Common Sense Addition
Explained by Hurwicz Optimism-Pessimism Criterion

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1. Common Sense Addition

• Suppose that we have two factors that affect the accuracy of a measuring instrument.

• One factor leads to errors $\pm 10\%$.

• This means that the resulting error component can take any value from $-10\%$ to $+10\%$.

• The second factor leads to errors of $\pm 0.1\%$.

• What is the overall error?

• From the purely mathematical viewpoint, the largest possible error is $10.1\%$.

• However, from the common sense viewpoint, an engineer would say: $10\%$. 
2. Common Sense Addition (cont-d)

- A similar common sense addition occurs in other situations as well.

- For example:
  - if we have a car that weighs 1 ton = 1000 kg,
  - and we place a coke can that weighs 0.35 kg in the car,
  - what will be now the weight of the car?

- Mathematics says 1000.35 kg, but common sense clearly says: still 1 ton.

- How can we explain this common sense addition?
3. Towards Precise Formulation of the Problem

- We know that the overall measurement error $\Delta x$ is equal to $\Delta x_1 + \Delta x_2$, where:
  - the value $\Delta x_1$ can take all possible values from the interval $[-\Delta_1, \Delta_1]$, and
  - the value $\Delta x_2$ can take all possible values from the interval $[-\Delta_2, \Delta_2]$.
- What can we say about the largest possible value $\Delta$ of the absolute value $|\Delta x|$ of the sum $\Delta x = \Delta x_1 + \Delta x_2$?
- Let us describe this problem in precise terms.
- For every pair $(x_1, x_2)$, let $\pi_1(x_1, x_2)$ denote $x_1$ and let $\pi_2(x_1, x_2)$ stand for $x_2$.
- Let $\Delta_1 > 0$ and $\Delta_2 > 0$ be two numbers.
- Without losing generality, we can assume $\Delta_1 \geq \Delta_2$. 
4. Towards Precise Formulation (cont-d)

• By \( S \), let us denote the class of all possible sets \( S \subseteq [-\Delta_1, \Delta_1] \times [-\Delta_2, \Delta_2] \) for which
  \[
  \pi_1(S) = [-\Delta_1, \Delta_1] \text{ and } \pi_2(S) = [-\Delta_2, \Delta_2].
  \]

• We are interested in the value
  \[
  \Delta(S) = \max\{|\Delta x_1 + \Delta x_2|: (\Delta x_1, \Delta x_2) \in S\}.
  \]

• Here, \( S \) is the actual (unknown) set.

• We do not know what is the actual set \( S \), we only know that \( S \in \mathcal{S} \).

• For different sets \( S \in \mathcal{S} \), we may get different \( \Delta(S) \).

• The only thing we know about \( \Delta(S) \) is that \( \Delta(S) \in [\Delta, \bar{\Delta}] \), where:
  \[
  \Delta = \min_{S \in \mathcal{S}} \Delta(S), \quad \bar{\Delta} = \max_{S \in \mathcal{S}} \Delta(S).
  \]

• Which value \( \Delta \) from this interval should we choose?
5. Hurwicz Optimism-Pessimism Criterion

- Often, we do not know the value of a quantity, we only know the interval of its possible values.

- In such situations, decision theory recommends using *Hurwicz optimism-pessimism criterion*.

- Namely, we select the value $\alpha \cdot \Delta + (1 - \alpha) \cdot \bar{\Delta}$ for some $\alpha \in [0, 1]$.

- A usual recommendation is to use $\alpha = 0.5$.

- Let us see what will be the result of applying this criterion to our problem.
6. Computing $\bar{\Delta}$

- For every set $S \in \mathcal{S}$, from $|\Delta x_1| \leq \Delta_1$ and $|\Delta x_2| \leq \Delta_2$, we conclude that $|\Delta x_1 + \Delta x_2| \leq \Delta_1 + \Delta_2$.

- Thus always $\Delta(S) \leq \Delta_1 + \Delta_2$ and hence,

$$\bar{\Delta} = \max \Delta(S) \leq \Delta_1 + \Delta_2.$$

- Let us take $S_0 = \{v, (\Delta_2/\Delta_1) \cdot v) : v \in [-\Delta_1, \Delta_1]\} \in \mathcal{S}$.

- For $S_0$, we have $\Delta x_1 + \Delta x_2 = \Delta x_1 \cdot (1 + \Delta_2/\Delta_1)$.

- Thus in this case, the largest possible value $\Delta(S_0)$ of $\Delta x_1 + \Delta x_2$ is equal to

$$\Delta(S_0) = \Delta_1 \cdot (1 + \Delta_2/\Delta_1) = \Delta_1 + \Delta_2.$$

- So, $\bar{\Delta} = \max \Delta(S') \geq \Delta(S_0) = \Delta_1 + \Delta_2$.

- Hence, $\bar{\Delta} = \Delta_1 + \Delta_2$. 
7. **Computing \( \Delta \)**

- For every \( S \in \mathcal{S} \), since \( \pi_1(S) = [-\Delta_1, \Delta_1] \), we have \( \Delta_1 \in \pi_1(S) \).

- Thus, there exists a pair \((\Delta_1, \Delta x_2) \in S\) corresponding to \( \Delta x_1 = \Delta_1 \).

- For this pair, we have
  \[
  |\Delta x_1 + \Delta x_2| \geq |\Delta x_1| - |\Delta x_2| = \Delta_1 - |\Delta x_2|.
  \]

- Here, \( |\Delta x_2| \leq \Delta_2 \), so \( |\Delta x_1 + \Delta x_2| \geq \Delta_1 - \Delta_2 \).

- Thus, for each \( S \in \mathcal{S} \), the largest possible value \( \Delta(S) \) of \( |\Delta x_1 + \Delta x_2| \) cannot be smaller than \( \Delta_1 - \Delta_2 \):
  \[
  \Delta(S) \geq \Delta_1 - \Delta_2.
  \]

- Hence, \( \Delta = \min_{S \in \mathcal{S}} \Delta(S) \geq \Delta_1 - \Delta_2 \).
8. Computing $\Delta$ (cont-d)

- Take $S_0 = \{v, \left(-\frac{\Delta_2}{\Delta_1}\right) \cdot v : v \in [-\Delta_1, \Delta_1]\} \in S$.
- For $S_0$, we have $\Delta x_1 + \Delta x_2 = \Delta x_1 \cdot \left(1 - \frac{\Delta_2}{\Delta_1}\right)$.
- Thus in this case, the largest possible value $\Delta(S_0)$ of $\Delta x_1 + \Delta x_2$ is equal to
  \[ \Delta(S_0) = \Delta_1 \cdot \left(1 - \frac{\Delta_2}{\Delta_1}\right) = \Delta_1 - \Delta_2. \]
- So, $\underline{\Delta} = \min_{S \in S} \Delta(S) \geq \Delta(S_0) = \Delta_1 - \Delta_2$.
- Thus, $\underline{\Delta} \leq \Delta_1 - \Delta_2$.
- Hence, $\underline{\Delta} = \Delta_1 - \Delta_2$. 

9. Let Us Apply Hurwicz Criterion

- Let us apply Hurwicz criterion with \( \alpha = 0.5 \) to the interval \( [\Delta, \Delta] = [\Delta_1 - \Delta_2, \Delta_1 + \Delta_2] \).

- Then, we get \( \Delta = 0.5 \cdot \Delta + 0.5 \cdot \Delta = \Delta_1 \).

- For example, for \( \Delta_1 = 10\% \) and \( \Delta_2 = 0.1\% \), we get \( \Delta = 10\% \), in full accordance with common sense.

- In other words, *Hurwicz criterion explains the above-described common-sense addition.*
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