What If We Only Know Hurwicz’s Optimism-Pessimism Parameter with Interval Uncertainty?

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1. Formulation of the Problem

- In many practical situations, we do not know the exact consequences of each possible action.

- As a result:
  - instead of single utility value $u$,
  - we can characterize each possible action by the interval $[u, \bar{u}]$ of possible utility values.

- In such cases, decision theory recommends an alternative for which the following combination is the largest:
  \[ \alpha \cdot \bar{u} + (1 - \alpha) \cdot u = u + \alpha \cdot (\bar{u} - u). \]

- The parameter $\alpha$ is known as Hurwicz’s optimism-pessimism parameter.

- It may be different from different people.
2. Formulation of the Problem (cont-d)

- The value $\alpha = 1$ corresponds to absolute optimists.
- The value $\alpha = 0$ describes a complete pessimist.
- Values between 0 and 1 describe reasonable decision makers.
- The parameter $\alpha$ needs to be determined based on a person’s preferences and decisions.
- Often, in different situations, the decisions of the same person correspond to different values $\alpha$.
- As a result, instead of a single value $\alpha$, we have the whole range $[\underline{\alpha}, \overline{\alpha}]$ of possible values.
- In this case, how should we make decisions?
3. **Our Solution**

- For each $\alpha \in [\underline{\alpha}, \bar{\alpha}]$, the interval $[u, \bar{u}]$ is equivalent to the value
  
  $$u + \alpha \cdot (\bar{u} - u).$$

- This expression is monotonic in $\alpha$.

- So, in general, the original interval $[u, \bar{u}]$ is equivalent to the following range of possible values
  
  $$[u_1, \bar{u}_1] = [u + \alpha \cdot (\bar{u} - u), u + \bar{\alpha} \cdot (\bar{u} - u)].$$

- Similarly, the interval $[u_1, \bar{u}_1]$ is equivalent to
  
  $$[u_2, \bar{u}_2] = [u_1 + \alpha \cdot (\bar{u}_1 - u_1), u_1 + \bar{\alpha} \cdot (\bar{u}_1 - u_1)].$$

- We can repeat this construction again and again:
  
  $$[u_{k+1}, \bar{u}_{k+1}] = [u_k + \alpha \cdot (\bar{u}_k - u_k), u_k + \bar{\alpha} \cdot (\bar{u}_k - u_k)].$$
4. Our Solution (cont-d)

• Reminder:

\[ [u_{k+1}, \bar{u}_{k+1}] = [u_k + \alpha \cdot (\bar{u}_k - u_k), u_k + \bar{\alpha} \cdot (\bar{u}_k - u_k)] \].

• At each step, the width of the original intervals decreases by the factor \( \bar{\alpha} - \alpha \):

\[ \bar{u}_{k+1} - \bar{u}_k = (\bar{\alpha} - \alpha) \cdot (\bar{u}_k - u_k). \]

• Thus, by induction, we conclude that:

\[ \bar{u}_k - u_k = (\bar{\alpha} - \alpha)^k \cdot (\bar{u} - u). \]

• So,

\[ u_{k+1} = u_k + \alpha \cdot (\bar{u}_k - u_k) = u_k + \alpha \cdot (\bar{\alpha} - \alpha)^k \cdot (\bar{u} - u). \]

• Hence,

\[ u_k = u + \alpha \cdot (\bar{u} - u) + \ldots + \alpha \cdot (\bar{u} - u) \cdot (\bar{\alpha} - \alpha)^k. \]
5. **Our Solution (cont-d)**

- Here, $u_k \leq u_{k+1} \leq \bar{u}_{k+1} \leq \bar{u}_k$ and $\bar{u}_k - u_k \to 0$.

- Thus, in the limit, the intervals $[u_k, \bar{u}_k]$ tend to a single point

$$ u = u + \alpha \cdot (\bar{u} - u) + \alpha \cdot (\bar{u} - u) \cdot (\bar{\alpha} - \alpha) + \alpha \cdot (\bar{u} - u) \cdot (\bar{\alpha} - \alpha)^2 + \ldots $$

- The corresponding geometric progression adds to

$$ u + \alpha \cdot (\bar{u} - u) \quad \text{for} \quad \alpha = \frac{\alpha}{1 - (\bar{\alpha} - \alpha)}.$$  

- This is the desired equivalent value of $\alpha$ for the case when we know $\alpha$ with interval uncertainty.

- This is how we should make decisions in this case.