“And”- and “Or”-Operations for “Double”, “Triple”, etc. Fuzzy Sets

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1. Outline

• In the traditional fuzzy logic:
  – the expert’s degree of confidence \( d(A \& B) \) in a complex statement \( A \& B \)
  – is uniquely determined by his/her degrees of confidence \( d(A) \) and \( d(B) \) in the statements \( A \) and \( B \).

• In practice, for the same degrees \( d(A) \) and \( d(B) \), we may have different degrees \( d(A \& B) \).

• The best way to take this relation into account is to explicitly elicit the corresponding degrees \( d(A \& B) \).

• If we only elicit information about pairs of statements, then we still need to estimate, e.g., the degree \( d(A \& B \& C) \).

• In this talk, we explain how to produce such “and”-operations for “double” fuzzy sets.
2. Traditional Fuzzy Techniques: A Brief Reminder

- Experts often describe their knowledge by using imprecise ("fuzzy") words like "small" or "fast".

- We need to describe this knowledge in computer understandable terms.

- A natural idea is to assign degrees of certainty \( d(S) \in [0, 1] \) to expert statements \( S \).

- We can ask an expert to mark his/her degree of certainty by a mark \( m \) on a scale from 0 to \( n \), and take \( d(S) = m/n \).

- We can also poll \( n \) experts; if \( m \) of them think that \( S \) is true, we take \( d(S) = m/n \).
3. Need for “And’- and “Or”-Operations

- We use expert knowledge to answer queries.
- The answer to a query $Q$ usually depends on several statements.
- What is $d(Q)$?
- For example, $Q$ holds if either $S_1$ and $S_2$ hold, or if $S_3$, $S_3$, and $S_5$ hold.
- Thus, to estimate $d(Q)$, we must estimate the degree of certainty in propositional combinations like
  $$(S_1 \& S_2) \lor (S_3 \& S_4 \& S_5).$$
- Ideally, we should ask the expert’s opinion about all such combinations.
- However, for $n$ statements, we have $2^n$ such combinations, so we cannot ask about all of them.
4. Need for “And’- and “Or”-Operations (cont-d)

- We cannot ask the expert about degree of certainty in all possible propositional combinations.

- It is therefore necessary to estimate \( d(A \& B) \) based on \( d(A) \) and \( d(B) \).

- The estimate \( f_\& (a, b) \) for \( d(A \& B) \) based on \( a = d(A) \) and \( b = d(B) \) is known as an “and”-operation (\( t \)-norm).

- Similarly, we need an “or”-operation \( f_\lor (a, b) \) and a negation operation \( f_\neg (a) \).

- The most widely used operations are:

\[
\begin{align*}
 f_\& (a, b) &= \min(a, b), & f_\& (a, b) &= a \cdot b, \\
 f_\lor (a, b) &= \max(a, b), & f_\lor (a, b) &= a + b - a \cdot b, \\
 f_\neg (a) &= 1 - a. 
\end{align*}
\]
5. Need to Go Beyond Traditional Fuzzy

- In the traditional fuzzy techniques, we base our estimate of $d(A \& B)$ only on $d(A)$ and $d(B)$.
- In reality, for the same degrees of belief in $A$ and $B$, we may have different degrees of belief in $A \& B$.
  - *Example 1*: if $d(A) = 0.5$, then $d(\neg A) = 1 - 0.5 = 0.5$.
  - For $B = A$, $d(A) = d(B) = 0.5$ and $d(A \& B) = d(A) = 0.5$.
  - For $B = \neg A$, $d(A) = d(B) = 0.5$ and $d(A \& B) = 0$.
  - *Example 2*: $d(50\text{-year-old is old}) = 0.1$, $d(60\text{-year-old is old}) = 0.8$, so $d_0 \overset{\text{def}}{=} d(50\text{-year-old is old} \& 60\text{-year-old is not old}) = \min(0.1, 1 - 0.2) > 0$ for $\min(a, b)$ and $a \cdot b$.
- However, intuitively, $d_0 = 0$. 
6. A Natural Idea

- A natural solution to the above problem is to explicitly elicit and store:
  - not only the expert’s degree of confidence $\mu_P(x)$ that a given value $x$ satisfies the property $P$
  - but also the degree of confidence $\mu_{PP}(x, x')$ that both $x$ and $x'$ satisfy the property $P$.

- In this approach, to describe a property, we need two functions:
  - a function $\mu_P : X \to [0, 1]$, and
  - a function $\mu_{PP} : X \times X \to [0, 1]$ for which
    $$\mu_{PP}(x, x') = \mu_{PP}(x', x) \quad \text{and} \quad \mu_{PP}(x, x') \leq \mu_P(x).$$

- Since we need two functions, it is natural to call such pairs $(\mu_P, \mu_{PP})$ double fuzzy sets.

- We can also ask about the triples $(x, x', x'')$ etc.
7. We Need to Extend “And”- and “Or”-Operations to “Double”, “Triple” etc. Fuzzy Sets

- If we explicitly elicit \( d(A \& B) \), we do not need the usual “and”-operation.

- However, we still need to estimate \( d(A \& B \& C) \) based on the available values:

  \[ d(A), \ d(B), \ d(C), \ d(A \& B), \ d(A \& C), \ d(B \& C). \]

- We will show that:
  - the ideas behind the most popular t-norms and t-conorms
  - can be used describe the desired “and”- and “or”-operations for the “double” fuzzy sets.
8. “And”-Operations in Traditional Fuzzy Logic: Reminder

- Traditionally, expert’s degrees of certainty are also called subjective probabilities.

- In probabilistic terms:
  - we know the probabilities \( p(s_1) \) and \( p(s_2) \) of two statements \( s_1 \) and \( s_2 \);
  - we want to estimate the probability \( p(s_1 \& s_2) \).

- Depending on the dependence between \( s_1 \) and \( s_2 \), we may have different values of \( p(s_1 \& s_2) \).

- There are two main approaches to deal with this non-uniqueness:
  - we can find the range of all possible values \( p(s_1 \& s_2) \);
  - or we can select a single “most probable” value \( p(s_1 \& s_2) \).
9. Inequalities (Linear Programming) Approach

- We need to know the probabilities of all basic combinations \( s_1 \& s_2, s_1 \& \neg s_2, \neg s_1 \& s_2, \) and \( \neg s_1 \& \neg s_2. \)
- We know \( d_1 = p(s_1) \) and \( d_2 = p(s_2); \) based on \( x \overset{\text{def}}{=} p(s_1 \& s_2), \) we get:

\[
\begin{align*}
p(s_1 \& \neg s_2) &= p(s_1) - p(s_1 \& s_2) = d_1 - x, \\
p(\neg s_1 \& s_2) &= p(s_2) - p(s_1 \& s_2) = d_2 - x, \quad \text{and} \\
p(\neg s_1 \& \neg s_2) &= 1 - p(s_1) - p(s_2) + p(s_1 \& s_2) = 1 - d_1 - d_2 + x.
\end{align*}
\]
- All the basic probabilities must be non-negative:

\[
x \geq 0; \quad d_1 - x \geq 0; \quad d_2 - x \geq 0; \quad 1 - d_1 - d_2 + x \geq 0, \quad \text{i.e.,} \\
x \geq 0; \quad x \leq d_1; \quad x \leq d_2; \quad x \geq d_1 + d_2 - 1.
\]
- So, the range of possible values is

\[
\max(d_1 + d_2 - 1, 0) \leq x \leq \min(d_1, d_2).
\]
- Both endpoints serve as possible t-norms.
10. Maximum Entropy (MaxEnt) Approach

- Often, we do not know the exact probabilities.
- It is reasonable not to hide uncertainty, i.e., select a distribution with the largest uncertainty.
- There are reasonable arguments that uncertainty of a probability distribution is best described by its entropy
  \[ S = -\sum p_i \cdot \ln(p_i). \]

- Here, \( p_i = x, d_1 - x, d_2 - x, \) and \( 1 - d_1 - d_2 + x, \) so
  \[ S = -x \cdot \ln(x) - (d_1 - x) \cdot \ln(d_1 - x) - (d_2 - x) \cdot \ln(d_2 - x) - (1 - d_1 - d_2 + x) \cdot \ln(1 - d_1 - d_2 + x). \]
- Maximizing \( S \) results in \( x = d_1 \cdot d_2. \)
- For “or”, inequalities approach leads to
  \[ \max(a, b) \leq x \leq \min(a + b, 1). \]
- For “or”, MaxEnt leads to \( d_1 + d_2 = d_2 \cdot d_2. \)
11. “And”-Operations for “Double” Fuzzy Sets

- We know $d_i = p(s_i)$ and $d_{ij} = p(s_i \& s_j)$, $1 \leq i, j \leq 3$.

- From $x = p(s_1 \& s_2 \& s_3)$, we can describe $d_{\varepsilon_1\varepsilon_2\varepsilon_3} \overset{\text{def}}{=} p(s_1^{\varepsilon_1} \& s_2^{\varepsilon_2} \& s_3^{\varepsilon_3})$, $\varepsilon_i = \pm (s^+ = s, s^- = \neg s)$, as

$$
\begin{align*}
    d_{++-} &= d_{12} - x, \\
    d_{+-+} &= d_{13} - x, \\
    d_{+++} &= d_{23} - x, \\
    d_{+-} &= d_1 - d_{12} - d_{23} + x, \\
    d_{--} &= d_2 - d_{12} - d_{23} + x, \\
    d_{++} &= d_3 - d_{13} - d_{23} + x, \\
    d_{--} &= 1 - d_1 - d_2 - d_3 + d_{12} + d_{13} + d_{23} - x.
\end{align*}
$$

- The requirement that $d_{\varepsilon_1\varepsilon_2\varepsilon_3} \geq 0$ leads to:

$$
\max(d_{12} + d_{13} - d_1, d_{12} + d_{23} - d_2, d_{13} + d_{23} - d_3, 0) \leq x \leq \\
\min(d_{12}, d_{13}, d_{23}, 1 - d_1 - d_2 - d_3 + d_{12} + d_{13} + d_{23}).
$$

- Both bounds can thus serve as appropriate “and”-operations.

- By using duality $A \lor B = \neg(\neg A \& \neg B)$, we can get the corresponding “or”-operations.
12. **MaxEnt Approach** 

\[ S = \sum p_i \cdot \ln(p_i) \rightarrow \text{max} \]

- We get 
  \[ p_i = x, \ d_{12} - x, \ d_{13} - x, \ d_{23} - x, \ d_1 - d_{12} - d_{13} + x, \]
  \[ d_2 - d_{12} - d_{23} + x, \ d_3 - d_{12} - d_{23} + x, \] and
  \[ 1 - d_1 - d_2 - d_3 + d_{12} + d_{23} + d_{13} - x. \]

- Equation \( \frac{dS}{dx} = 0 \) leads to
  \[
  - \ln(x) + \ln(d_{12} - x) + \ln(d_{13} - x) + \ln(d_{23} - x) + \ln(d_1 - d_{12} - d_{13} + x) - \ln(d_2 - d_{12} - d_{23} + x) - \ln(d_3 - d_{13} - d_{23} + x) + \ln(1 - d_1 - d_2 - d_3 + d_{12} + d_{23} + d_{13} - x) = 0.
  \]

- If we raise \( e \) to the power of both side, we get a 4-th order equation.

- It is actually 3rd order since terms \( x^4 \) cancel out.

- By using duality \( A \lor B = \neg(\neg A \land \neg B) \), we can get the corresponding “or”-operations.
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