Towards a More Natural Proof of Metrization Theorem for Space-Times

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1. Urysohn’s Lemma and Urysohn’s Metrization Theorem: Reminder

- **Who, when:** early 1920s, Pavel Urysohn.

- **Claim for fame:** Urysohn’s Lemma is “first non-trivial result of point set topology”.

- **Condition:** $X$ is a normal topological space $X$, $A$ and $B$ are disjoint closed sets.

- **Conclusion:** there exists $f : X \to [0, 1]$ s.t. $f(A) = \{0\}$ and $f(B) = \{1\}$.

- **Reminder:** normal means that every two disjoint closed sets have disjoint open neighborhoods.

- **Application:** every normal space with countable base is metrizable.

- **Comment:** actually, every regular Hausdorff space with countable base is metrizable.
2. Extension to Space-Times: Urysohn’s Problem

- **Fact:** a few years before that, in 1919, Einstein’s GRT has been experimentally confirmed.

- **Corresponding structure:** topological space with an order (casuality).

- **Urysohn’s problem:** extend his lemma and metrization theorem to (causality-)ordered topological spaces.

- **Tragic turn of events:** Urysohn died in 1924.

- **Follow up:** Urysohn’s student Vadim Efremovich; Efremovich’s student Revolt Pimenov; Pimenov’s students.

- **Other researchers:** H. Busemann (US), E. Kronheimer and R. Penrose (UK).

- **Result:** by the 1970s, space-time versions of Uryson’s lemma and metrization theorem have been proven.
3. Causality: A Reminder

\[ x = -c \cdot t \]

\[ x = c \cdot t \]
4. Space-Time Metrization Results: A Challenge

- **Objective:** come up with useful applications to physics.
- **Conclusion:** we need proofs that directly follow from the analysis of the main notions and ideas.
- **Fact:** the original 1970s proofs look like the use of clever tricks.
- **Conclusion:** we must make these proofs more natural.
- **How:** we use Zadeh’s ideas of applying fuzzy to causality.
- **We show:** that fuzzy logic indeed leads to such more natural proofs.
- **Not yet:** we are still far away from practical applications.
- **We believe** that our result has brought us one step closer to these future applications.
5. Space-Time Models: Reminder

- *Theoretical relation:* (transitive) causality $a \preceq b$.
- *Problem:* events are not located exactly: $\tilde{a} \approx a$, $\tilde{b} \approx b$.
- *Practical relation:* kinematic causality $a \prec b$.
- *Meaning:* every event in some small neighborhood of $b$ causally follows $a$, i.e., $b \in \text{Int}(a^+)$.  

- *Properties of $\prec$:*$\prec$ is transitive; $a \not\prec a$;
  \[ \forall a \exists a, \bar{a} \left(a \prec a \prec \bar{a}\right); \quad a \prec b \Rightarrow \exists c \left(a \prec c \prec b\right); \]
  \[ a \prec b, c \Rightarrow \exists d \left(a \prec d \prec b, c\right); \quad b, c \prec a \Rightarrow \exists d \left(b, c \prec d \prec a\right). \]

- *Alexandrov topology:* with intervals as the base:
  \[ (a, b) \overset{\text{def}}{=} \{c : a \prec c \prec b\}. \]

- *Description of causality:* $a \preceq b \overset{\text{def}}{=} b \in \overline{a^+}$.

- *Additional property:* $b \in \overline{a^+} \iff a \in \overline{b^-}$.  


6. Space-Time Analog of a Metric

- **Traditional metric**: a function $\rho : X \times X \rightarrow R^+_0$ s.t.
  \[
  \rho(a, b) = 0 \iff a = b;
  \rho(a, b) = \rho(b, a);
  \rho(a, c) \leq \rho(a, b) + \rho(b, c).
  \]

- **Physical meaning**: the length of the shortest path between $a$ and $b$.

- **Kinematic metric**: a function $\tau : X \times X \rightarrow R^+_0$ s.t.
  \[
  \tau(a, b) > 0 \iff a \prec b;
  a \prec b \prec c \Rightarrow \tau(a, c) \geq \tau(a, b) + \tau(b, c).
  \]

- **Physical meaning**: the longest (= proper) time from event $a$ to event $b$.

- **Explanation**: when we speed up, time slows down.
7. Space-Time Analogs of Urysohn’s Lemma and Metrization Theorem

- **Main condition:** the kinematic space is separable, i.e., there exists a countable dense set \( \{x_1, x_2, \ldots, x_n, \ldots\} \).

- **Condition of the lemma:** \( X \) is separable, and \( a \prec b \).

- **Lemma:** \( \exists \) a cont. \( \preceq \)-increasing f-n \( f_{(a,b)} : X \rightarrow [0, 1] \) s.t. \( f_{(a,b)}(x) = 0 \) for \( a \not\prec x \) and \( f_{(a,b)}(x) = 1 \) for \( b \preceq x \).

- **Relation to the original Urysohn’s lemma:** \( f_{(a,b)} \) separates disjoint closed sets \( -a^+ \) and \( b^+ \).

- **Condition of the theorem:** \( (X, \prec) \) is a separable kinematic space.

- **Theorem:** there exists a continuous metric \( \tau \) which generates the corresponding relation \( \prec \).

- **Corollary:** \( \tau \) also generates the corresponding topology.
8. How the Space-Time Metrization Theorem Is Proved

Now

- **First lemma:** for every $x$, there exists a $\prec$-monotonic function $f_x : X \to [0, 1]$ for which $f_x(b) > 0 \iff x \prec b$.

  - **Proof:** $\exists y_i \searrow x$; take $f_x(b) = \sum_{i=1}^{\infty} 2^{-i} \cdot f(x, y_i)(b)$.

- **Second lemma:** for every $x$, there exists a $\prec$-monotonic function $g_x : X \to [0, 1]$ for which $g_x(a) > 0 \iff a \prec x$.

  - **Proof:** similar.

- **Resulting metric:** for a countable everywhere dense sequence $\{x_1, x_2, \ldots, x_n, \ldots\}$, take

$$
\tau(a, b) = \sum_{i=1}^{\infty} 2^{-i} \cdot \min(g_{x_i}(a), f_{x_i}(b)).
$$
9. Towards a Fuzzy Interpretation of Space-Time Ideas

- **Intuitive idea**: $\tau(a, b)$ is a “degree of causality”.
- **Why fuzzy logic**: it was specifically designed to generate and process such degrees.
- **Simplest case**: we only have a lower bound $x$ and an upper bound $\bar{x}$ for a quantity $x$.
- **Description**: the value $x$ belongs to the interval $[x, \bar{x}] \overset{\text{def}}{=} \{ x : x \leq x \leq \bar{x} \}$.
- **Space-time case**: we know:
  - an event $x$ that influenced $x$ (i.e., that causally precedes $x$), and
  - an event $\bar{x}$ that was influenced by $x$ (i.e., that causally follows from $x$).
- **Description**: $[x, \bar{x}] \overset{\text{def}}{=} \{ x : x \leq x \leq \bar{x} \}$.
10. Degree Interpretation of Urysohn’s Lemma

- **We know:** that the event $e$ is in the interval $[a, b]$.
- **We want to estimate:** the degree $f_{[a,b]}(x)$ to which it is possible that $e \leq x$.
- **1st property:** if $b \leq x$, then $e \leq x$ hence $f_{[a,b]}(x) = 1$.
- **2nd property:** is $a \not< x$, then $e \not< x$ hence $f_{[a,b]}(x) = 0$.
- **Fact:** if $x \leq x'$ and $e \leq x$, then, of course, $e \leq x'$.
- **Thus:** our degree of possibility that $e \leq x'$ is larger (or equal) than the degree that $e \leq x$: $f_{[a,b]}(x) \leq f_{[a,b]}(x')$.
- **3rd property:** in mathematical terms, this means that the function $f_{[a,b]}(x)$ is $\leq$-monotonic.
- When we change $x$ slightly, our degree $f_{[a,b]}(x)$ should change only slightly, i.e., $f_{[a,b]}$ should be *continuous*.
- These are exactly the properties of a function existing due to the space-time analogue of Urysohn’s lemma.
11. Towards a More Natural Proof of the Lemmas

- **Ideal case:** $x \preceq b \iff b$ is influenced by a signal emitted at the moment $x$.

- **In practice:** there is always a delay between the decision $x$ to emit the signal and the actual emission $y$.

- Influence $x \preceq b$ is confirmed if $b$ follows from some event $y \in [x, y_1]$, for a known upper bound $y_1$.

- By using more and more accurate technologies, we can make this delay smaller and smaller, with $y_i$ such that

  $$x \prec \ldots \prec y_3 \prec y_2 \prec y_1 \text{ and } y_i \rightarrow x.$$ 

- Thus, $x \preceq b \iff \exists i (y \prec b \text{ for some } y \in [x, y_i])$.

- **In practice:** large $i$ may require technology which is not yet available, so

  $$\exists i ((i \text{ is not too large}) \& (y \prec b \text{ for some } y \in [x, y_i])).$$
12. Let Us Use the Simplest Fuzzy Translations

- \( \exists i ((i \text{ is not too large}) \& (y \prec b \text{ for some } y \in [x, y_i])) \).
- We know: the degree of belief \( f_{[x,y_i]}(b) \) that \( y \prec b \) for some \( y \in [x, y_i] \).
- We need to describe:
  - the degree of belief \( N(i) \) that \( i \) is not too large,
  - the degree of belief that \( y \prec b \) for some \( y \in [x, y_i] \),
  - t-norm ("and") and t-conorm ("or") operations \& and \( \lor \):

\[
d(A), d(B) \rightarrow d(A \& B) \approx d(A) \& d(B), d(A \lor B) \approx d(A) \lor d(B).
\]
- Our selection: computationally simplest \( a \& b = a \cdot b \) and \( a \lor b = \min(a + b, 1) \) (in our case \( a \lor b = a + b \)).
- Comment: our proof works for other choices as well.
- Remaining problem: describe the degrees \( N(i) \).
13. Towards a More Natural Proof of Lemma 1

- If \((i \text{ is not too large})\) and \((j \text{ is not too large})\) then \((i + j \text{ is not too large})\).
- **Description:** \(N(i + j) = N(i) \& N(j) = N(i) \cdot N(j)\).
- Thus, we get \(N(i) = N(i - 1) \cdot N(1)\) and \(N(i) = N(1)^i\).
- **Simplest case:** \(N(1) = 1/2\), then \(N(i) = 2^{-i}\).
- \(\exists i ((i \text{ is not too large}) \& (y < b \text{ for some } y \in [x, y_i]))\).
- **Reminder:** \(a \& b\) is \(a \cdot b\); \(\exists i P(i)\) is \(A_1 \lor A_2 \lor \ldots\); \(a \lor b\) is \(a + b\); and \(d(y < b \text{ for some } y \in [x, y_i]) = f_{[x,y_i]}(b)\).
- **Conclusion:** \(f_x(b) = \sum_{i=1}^{\infty} 2^{-i} \cdot f_{[x,y_i]}(b)\).
- **Easy to prove:** Lemma 1, i.e., \(f_x\) is \(\preceq\)-monotonic and \(f_x(b) > 0 \Leftrightarrow x < b\).
- **Comment:** Lemma 2 can be similarly proven.
14. Towards a More Natural Proof of the Metrization Theorem for Space-Times

- **Idea:** $a \preceq b \iff$ a signal emitted at $a$ is detected at $b$.
- **Direct implementation:** every event directly sends signals to everyone else.
- **Problem:** too much energy needed.
- **Solution:** retransmissions at $x_1, x_2, \ldots$:
  \[ a \prec b \iff \exists i ((a \prec x_i) \& (x_i \prec b)). \]
- **In practice:** $\exists i ((i$ is not too large$) \& (a \prec x_i)$.
- **Result** of using fuzzy translation:
  \[ \tau(a, b) = \sum_{i=1}^{\infty} 2^{-i} \cdot g_{x_i}(a) \cdot f_{x_i}(b). \]
- **It is** (relatively) easy to prove: that this expression satisfies both properties of the kinematic metric.
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