Why We Mostly Use 2-, 3- And 5-Based Number Systems?

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1. What Number Systems Do We Use?

- Officially, we only use the decimal system, with base
  \[10 = 2 \cdot 5\].

- However, in practice, when we count, we also use dozens
  \[12 = 2 \cdot 2 \cdot 3\], half-dozens \(6 = 2 \cdot 3\), etc.

- Languages show us that in the past, some of used other bases.

- For example, in French and in Spanish, 20 is described by a different word than all other multiples of 10.

- This shows that in the past, people used \(20 = 2 \cdot 2 \cdot 5\) as the base.
2. **What Number Systems Do We Use (cont-d)**

- In Russian, 40 is described by a different word “sorok”.
- There is even an expression “sorok sorokov” (40 of 40s) for $40 \cdot 40$.
- This shows that the number $40 = 2 \cdot 2 \cdot 2 \cdot 5$ was indeed used as a number base.
- Historical documents show other number bases:
  - Mayan used base 20,
  - Babylonians used base $60 = 2 \cdot 2 \cdot 3 \cdot 5$, etc.
3. Formulation of the Problem

- In all these cases, we use numbers formed by multiplying the first three prime numbers: 2, 3, and 5.
- Why? Why not 7?
- We use 7 often: e.g., we combine days into 7-day weeks.
- However, there does not seem to be a widely spread tradition of using base-7 numbers for computing.
- There is even less evidence of using 11, 13, and larger prime numbers.
- How can we explain this?
4. Possible Explanation: Main Idea

• One possible explanation comes from the need to consider areas and volumes.
• We measure areas – e.g., when buying and selling land.
• Then, for each base $b$, in addition to the original unit, we have a $b^2$ times larger unit.
• For example, in the US system, 1 yard is equal to 3 feet.
• If we want to measure distance and the foot is too small a unit, we can use yards.
• Similarly, if we measure area and the square foot is too small a unit, we can use square yards.
• One square yard is equal to $3^2$ square feet.
5. Main Idea (cont-d)

• Similarly, when we measure volumes – e.g., when buying or selling wine or olive oil – then:
  – with each original unit of volume,
  – we get a new unit which is $b^3$ times larger.

• For example, a cubic yard is equal to $3^3$ cubic feet.

• Sometimes, we buy area-related things and sell volume-related things in return.

• For example, a farmer may want to sell his olive oil crop and use this money to buy some extra land.
6. Main Idea (cont-d)

- In such exchanges, it would be convenient to make sure that the cube of the corresponding base is:
  - either equal to the exact square of some number
  - or, if this is not possible, at least be close to some square,
  - so that the negotiations can succeed with one side paying a small difference of 1 or 2 units.

- In precise terms, we look for numbers \( b \) for which \( b^3 \) is close to some value \( v^2 \), i.e., for which \( |b^3 - v^2| \leq 2 \).
7. **Explanation: Details**

- The cases when this difference is 0, i.e., when $b^3 = v^2$, are easy to describe.
- These are the cases when for some integer $t$, we have $b = t^2$ and $v = t^3$.
- For example, we can take $t = 2$, then $b = 4$ and $v = 8$.
- We can take $t = 3$, then $b = 9$ and $v = 27$.
- In all these cases, we have numbers formed from 2, 3, and 5.
- To use another prime number – the smallest of which is 7 – we need $v = 7^3 = 343$.
- This number is too large to serve as a base for a number system.
- To find all the cases when the difference is ±1 or ±2, we used a program to check all pairs $(b, v)$. 
8. **Explanation: Details (cont-d)**

- To be on the safe side, we tested all the pairs for which both $b$ and $v$ do not exceed 10,000.

- Interestingly, among such pairs, only for two pairs the absolute value of the difference does not exceed 2: namely:
  - we have $3^2 - 2^3 = 9 - 8 = 1$ and  
  - we have $3^3 - 5^2 = 27 - 25 = 2$.

- Thus, from this viewpoint, reasonable bases are 2, 3, and 5.

- This explains why such bases are mostly used.
9. **Explanation: Details (cont-d)**

- This also explains why:
  - in spite of the prevalence of the decimal system that only uses 2 and 5,
  - we also continue to count in dozens and half-dozens (that use 3).
- Indeed, the closest values to $2^3$ and to $5^2$ are powers of 3.