How to Assign Weights to Different Factors in Vulnerability Analysis: Towards a Justification of a Heuristic Technique

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1. Need for Vulnerability Analysis

- Many important systems are vulnerable – to a storm, to a terrorist attack, to hackers’ attack, etc.
- We need to protect them.
- Usually, there are many different ways to protect the same system.
- It is desirable to select the protection scheme with the largest degree of protection within the given budget.
- The corresponding analysis of different vulnerability aspects is known as *vulnerability analysis*.
2. **Vulnerability Analysis: Reminder**

- There are many different aspects of vulnerability.
- Usually, it is known how to gauge the vulnerability $v_i$ of each aspect $i$.
- Thus, each alternative can be characterized by the corresponding vulnerability values $(v_1, \ldots, v_n)$.
- To compare alternatives, we need to combine the values $v_i$ into a single index $v = f(v_1, \ldots, v_n)$.
- If one of the vulnerabilities $v_i$ increases, then the overall vulnerability index $v$ must also increase.
- Thus, $f(v_1, \ldots, v_n)$ must be increasing in each $v_i$.
- Usually, vulnerabilities $v_i$ are reasonably small.
- Thus, we can expand $f(v_1, \ldots, v_n)$ in Taylor series in $v_i$ and keep only linear terms: $v = c_0 + \sum_{i=1}^{n} c_i \cdot v_i$. 
3. Vulnerability Analysis (cont-d)

- Comparison does not change if we subtract the same constant $c_0$ from all the combined values:
  \[ v < v' \iff v - c_0 < v' - c_0. \]

- So, we can safely assume $c_0 = 0$ and $v = \sum_{i=1}^{n} c_i \cdot v_i$.

- Similarly, comparison does not change if we re-scale all the values, e.g., divide them by $\sum_{i=1}^{n} c_i$.

- This is equivalent to considering a new (re-scaled) combined function $f(v_1, \ldots, v_n) = \sum_{i=1}^{n} w_i \cdot v_i$ with $\sum_{i=1}^{n} w_i = 1$.

- The important challenge is how to compute the corresponding weights $w_i$. 
4. How to Find Weights? Heuristic Solution

- For each aspect \( i \), we know the frequency \( f_i \) with which this aspect is mentioned in the corr. requirements.

- Sometimes, this is the only information that we have.

- Then, it is reasonable to determine \( w_i \) based on \( f_i \), i.e., to take \( w_i = F(f_i) \) for some function \( F(f) \).

- The following empirical idea works well: take \( w_i = c \cdot f_i \).

- A big problem is that this idea does not have a solid theoretical explanation.

- In this talk, we provide a possible theoretical explanation for this empirically successful idea.
5. Towards a Theoretical Explanation

- The more frequently the aspect is mentioned, the more important it is: $f_i > f_j \Rightarrow w_i = F(f_i) > F(f_j) = w_j$.
- So, $F(f)$ must be increasing.
- For every combination of frequencies $f_1, \ldots, f_n$ for which $\sum_{i=1}^{n} f_i = 1$, the resulting weights must add up to 1:
  \[
  \sum_{i=1}^{n} w_i = \sum_{i=1}^{n} F(f_i) = 1.
  \]
- **Proposition.** Let $F : [0, 1] \rightarrow [0, 1]$ be an increasing f-n for which $\sum_{i=1}^{n} f_i = 1$ implies $\sum_{i=1}^{n} F(f_i) = 1$. Then,
  \[
  F(x) = x.
  \]
- This justifies the empirically successful heuristic idea.
6. Towards a More General Approach

- So far, we assumed that the $i$-th weight $w_i$ depends only on the $i$-th frequency $f_i$.
- Alternatively, we can normalize the “pre-weights” $F(f_i)$ so that they add up to one: 
  \[ w_i = \frac{F(f_i)}{\sum_{k=1}^{n} F(f_k)}. \]
- In this more general approach, how to select $F(f)$?
- **Example:** we have four aspects, each mentioned $n_i$ times, then
  \[ f_i = \frac{n_i}{n_1 + n_2 + n_3 + n_4}. \]
- For some problems, the fourth aspect is irrelevant, so $v_4 = 0$ and
  \[ v = w_1 \cdot v_1 + w_2 \cdot v_2 + w_3 \cdot v_3. \]
- On the other hand, since the 4th aspect is irrelevant, it makes sense to only consider $n_1$, $n_2$, and $n_3$:
  \[ f'_i = \frac{n_i}{n_1 + n_2 + n_3}. \]
7. General Approach (cont-d)

- Based on the new frequencies $f_i'$, we can compute the new weights $w_i'$ and

$$v' = w_1' \cdot v_1 + w_2' \cdot v_2 + w_3' \cdot v_3.$$ 

- Whether we use $v$ or $v'$, the selection should be the same.

- To make sure that the selections are the same, we must guarantee that $$\frac{w_i'}{w_j'} = \frac{w_i}{w_j}.$$ 

- The new frequencies $f_i'$ can be obtained from the previous ones by multiplying by the same constant:

$$f_i' = \frac{n_i}{n_1 + n_2 + n_3} = \frac{n_1 + n_2 + n_3 + n_4}{n_1 + n_2 + n_3} \cdot \frac{n_i}{n_1 + n_2 + n_3 + n_4} = k \cdot f_i.$$ 

- Thus, the requirement takes the form $$\frac{F(k \cdot f_i)}{F(k \cdot f_j)} = \frac{F(f_i)}{F(f_j)}.$$
8. General Approach: Main Result

- **Proposition.** For an increasing f-n $F : [0, 1] \rightarrow [0, 1]$: 
  \[
  \frac{F(k \cdot f_i)}{F(k \cdot f_j)} = \frac{F(f_i)}{F(f_j)} \quad \text{for all } k, f_i, f_j \Leftrightarrow F(f) = C \cdot f^\alpha \quad \text{for } \alpha > 0.
  \]

- So, we should take $F(f) = C \cdot f^\alpha$.

- **Discussion:**
  - The previous case corresponds to $\alpha = 1$.
  - If we multiply all the values $F(f_i)$ by a constant $C$, then the resulting weights do not change.
  - Thus, from the viewpoint of application to vulnerability, it is sufficient to consider only functions 
    \[
    F(f) = f^\alpha.
    \]
9. Possible Probabilistic Interpretation of $w_i = f_i$

- Let us assume that the actual weights of two aspects are $w_1$ and $w_2 = 1 - w_1$.

- Let us also assume that vulnerabilities $v_i$ are independent identically distributed random variables.

- A document mentions the 1st aspect if this aspect is more important (i.e., $w_1 \cdot v_1 > w_2 \cdot v_2$), so:

  $$f_1 = P(w_1 \cdot v_1 > w_2 \cdot v_2).$$

- In a reasonable situation when both vulnerabilities are exponentially distributed, we have

  $$w_1 = P(w_1 \cdot v_1 > w_2 \cdot v_2), \text{ i.e., } w_i = f_i.$$
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11. Appendix: Proof of the First Result

- We require that \( \sum_{i=1}^{n} f_i = 1 \) implies \( \sum_{i=1}^{n} F(f_i) = 1 \).
- We want to prove that \( F(f) = f \) for all \( f \).
- For \( n = 1 \) and \( f_1 = 1 \), we get \( F(f_1) = F(1) = 1 \).
- For \( f_1 = 0 \) and \( f_2 = 1 \), we get \( F(0) + F(1) = 1 \) hence \( F(0) = 1 - F(1) = 1 - 1 = 0 \).
- For every \( m \geq 2 \), for \( f_1 = \ldots = f_m = \frac{1}{m} \), we get \( \sum_{i=1}^{m} F(f_i) = m \cdot F\left(\frac{1}{m}\right) = 1 \), hence \( F\left(\frac{1}{m}\right) = \frac{1}{m} \).
- For every \( k \leq m \), for \( f_1 = \frac{k}{m} \) and \( f_2 = \ldots = f_{m-k+1} = \frac{1}{m} \), we get \( F\left(\frac{k}{m}\right) + (m - k) \cdot F\left(\frac{1}{m}\right) = 1 \), hence
\[
F\left(\frac{k}{m}\right) = 1 - (m - k) \cdot F\left(\frac{1}{m}\right) = 1 - (m - k) \cdot \frac{1}{m} = \frac{k}{m}.
\]
12. Proof (cont-d)

- We have proved that \( F \left( \frac{k}{m} \right) = \frac{k}{m} \) for any rational number \( \frac{k}{m} \).
- Any real number \( f \) can be approximated by rational numbers: \( \frac{k}{m} \leq f < \frac{k + 1}{m} \).
- When \( m \to \infty \), we have \( \frac{k}{m} \to f \) and \( \frac{k + 1}{m} \to f \).
- Due to monotonicity,
  \[
  \frac{k}{m} = F \left( \frac{k}{m} \right) \leq F(f) < F \left( \frac{k + 1}{m} \right) = \frac{k + 1}{m}.
  \]
- In the limit \( m \to \infty \), we conclude that \( F(f) = f \) for any real number \( f \). Q.E.D.