(\omega_i) + b(\omega_i) \cdot Y(\omega_i)).
  \]

- Thus, \( p(\chi) = \exp(P(\ln(\chi))) = \prod_i |\chi(\omega_i)|^{a_i}. \)
12. Third Idea: Maximal Set of Possible Future States

- Reminder: \( p(\chi) = \prod_i |\chi(\omega_i)|^{a_i} \).
- Reminder: each value \( \chi(\omega_i) \) is equal to the Feynman sum, with \( \hbar_i = 1/\omega_i \).
- Question: why only one such term?
- In classical physics: once we know the initial state \( \gamma \), we can uniquely predict all future states \( \gamma' \).
- In quantum physics: we can only predict probabilities.
- Sometimes: \( \psi(x) = 0 \), so the state \( x \) is not possible.
- Idea: select a theory for which the set \( I \) of inaccessible states \( \gamma' \) is the smallest possible.
- Fact: a state is inaccessible if \( \chi(\omega_i) = 0 \) for some \( i \).
- Conclusion: the set \( I \) is the smallest when we have only one such term: \( p(\chi) = |\chi(\omega)|^a \).
13. Fourth Idea: Analyticity and Simplicity

- **Reminder**: \( p(\chi) = |\chi(\omega)|^a \) for some \( a \).

- **In physics**: most dependencies are real-analytical, i.e., expandable in convergent Taylor series.

- **For** \( z = x + i \cdot y \): we have
  \[
  |z|^a = \left( \sqrt{x^2 + y^2} \right)^a = (x^2 + y^2)^{a/2}.
  \]

- **This expression** is analytical at \( z = 0 \) if and only if \( a \) is an even natural number (\( a = 0, 2, 4, \ldots \))

- **Fact**: case \( a = 0 \) is trivial: all transition probabilities are the same.

- **Conclusion**: the simplest non-trivial case is \( a = 2 \).

- **Thus**: we have indeed justified Feynman integration.
14. Acknowledgments

This work was supported in part by:

- by National Science Foundation grants HRD-0734825 and DUE-0926721,

- by Grant 1 T36 GM078000-01 from the National Institutes of Health, and

- by Grant 5015 from the Science and Technology Centre in Ukraine (STCU), funded by European Union.