Towards Decision Making under Interval, Set-Valued, Fuzzy, and Z-Number Uncertainty: A Fair Price Approach

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1. Need for Decision Making

- In many practical situations:
  - we have several alternatives, and
  - we need to select one of these alternatives.
- Examples:
  - a person saving for retirement needs to find the best way to invest money;
  - a company needs to select a location for its new plant;
  - a designer must select one of several possible designs for a new airplane;
  - a medical doctor needs to select a treatment for a patient.
2. Need for Decision Making Under Uncertainty

• Decision making is easier if we know the exact consequences of each alternative selection.

• Often, however:
  – we only have an incomplete information about consequences of different alternative, and
  – we need to select an alternative under this uncertainty.
3. How Decisions Under Uncertainty Are Made Now

- Traditional decision making assumes that:
  - for each alternative $a$,
  - we know the probability $p_i(a)$ of different outcomes $i$.

- It can be proven that:
  - preferences of a rational decision maker can be described by utilities $u_i$ so that
  - an alternative $a$ is better if its expected utility $\overline{u}(a) \overset{\text{def}}{=} \sum_i p_i(a) \cdot u_i$ is larger.
4. Hurwicz Optimism-Pessimism Criterion

• Often, we do not know these probabilities $p_i$.

• For example, sometimes:
  • we only know the range $[u, \bar{u}]$ of possible utility values, but
  • we do not know the probability of different values within this range.

• It has been shown that in this case, we should select an alternative s.t. $\alpha_H \cdot \bar{u} + (1 - \alpha_H) \cdot u \to \max$.

• Here, $\alpha_H \in [0, 1]$ described the optimism level of a decision maker:
  • $\alpha_H = 1$ means optimism;
  • $\alpha_H = 0$ means pessimism;
  • $0 < \alpha_H < 1$ combines optimism and pessimism.
5. **What If We Have Fuzzy Uncertainty? Z-Number Uncertainty?**

- There are many semi-heuristic methods of decision making under fuzzy uncertainty.
- These methods have led to many practical applications.
- However, often, different methods lead to different results.
- R. Aliev proposed a utility-based approach to decision making under fuzzy and Z-number uncertainty.
- However, there still are many practical problems when it is not fully clear how to make a decision.
- In this talk, we provide foundations for the new methodology of decision making under uncertainty.
- This methodology which is based on a natural idea of a *fair price.*
6. Fair Price Approach: An Idea

- When we have a full information about an object, then:
  - we can express our desirability of each possible situation
  - by declaring a price that we are willing to pay to get involved in this situation.

- Once these prices are set, we simply select the alternative for which the participation price is the highest.

- In decision making under uncertainty, it is not easy to come up with a fair price.

- A natural idea is to develop techniques for producing such fair prices.

- These prices can then be used in decision making, to select an appropriate alternative.
7. Case of Interval Uncertainty

- **Ideal case:** we know the exact gain $u$ of selecting an alternative.

- **A more realistic case:** we only know the lower bound $\underline{u}$ and the upper bound $\overline{u}$ on this gain.

- **Comment:** we do not know which values $u \in [\underline{u}, \overline{u}]$ are more probable or less probable.

- This situation is known as *interval uncertainty*.

- We want to assign, to each interval $[\underline{u}, \overline{u}]$, a number $P([\underline{u}, \overline{u}])$ describing the fair price of this interval.

- Since we know that $u \leq \overline{u}$, we have $P([\underline{u}, \overline{u}]) \leq \overline{u}$.

- Since we know that $\underline{u}$, we have $\underline{u} \leq P([\underline{u}, \overline{u}])$. 
8. Case of Interval Uncertainty: Monotonicity

- **Case 1**: we keep the lower endpoint \( u \) intact but increase the upper bound.
- This means that we:
  - keeping all the previous possibilities, but
  - we allow new possibilities, with a higher gain.
- In this case, it is reasonable to require that the corresponding price not decrease:
  \[
  \text{if } u = v \text{ and } \bar{u} < \bar{v} \text{ then } P([u, \bar{u}]) \leq P([v, \bar{v}]).
  \]
- **Case 2**: we dismiss some low-gain alternatives.
- This should increase (or at least not decrease) the fair price:
  \[
  \text{if } u < v \text{ and } \bar{u} = \bar{v} \text{ then } P([u, \bar{u}]) \leq P([v, \bar{v}]).
  \]
9. Additivity: Idea

- Let us consider the situation when we have two consequent independent decisions.
- We can consider two decision processes separately.
- We can also consider a single decision process in which we select a pair of alternatives:
  - the 1st alternative corr. to the 1st decision, and
  - the 2nd alternative corr. to the 2nd decision.
- If we are willing to pay:
  - the amount \( u \) to participate in the first process, and
  - the amount \( v \) to participate in the second decision process,
- then we should be willing to pay \( u + v \) to participate in both decision processes.
10. Additivity: Case of Interval Uncertainty

- About the gain $u$ from the first alternative, we only know that this (unknown) gain is in $[u, \bar{u}]$.
- About the gain $v$ from the second alternative, we only know that this gain belongs to the interval $[v, \bar{v}]$.
- The overall gain $u + v$ can thus take any value from the interval

$$[u, \bar{u}] + [v, \bar{v}] \overset{\text{def}}{=} \{u + v : u \in [u, \bar{u}], v \in [v, \bar{v}]\}.$$

- It is easy to check that

$$[u, \bar{u}] + [v, \bar{v}] = [u + v, \bar{u} + \bar{v}].$$

- Thus, the additivity requirement about the fair prices takes the form

$$P([u + v, \bar{u} + \bar{v}]) = P([u, \bar{u}]) + P([v, \bar{v}]).$$
11. Fair Price Under Interval Uncertainty

- By a fair price under interval uncertainty, we mean a function $P([u, \bar{u}])$ for which:
  - $u \leq P([u, \bar{u}]) \leq \bar{u}$ for all $u$ (conservativeness);
  - if $u = v$ and $\bar{u} < \bar{v}$, then $P([u, \bar{u}]) \leq P([v, \bar{v}])$ (monotonicity);
  - (additivity) for all $u$, $\bar{u}$, $v$, and $\bar{v}$, we have
    \[ P([u + v, \bar{u} + \bar{v}]) = P([u, \bar{u}]) + P([v, \bar{v}]). \]

- **Theorem:** Each fair price under interval uncertainty has the form
  \[ P([u, \bar{u}]) = \alpha_H \cdot \bar{u} + (1 - \alpha_H) \cdot u \]
  for some $\alpha_H \in [0, 1]$.

- **Comment:** we thus get a new justification of Hurwicz optimism-pessimism criterion.
12. Proof: Main Ideas

- Due to monotonicity, \( P([u, u]) = u \).
- Due to monotonicity, \( \alpha_H \overset{\text{def}}{=} P([0, 1]) \in [0, 1] \).
- For \([0, 1] = [0, 1/n] + \ldots + [0, 1/n] \) \((n\) times\), additivity implies \( \alpha_H = n \cdot P([0, 1/n]) \), so \( P([0, 1/n]) = \alpha_H \cdot (1/n) \).
- For \([0, m/n] = [0, 1/n] + \ldots + [0, 1/n] \) \((m\) times\), additivity implies \( P([0, m/n]) = \alpha_H \cdot (m/n) \).
- For each real number \( r \), for each \( n \), there is an \( m \) s.t. \( m/n \leq r \leq (m + 1)/n \).
- Monotonicity implies \( \alpha_H \cdot (m/n) = P([0, m/n]) \leq P([0, r]) \leq P([0, (m + 1)/n]) = \alpha_H \cdot ((m + 1)/n) \).
- When \( n \to \infty \), \( \alpha_H \cdot (m/n) \to \alpha_H \cdot r \) and \( \alpha_H \cdot ((m + 1)/n) \to r \), hence \( P([0, r]) = \alpha_H \cdot r \).
- For \([u, \bar{u}] = [u, u] + [0, \bar{u} - u] \), additivity implies \( P([u, \bar{u}]) = u + \alpha_H \cdot (\bar{u} - u) \). Q.E.D.
13. Case of Set-Valued Uncertainty

- In some cases:
  - in addition to knowing that the actual gain belongs to the interval \([u, \bar{u}]\),
  - we also know that some values from this interval cannot be possible values of this gain.

- For example:
  - if we buy an obscure lottery ticket for a simple prize-or-no-prize lottery from a remote country,
  - we either get the prize or lose the money.

- In this case, the set of possible values of the gain consists of two values.

- Instead of a (bounded) interval of possible values, we can consider a general bounded set of possible values.
14. Fair Price Under Set-Valued Uncertainty

- We want a function $P$ that assigns, to every bounded closed set $S$, a real number $P(S)$, for which:
  - $P([u, \bar{u}]) = \alpha_H \cdot \bar{u} + (1 - \alpha_H) \cdot u$ (conservativeness);
  - $P(S + S') = P(S) + P(S')$, where $S + S' \overset{\text{def}}{=} \{s + s' : s \in S, s' \in S'\}$ (additivity).

- **Theorem:** Each fair price under set uncertainty has the form $P(S) = \alpha_H \cdot \sup S + (1 - \alpha_H) \cdot \inf S$.

- **Proof:** Idea.
  - $\{\underline{s}, \bar{s}\} \subseteq S \subseteq [\underline{s}, \bar{s}]$, where $\underline{s} \overset{\text{def}}{=} \inf S$ and $\bar{s} \overset{\text{def}}{=} \sup S$;
  - thus, $[2\underline{s}, 2\bar{s}] = \{\underline{s}, \bar{s}\} + [\underline{s}, \bar{s}] \subseteq S + [\underline{s}, \bar{s}] \subseteq [\underline{s}, \bar{s}] + [\underline{s}, \bar{s}] = [2\underline{s}, 2\bar{s}]$;
  - so $S + [\underline{s}, \bar{s}] = [2\underline{s}, 2\bar{s}]$, hence $P(S) + P([\underline{s}, \bar{s}]) = P([2\underline{s}, 2\bar{s}])$, and
  $$P(S) = (\alpha_H \cdot (2\bar{s}) + (1 - \alpha_H) \cdot (2\underline{s})) - (\alpha_H \cdot \bar{s} + (1 - \alpha_H) \cdot \underline{s}).$$
15. Crisp Z-Numbers, Z-Intervals, and Z-Sets

- Until now, we assumed that we are 100% certain that the actual gain is contained in the given interval or set.
- In reality, mistakes are possible.
- Usually, we are only certain that $u$ belongs to the interval or set with some probability $p \in (0, 1)$.
- A pair of information and a degree of certainty about this is what L. Zadeh calls a Z-number.
- We will call a pair $(u, p)$ consisting of a (crisp) number and a (crisp) probability a crisp Z-number.
- We will call a pair $([u, \bar{u}], p)$ consisting of an interval and a probability a Z-interval.
- We will call a pair $(S, p)$ consisting of a set and a probability a Z-set.
16. Additivity for Z-Numbers

- **Situation:**
  - for the first decision, our degree of confidence in the gain estimate $u$ is described by some probability $p$;
  - for the 2nd decision, our degree of confidence in the gain estimate $v$ is described by some probability $q$.
- The estimate $u + v$ is valid only if both gain estimates are correct.
- Since these estimates are independent, the probability that they are both correct is equal to $p \cdot q$.
- Thus, for crisp Z-numbers $(u, p)$ and $(v, q)$, the sum is equal to $(u + v, p \cdot q)$.
- Similarly, for Z-intervals $([u, \bar{u}], p)$ and $([v, \bar{v}], q)$, the sum is equal to $([u + v, \bar{u} + \bar{v}], p \cdot q)$.
- For Z-sets, $(S, p) + (S', q) = (S + S', p \cdot q)$. 
17. Fair Price for Z-Numbers and Z-Sets

- We want a function $P$ that assigns, to every crisp Z-number $(u, p)$, a real number $P(u, p)$, for which:
  - $P(u, 1) = u$ for all $u$ (conservativeness);
  - for all $u$, $v$, $p$, and $q$, we have $P(u + v, p \cdot q) = P(u, p) + P(v, q)$ (additivity);
  - the function $P(u, p)$ is continuous in $p$ (continuity).

- **Theorem:** Fair price under crisp Z-number uncertainty has the form $P(u, p) = u - k \cdot \ln(p)$ for some $k$.

- **Theorem:** For Z-intervals and Z-sets,
  \[ P(S, p) = \alpha_H \cdot \sup S + (1 - \alpha_H) \cdot \inf S - k \cdot \ln(p). \]

- **Proof:** $(u, p) = (u, 1) + (0, p)$; for continuous $f(p) \overset{\text{def}}{=} (0, p)$, additivity means $f(p \cdot q) = f(p) + f(q)$, so
  \[ f(p) = -k \cdot \ln(p). \]
18. Case When Probabilities Are Known With Interval Or Set-Valued Uncertainty

- We often do not know the exact probability $p$.
- Instead, we may only know the interval $[\underline{p}, \overline{p}]$ of possible values of $p$.
- More generally, we know the set $\mathcal{P}$ of possible values of $p$.
- If we only know that $p \in [\underline{p}, \overline{p}]$ and $q \in [\underline{q}, \overline{q}]$, then possible values of $p \cdot q$ form the interval

$$[\underline{p} \cdot \underline{q}, \overline{p} \cdot \overline{q}].$$

- For sets $\mathcal{P}$ and $\mathcal{Q}$, the set of possible values $p \cdot q$ is the set

$$\mathcal{P} \cdot \mathcal{Q} \overset{\text{def}}{=} \{p \cdot q : p \in \mathcal{P} \text{ and } q \in \mathcal{Q}\}.$$
19. Fair Price When Probabilities Are Known With Interval Uncertainty

- We want a function $P$ that assigns, to every Z-number $(u, [\underline{p}, \overline{p}])$, a real number $P(u, [\underline{p}, \overline{p}])$, so that:
  - $P(u, [p, p]) = u - k \cdot \ln(p)$ (conservativeness);
  - $P(u + v, [\underline{p} \cdot q, \overline{p} \cdot q]) = P(u, [\underline{p}, \overline{p}]) + P(v, [q, q])$ (additivity);
  - $P(u, [\underline{p}, \overline{p}])$ is continuous in $\underline{p}$ and $\overline{p}$ (continuity).

- **Theorem:** Fair price has the form
  
  $$P(u, [\underline{p}, \overline{p}]) = u - (k - \beta) \cdot \ln(\overline{p}) - \beta \cdot \ln(\underline{p})$$

  for some $\beta \in [0, 1]$.

- For set-valued probabilities, we similarly have
  
  $$P(u, \mathcal{P}) = u - (k - \beta) \cdot \ln(\sup \mathcal{P}) - \beta \cdot \ln(\inf \mathcal{P})$$

- For Z-sets and Z-intervals, we have
  
  $$P(S, \mathcal{P}) = \alpha_H \cdot \sup S + (1 - \alpha_H) \cdot \inf S - (k - \beta) \cdot \ln(\sup \mathcal{P}) - \beta \cdot \ln(\inf \mathcal{P})$$
20. Proof

- By additivity, $P(S, \mathcal{P}) = P(S, 1) + P(0, \mathcal{P})$, so it is sufficient to find $P(0, \mathcal{P})$.

- For intervals, $P(0, [\underline{p}, \bar{p}]) = P(0, \bar{p}) + P(0, [p, 1])$, for $p \overset{\text{def}}{=} \underline{p}/\bar{p}$.

- For $f(p) \overset{\text{def}}{=} P(0, [p, 1])$, additivity means $f(p \cdot q) = f(p) \cdot f(q)$.

- Thus, $f(p) = -\beta \cdot \ln(p)$ for some $\beta$.

- Hence, $P(0, [\underline{p}, \bar{p}]) = -k \cdot \ln(\bar{p}) - \beta \cdot \ln(p)$.

- Since $\ln(p) = \ln(\bar{p}) - \ln(\underline{p})$, we get the desired formula.

- For sets $\mathcal{P}$, with $p \overset{\text{def}}{=} \inf \mathcal{P}$ and $\bar{p} \overset{\text{def}}{=} \sup \mathcal{P}$, we have $\mathcal{P} \cdot [\underline{p}, \bar{p}] = [p^2, \bar{p}^2]$, so $P(0, \mathcal{P}) + P(0, [p, \bar{p}]) = P(0, [p^2, \bar{p}^2])$.

- Thus, from known formulas for intervals $[\underline{p}, \bar{p}]$, we get formulas for sets $\mathcal{P}$. 
21. Case of Fuzzy Numbers

- An expert is often imprecise (“fuzzy”) about the possible values.
- For example, an expert may say that the gain is small.
- To describe such information, L. Zadeh introduced the notion of fuzzy numbers.
- For fuzzy numbers, different values \( u \) are possible with different degrees \( \mu(u) \in [0, 1] \).
- The value \( w \) is a possible value of \( u + v \) if:
  - for some values \( u \) and \( v \) for which \( u + v = w \),
  - \( u \) is a possible value of 1st gain, and
  - \( v \) is a possible value of 2nd gain.
- If we interpret “and” as min and “or” (“for some”) as max, we get Zadeh’s extension principle:
  \[
  \mu(w) = \max_{u,v:u+v=w} \min(\mu_1(u), \mu_2(v)).
  \]
22. Case of Fuzzy Numbers (cont-d)

- **Reminder:** \( \mu(w) = \max_{u,v: u+v=w} \min(\mu_1(u), \mu_2(v)) \).

- This operation is easiest to describe in terms of \( \alpha \)-cuts
  \[ u(\alpha) = [u^{-}(\alpha), u^{+}(\alpha)] \overset{\text{def}}{=} \{ u : \mu(u) \geq \alpha \}. \]

- Namely, \( w(\alpha) = u(\alpha) + v(\alpha) \), i.e.,
  \[ w^{-}(\alpha) = u^{-}(\alpha) + v^{-}(\alpha) \text{ and } w^{+}(\alpha) = u^{+}(\alpha) + v^{+}(\alpha). \]

- For product (of probabilities), we similarly get
  \[ \mu(w) = \max_{u,v: u \cdot v=w} \min(\mu_1(u), \mu_2(v)). \]

- In terms of \( \alpha \)-cuts, we have \( w(\alpha) = u(\alpha) \cdot v(\alpha) \), i.e.,
  \[ w^{-}(\alpha) = u^{-}(\alpha) \cdot v^{-}(\alpha) \text{ and } w^{+}(\alpha) = u^{+}(\alpha) \cdot v^{+}(\alpha). \]
23. Fair Price Under Fuzzy Uncertainty

- We want to assign, to every fuzzy number $s$, a real number $P(s)$, so that:
  - if a fuzzy number $s$ is located between $u$ and $\bar{u}$, then $u \leq P(s) \leq \bar{u}$ (*conservativeness*);
  - $P(u + v) = P(u) + P(v)$ (*additivity*);
  - if for all $\alpha$, $s^-(\alpha) \leq t^-(\alpha)$ and $s^+(\alpha) \leq t^+(\alpha)$, then we have $P(s) \leq P(t)$ (*monotonicity*);
  - if $\mu_n$ uniformly converges to $\mu$, then $P(\mu_n) \to P(\mu)$ (*continuity*).

- *Theorem.* The fair price is equal to

$$P(s) = s_0 + \int_0^1 k^-(\alpha) \, ds^-(\alpha) - \int_0^1 k^+(\alpha) \, ds^+(\alpha)$$

for some $k^\pm(\alpha)$.
24. Discussion

- \( \int f(x) \cdot dg(x) = \int f(x) \cdot g'(x) \, dx \) for a \textit{generalized function} \( g'(x) \), hence for generalized \( K^\pm(\alpha) \), we have:

\[
P(s) = \int_0^1 K^-(\alpha) \cdot s^- (\alpha) \, d\alpha + \int_0^1 K^+(\alpha) \cdot s^+ (\alpha) \, d\alpha.
\]

- Conservativeness means that

\[
\int_0^1 K^-(\alpha) \, d\alpha + \int_0^1 K^+(\alpha) \, d\alpha = 1.
\]

- For the interval \([u, \bar{u}]\), we get

\[
P(s) = \left( \int_0^1 K^-(\alpha) \, d\alpha \right) \cdot u + \left( \int_0^1 K^+(\alpha) \, d\alpha \right) \cdot \bar{u}.
\]

- Thus, Hurwicz optimism-pessimism coefficient \( \alpha_H \) is equal to \( \int_0^1 K^+(\alpha) \, d\alpha \).

- In this sense, the above formula is a generalization of Hurwicz’s formula to the fuzzy case.
25. Proof

- Define \( \mu_{\gamma,u}(0) = 1, \mu_{\gamma,u}(x) = \gamma \) for \( x \in (0,u] \), and \( \mu_{\gamma,u}(x) = 0 \) for all other \( x \).

- \( s_{\gamma,u}(\alpha) = [0,0] \) for \( \alpha > \gamma \), \( s_{\gamma,u}(\alpha) = [0,u] \) for \( \alpha \leq \gamma \).

- Based on the \( \alpha \)-cuts, one check that \( s_{\gamma,u+v} = s_{\gamma,u} + s_{\gamma,v} \).

- Thus, due to additivity, \( P(s_{\gamma,u+v}) = P(s_{\gamma,u}) + P(s_{\gamma,v}) \).

- Due to monotonicity, \( P(s_{\gamma,u}) \uparrow \) when \( u \uparrow \).

- Thus, \( P(s_{\gamma,u}) = k^+(\gamma) \cdot u \) for some value \( k^+(\gamma) \).

- Let us now consider a fuzzy number \( s \) s.t. \( \mu(x) = 0 \) for \( x < 0 \), \( \mu(0) = 1 \), then \( \mu(x) \) continuously \( \downarrow \) 0.

- For each sequence of values \( \alpha_0 = 1 < \alpha_1 < \alpha_2 < \ldots < \alpha_{n-1} < \alpha_n = 1 \), we can form an approximation \( s_n \):
  - \( s_n^-(\alpha) = 0 \) for all \( \alpha \); and
  - when \( \alpha \in [\alpha_i, \alpha_{i+1}) \), then \( s_n^+(\alpha) = s^+(\alpha_i) \).
26. Proof (cont-d)

- Here, \( s_n = s_{\alpha_{n-1}, s^+} + s_{\alpha_{n-2}, s^+} - s^+ - s_{\alpha_{n-1}} + \ldots + s_{\alpha_1, \alpha_1 - \alpha_2} \).

- Due to additivity, \( P(s_n) = k^+_{\alpha_{n-1}} \cdot s^+ + k^+_{\alpha_{n-2}} \cdot (s^+ - s^+) + \ldots + k^+_{\alpha_1} \cdot (\alpha_1 - \alpha_2) \).

- This is minus the integral sum for \( \int_0^1 k^+(\gamma) \, ds^+(\gamma) \).

- Here, \( s_n \to s \), so \( P(s) = \lim P(s_n) = \int_0^1 k^+(\gamma) \, ds^+(\gamma) \).

- Similarly, for fuzzy numbers \( s \) with \( \mu(x) = 0 \) for \( x > 0 \), we have \( P(s) = \int_0^1 k^-(\gamma) \, ds^-(\gamma) \) for some \( k^-(\gamma) \).

- A general fuzzy number \( g \), with \( \alpha \)-cuts \([g^-(\alpha), g^+(\alpha)]\) and a point \( g_0 \) at which \( \mu(g_0) = 1 \), is the sum of \( g_0 \),
  - a fuzzy number with \( \alpha \)-cuts \([0, g^+(\alpha) - g_0] \), and
  - a fuzzy number with \( \alpha \)-cuts \([g_0 - g^-(\alpha), 0] \).

- Additivity completes the proof.
27. Case of General Z-Number Uncertainty

• In this case, we have two fuzzy numbers:
  • a fuzzy number $s$ which describes the values, and
  • a fuzzy number $p$ which describes our degree of confidence in the piece of information described by $s$.

• We want to assign, to every pair $(s, p)$ s.t. $p$ is located on $[p_0, 1]$ for some $p_0 > 0$, a number $P(s, p)$ so that:
  • $P(s, 1)$ is as before (conservativeness);
  • $P(u + v, p \cdot q) = P(u, p) + P(v, q)$ (additivity);
  • if $s_n \to s$ and $p_n \to p$, then $P(s_n, p_n) \to P(s, p)$ (continuity).

• Thm: $P(s, p) = \int_0^1 K^-(\alpha) \cdot s^-(\alpha) \, d\alpha + \int_0^1 K^+(\alpha) \cdot s^+(\alpha) \, d\alpha + \int_0^1 L^-(\alpha) \cdot \ln(p^- (\alpha)) \, d\alpha + \int_0^1 L^+(\alpha) \cdot \ln(p^+(\alpha)) \, d\alpha$. 
28. Conclusions and Future Work

- In many practical situations:
  - we need to select an alternative, but
  - we do not know the exact consequences of each possible selection.

- We may also know, e.g., that the gain will be *somewhat larger* than a certain value $u_0$.

- We propose to make decisions by comparing the *fair price* corresponding to each uncertainty.

- *Future work*:
  - apply to practical decision problems;
  - generalize to type-2 fuzzy sets;
  - generalize to the case when we have several pieces of information $(s, p)$. 
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