Experimental Determination of Mechanical Properties Is, In General, NP-Hard – Unless We Measure Everything

Yan Wang\textsuperscript{1}, Oscar Galindo\textsuperscript{2}, Michael Baca\textsuperscript{2}, Jake Lasley\textsuperscript{2}, and Vladik Kreinovich\textsuperscript{2}

\textsuperscript{1}School of Mechanical Engineering, Georgia Institute of Technology, Atlanta, GA 30332-0405, USA, yan.wang@me.gatech.edu
\textsuperscript{1}University of Texas at El Paso, El Paso, TX 79968, USA
ogalindomo@miners.utep.edu, mvbaca@miners.utep.edu
jlasley@miners.utep.edu, vladik@utep.edu
1. Linear Elasticity: a Brief Reminder

- A force applied to a rubber band extends it or curves it.
- In general, a force applied to different parts of an elastic body changes the mutual location of its points.
- Once we know the forces applied at different locations, we can determine the deformations.
- Vice versa, we can determine the forces once we know all the deformations.
- In general, the dependence on forces $f_\alpha$ at different locations $\alpha$ on different displacement $\varepsilon_\beta$ is non-linear.
- However, usually, displacements are small.
- We can ignore terms quadratic or higher order in terms of $\varepsilon_\beta$. 
2. Linear Elasticity (cont-d)

- Thus, we can safely assume that the dependence of each component $f_\alpha$ on $\varepsilon_\beta$ is linear.

- Taking into account that in the absence of forces, there is no displacement, we conclude that $f_\alpha = \sum_\beta K_{\alpha,\beta} \cdot \varepsilon_\beta$.

- The coefficients $K_{\alpha,\beta}$ describe the mechanical properties of the body.

- It is therefore desirable to experimentally determine these coefficients.
3. Ideal Case

- In the ideal case, we measure displacements $\varepsilon_\beta$ and forces $f_\alpha$ at all possible locations.
- Each measurement results in an equation which is linear in terms of the unknowns $K_{\alpha,\beta}$:
  \[
  f_\alpha = \sum_\beta K_{\alpha,\beta} \cdot \varepsilon_\beta
  \]
- Thus, after performing sufficiently many measurements, we get an easy-to-solve system of linear equations.
- Solving this system enables us to find the values $K_{\alpha,\beta}$. 
4. In Practice, We Only Measure Some Values

- In reality, we only measure displacements and forces at some locations.
- So, we know only some values $f_\alpha$ and $\varepsilon_\beta$.
- Since both $K_{\alpha,\beta}$ and some $\varepsilon_\beta$ are unknown, the corresponding system of equations becomes quadratic.
- After sufficiently many measurements, we may still uniquely determine $K_{\alpha,\beta}$.
- However, the reconstruction is more complex.
5. How Complex: What We Prove

• How complex is the corresponding computational problem?

• In this talk, we prove that the corresponding reconstruction problem is, in general, NP-hard.

• This means that, if – as most computer scientists believe – \( NP \neq P \),
  
  – no feasible algorithm is possible
  – that would always reconstruct the mechanical properties \( K_{\alpha,\beta} \) based on the experimental results.

• We will prove NP-hardness even for the following:
  
  – given \( \alpha_0, \beta_0, \) and \( K_0 \),
  
  – check whether for some solution, \( K_{\alpha_0,\beta_0} = K_0 \).
6. Definition

- From the computational viewpoint, the above problem can be formulated as follows.

- Let $K$ be a natural number. This number will be called the number of experiments.

- By a problem of experimentally determining mechanical properties, we mean the following problem.
  
  - We know that for every $n$ from 1 to $N$, we have $f^{(n)}_{\alpha} = \sum_{\beta} K_{\alpha,\beta} \cdot \varepsilon^{(n)}_{\beta}$ for some values $f^{(n)}_{\alpha}$ and $\varepsilon^{(n)}_{\beta}$.
  
  - For each $n$, we know some of the values $f^{(n)}_{\alpha}$ and $\varepsilon^{(n)}_{\beta}$.
  
  - We need to check if for given $\alpha_0$, $\beta_0$, and $K_0$, we can have $K_{\alpha_0,\beta_0} = K_0$. 

7. Main Result

**Proposition.** The problem of experimentally determining mechanical properties is NP-hard.
8. Proof

- By definition, NP-hard means that all the problems from a certain class NP can be reduced to this problem.

- It is known that the following subset sum problem is NP-hard:
  
  - given \( m+1 \) natural numbers \( s_1, \ldots, s_m, S \),
  
  - check whether it is possible to find the values \( x_i \in \{0,1\} \) for which
    
    \[
    \sum_{i=1}^{m} s_i \cdot x_i = S.
    \]

  - We check whether there is a subset of the values \( s_1, \ldots, s_m \) whose sum is equal to the given value \( S \).

  - The subset sum problem is NP-hard.

  - This means that every problem from the class NP can be reduced to subset sum.
9. Proof (cont-d)

- So, if we reduce the subset problem to our problem, that would mean, by transitivity of reduction, that
  - every problem from the class NP
  - can be reduced to our problem as well.
- So, our problem is indeed NP-hard.
- Let \( s_1, \ldots, s_m, S \) be the values that describe an instance of the subset sum problem.
- Let us reduce it to the following instance of our problem.
- In this instance, we have \( 2^m + 1 \) variables \( \varepsilon_0, \varepsilon_1, \ldots, \varepsilon_m, \varepsilon_{m+1}, \ldots, \varepsilon_{2^m} \).
- We also have \( m + 1 \) different values \( f_\alpha, \alpha = 0, 1, \ldots, m \).
10. First Series of Experiments

- For each $i = 1, \ldots, m$, we have $\varepsilon_i^{(i)} = 1$, $\varepsilon_{m+i}^{(i)} = -1$, and $\varepsilon_j^{(i)} = 0$ for all $j \neq i$.

- The only value of $f_\alpha$ that we measure in each of these experiments is the value $f_0^{(i)} = 0$; then

$$0 = f_0^{(i)} = \sum_{\beta} K_{0,\beta} \cdot \varepsilon_{\beta}^{(i)} = K_{0,i} - K_{0,m+i}.$$ 

- We conclude that $K_{0,m+i} = K_{0,i}$. 

11. Second Series of Experiments

• For each \( n = m + i \), we measure \( \varepsilon_j^{(m+i)} = 0 \) for all \( j \neq n \), and we measure \( f_0^{(m+i)} = f_i^{(m+i)} = 1 \).

• From the corresponding equations, we conclude that 
  \[ 1 = K_{0,m+i} \cdot \varepsilon_{m+i} \] and 
  \[ 1 = K_{i,m+i} \cdot \varepsilon_{m+i} . \]

• We do not know the value \( \varepsilon_{m+i} \).

• However, we can find it from the first equation and substitute into the second one.

• As a result, we conclude that \( K_{0,m+i} = K_{i,m+i} \).

• We know that \( K_{0,i} = K_{0,m+i} \), thus \( K_{0,i} = K_{i,m+i} \).
12. Third Series of Experiments

• For each $i$, we measure $\varepsilon_i^{(2m+i)} = 1$, $\varepsilon_j^{(2m+i)} = 0$ for all other $j$, and we measure $f_i^{(2m+i)} = 1$.

• The corresponding equation implies that $K_{i,i} = 1$. 
13. Fourth Series of Experiments

- We measure the values $\varepsilon_{m+i}^{(3m+i)} = -1$ and $\varepsilon_{j}^{(3m+i)} = 0$ for all $j \neq i, m+i$.

- We also measure the values $f_{0}^{(3m+i)} = f_{i}^{(3m+i)} = 0$.

- In this case, we get $K_{0,i} \cdot \varepsilon_{i}^{(3m+i)} - K_{0,m+i} = 0$ and $K_{i,i} \cdot \varepsilon_{i}^{(3m+i)} - K_{i,m+i} = 0$.

- Since $K_{i,i} = 1$, we have $\varepsilon_{i}^{(3m+i)} = K_{i,m+i}$.

- Since $K_{i,m+i} = K_{0,i}$, this implies $\varepsilon_{i}^{(3m+i)} = K_{0,i}$.

- Let’s substitute this expression for $\varepsilon_{i}^{(3m+i)}$ into $K_{0,i} \cdot \varepsilon_{i}^{(3m+i)} - K_{0,m+i} = 0$.

- Taking into account that $K_{0,m+i} = K_{0,i}$, we get $K_{0,i}^{2} - K_{0,i} = 0$.

- Thus, for each $i$ from 1 to $m$, we have $K_{0,i} \in \{0, 1\}$. 
14. Final (Fifth) Series: A Single Experiment

- We measure \( \varepsilon_0^{(4m+1)} = -S, \varepsilon_1^{(4m+1)} = s_1, \ldots, \varepsilon_m^{(4m+1)} = s_m, \) and \( \varepsilon_{m+i}^{(4m+1)} = 0 \) for all \( i = 1, \ldots, m \).

- We also measure \( f_0^{(4m+1)} = 0 \).

- We want to check whether it is possible that \( K_{0,0} = 1 \).

- For \( K_{0,0} = 1 \), the corresponding equation takes the form \( -S + K_{0,1} \cdot s_1 + \ldots + K_{0,m} \cdot s_m = 0 \).

- So, \( K_{0,1} \cdot s_1 + \ldots + K_{0,m} \cdot s_m = S \) for some \( K_{0,i} \in \{0, 1\} \).

- Suppose that the original instance of the subset sum problem has a solution \( x_i \in \{0, 1\} \).

- Then the above equality holds for \( K_{0,i} = x_i \).
15. Final Series (cont-d)

- Vice versa, suppose that there exist values $K_{0,i} \in \{0, 1\}$ that satisfy the formula

$$K_{0,1} \cdot s_1 + \ldots + K_{0,m} \cdot s_m = S.$$

- Then the values $x_i = K_{0,i}$ solve the original subset sum problem:

$$\sum_{i=1}^{m} s_i \cdot x_i = S.$$

- Thus, we indeed have a reduction – and hence, our problem is indeed NP-hard.
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