Partial Orders for Representing Uncertainty, Causality, and Decision Making: General Properties, Operations, and Algorithms

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1. Partial Orders are Important

- One of the main objectives of science and engineering is to select the most beneficial decisions. For that:
  - we must know people's preferences,
  - we must have the information about different events (possible consequences of different decisions), and
  - since information is never absolutely accurate, we must have information about uncertainty.
- All these types of information naturally lead to partial orders:
  - For preferences, $a \le b$ means that b is preferable to a. This relation is used in decision theory.
  - For events, $a \le b$ means that a can influence b. This causality relation is used in space-time physics.
  - For uncertain statements, $a \le b$ means that a is less certain than b (fuzzy logic etc.).

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2. What We Plan to Do

- In each of the three areas, there is a lot of research about studying the corresponding partial orders.
- This research has revealed that some ideas are common in all three applications of partial orders.
- In our research, we plan to analyze:
  - general properties, operations, and algorithms
  - related to partial orders for representing uncertainty, causality, and decision making.
- In our analysis, we will be most interested in uncertainty – the computer-science aspect of partial orders.
- In our presentation:
  - we first give a general outline,
  - then present two results in detail (if time allows).

3. Uncertainty is Ubiquitous in Applications of Partial Orders

- Uncertainty is explicitly mentioned only in the computer-science example of partial orders.
- However, uncertainty is ubiquitous in describing our knowledge about all three types of partial orders.
- For example, we may want to check what is happening exactly 1 second after a certain reaction.
- However, in practice, we cannot measure time exactly.
- So, we can only observe an event which is close to b – e.g., that occurs 1 ± 0.001 sec after the reaction.
- In general, we can only guarantee that the observed event is within a certain neighborhood U_b of the event b.
- In decision making, we similarly know the user's preferences only with some accuracy.

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4. Uncertainty-Motivated Experimentally Confirmable Relation

- Because of the uncertainty:
  - the only possibility to experimentally confirm that a precedes b (e.g., that a can causally influence b)
  - is when for some neighborhood U_b of the event b, we have $a \le \widetilde{b}$ for all $\widetilde{b} \in U_b$.
- In topological terms, this "experimentally confirmable" relation $a \prec b$ means that:
  - the element b is contained in the future cone $C_a^+ = \{c : a \le c\}$ of the event a
  - together with some neighborhood.
- In other words, b belongs to the *interior* K_a^+ of the closed cone C_a^+.
- Such relation, in which future cones are open, are called *open*.

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5. Uncertainty-Motivated Experimentally Confirmable Relation (cont-d)

- In usual space-time models:
  - once we know the open cone K_a^+,
  - we can reconstruct the original cone C_a^+ as the closure of K_a^+: $C_a^+ = \overline{K_a^+}$.
- A natural question is: vice versa,
  - can we uniquely reconstruct an open order
  - if we know the corresponding closed order?
- In our paper (Zapata Kreinovich to appear), we show that this reconstruction is possible.
- This result provides a partial solution to a known open problem.

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6. From Potentially Experimentally Confirmable (EC) Relation to Actually EC One

- It is also important to check what can be confirmed when we only have observations with a given accuracy.
- For example:
  - instead of the knowing the exact time location of an an event a,
  - we only know an event $\underline{a}$ that preceded a and an event $\overline{a}$ that follows a.
- In this case, the only information that we have about the actual event a is that it belongs to the interval

$$[\underline{a}, \overline{a}] \stackrel{\text{def}}{=} \{a : \underline{a} \le a \le \overline{a}\}.$$

- It is desirable to describe possible relations between such intervals.

7. From Potentially Experimentally Confirmable (EC) Relation to Actually EC One (cont-d)

- It is desirable to describe possible relations between such intervals.
- Such a description has already been done for intervals on the real line.
- The resulting description is known as Allen's algebra.
- In these terms, what we want is to generalize Allen's algebra to intervals over an arbitrary poset.
- We are currently working on a paper about intervals.
- Instead of intervals, we can also consider more general sets.
- Our preliminary results about general sets are described in a paper (Zapata Ramirez et al. 2011).

8. Properties of Ordered Spaces

- Once a new ordered set is defined, we may be interested in its properties.
- For example, we may want to know when such an order is a lattice, i.e., when:
  - for every two elements,
  - there is the greatest lower bound and the least upper bound.
- If this set is not a lattice, we may want to know:
  - when the order is a *semi-lattice*, i.e., e.g.,
  - when every two elements have the least upper bound.
- For the class of all subsets, we prove the lattice property in (Zapata Ramirez et al. 2011).
- We also describe when special relativity-type ordered spaces are lattices (Künzi et al. 2011).

9. Towards Combining Ordered Spaces: Fuzzy Logic

- In the traditional 2-valued logic, every statement is either true or false.
- Thus, the set of possible truth values consists of two elements: true (1) and false (0).
- Fuzzy logic takes into account that people have different degrees of certainty in their statements.
- Traditionally, fuzzy logic uses values from the interval $[0, 1]$ to describe uncertainty.
- In this interval, the order is *total* (*linear*) in the sense that for every $a, a' \in [0, 1]$, either $a \leq a'$ or $a' \leq a$.
- However, often, *partial* orders provide a more adequate description of the expert's degree of confidence.

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10. Towards General Partial Orders

- For example, an expert cannot describe her degree of certainty by an exact number.
- Thus, it makes sense to describe this degree by an *interval* $[\underline{d}, \overline{d}]$ of possible numbers.
- Intervals are only partially ordered; e.g., the intervals $[0.5, 0.5]$ and $[0, 1]$ are not easy to compare.
- More complex sets of possible degrees are also sometimes useful.
- Not to miss any new options, in this research, we consider general partially ordered spaces.

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11. Need for Product Operations

- Often, two (or more) experts evaluate a statement S.
- Then, our certainty in S is described by a pair (a_1, a_2), where $a_i \in A_i$ is the i-th expert's degree of certainty.
- To compare such pairs, we must therefore define a partial order on the set $A_1 \times A_2$ of all such pairs.
- One example of a partial order on $A_1 \times A_2$ is a *Cartesian* product: $(a_1, a_2) \le (a'_1, a'_2) \Leftrightarrow ((a_1 \le a'_1) \,\&\, (a_2 \le a'_2))$.
- This is a *cautious* approach, when our confidence in S' is higher than in $S \Leftrightarrow$ it is higher for both experts.
- *Lexicographic* product: $(a_1, a_2) \le (a'_1, a'_2) \Leftrightarrow$

$$((a_1 \le a'_1) \,\&\, a_1 \ne a'_1) \vee ((a_1 = a'_1) \,\&\, (a_2 \le a'_2))).$$

- Here, we are absolutely confident in the 1st expert – and only use the 2nd when the 1st is not sure.

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12. Natural Questions

- *Question:* when does the resulting partially ordered set $A_1 \times A_2$ satisfy a certain property?
- *Examples:* is it a total order? is it a lattice order?
- *It is desirable* to reduce the question about $A_1 \times A_2$ to questions about properties of component spaces A_i.
- *Some such reductions are known*; e.g.:
  - A Cartesian product is a total order $\Leftrightarrow$ one of A_i is a total order, and the other has only one element.
  - A lexicographic product is a total order if and only if both components are totally ordered.
- *In this talk*, we provide a general algorithm for such reduction.

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13. Similar Questions in Other Areas

- Similar questions arise in *other applications* of ordered sets.
- *Example:* in space-time geometry, $a \le b$ means that an event a can influence the event b.
- *Our algorithm* does not use the fact that the original relations are orders.
- Thus, our algorithm is applicable to a *general binary relation* – equivalence, similarity, etc.
- Moreover, this algorithm can be applied to the case when we have a space with *several* binary relations.
- *Example:* we may have an order relation and a similarity relation.

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14. Definitions

- *By a* space, *we mean a set A with m binary relations $P_1(a,a'), \ldots, P_m(a,a')$.*
- *By a* 1st order property, *we mean a formula F obtained from $P_i(x,x')$ by using logical $\vee$, $\&$, $\neg$, $\rightarrow$, $\exists x$ and $\forall x$.*
- *Note:* most properties of interest are 1st order; e.g. to be a total order means $\forall a \forall a' \, ((a \le a') \vee (a' \le a))$.
- *By a* product operation, *we mean a collection of m propositional formulas that*
  - *describe the relation $P_i((a_1,a_2),(a'_1,a'_2))$ between the elements $(a_1,a_2),(a'_1,a'_2) \in A_1 \times A_2$*
  - *in terms of the relations between the components $a_1, a'_1 \in A_1$ and $a_2, a'_2 \in A_2$ of these elements.*
- *Note:* both Cartesian and lexicographic order are product operations in this sense.

15. Main Result

- **Main Result.** *There exists an algorithm that, given*
  - *a product operation and*
  - *a property F,*

  generates a list of properties F_{11}, F_{12}, ..., F_{p1}, F_{p2} s.t.:

$$F(A_1 \times A_2) \Leftrightarrow ((F_{11}(A_1) \,\&\, F_{12}(A_2)) \vee \ldots \vee (F_{p1}(A_1) \,\&\, F_{p2}(A_2))).$$

- *Example:* For Cartesian product and total order F, we have

  $$F(A_1 \times A_2) \Leftrightarrow ((F_{11}(A_1) \,\&\, F_{12}(A_2)) \vee (F_{21}(A_1) \,\&\, F_{22}(A_2))) :$$

  - $F_{11}(A_1)$ means that A_1 is a total order,
  - $F_{12}(A_2)$ means that A_2 is a one-element set,
  - $F_{21}(A_1)$ means that A_1 is a one-element set, and
  - $F_{22}(A_2)$ means that A_2 is a total order.

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16. Auxiliary Results

- *Generalization:*
  - A similar algorithm can be formulated for a product of three or more spaces.
  - A similar algorithm can be formulated for the case when we allow ternary and higher order operations.
- *Specifically for partial orders:*
  - The only product operations that always leads to a partial order on $A_1 \times A_2$ for which

$$(a_1 \leq_1 a_1' \,\&\, a_2 \leq_2 a_2') \to (a_1, a_2) \leq (a_1', a_2')$$

    are Cartesian and lexicographic products.

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17. Proof of the Main Result

- The desired property $F(A_1 \times A_2)$ uses:
  - relations $P_i(a, a')$ between elements $a, a' \in A_1 \times A_2$;
  - quantifiers $\forall a$ and $\exists a$ over elements $a \in A_1 \times A_2$.
- Every element $a \in A_1 \times A_2$ is, by definition, a pair (a_1, a_2) in which $a_1 \in A_1$ and $a_2 \in A_2$.
- Let us explicitly replace each variable with such a pair.
- By definition of a product operation:
  - each relation $P_i((a_1, a_2), (a'_1, a'_2))$
  - is a propositional combination of relations betw. elements $a_1, a'_1 \in A_1$ and betw. elements $a_2, a'_2 \in A_2$.
- Let us perform the corresponding replacement.
- Each quantifier can be replaced by quantifiers corresponding to components: e.g., $\forall (a_1, a_2) \Leftrightarrow \forall a_1 \forall a_2$.

18. Proof of the Main Result (cont-d)

- So, we get an equivalent reformulation of F s.t.:
  - elementary formulas are relations between elements of A_1 or between A_2, and
  - quantifiers are over A_1 or over A_2.
- We use induction to reduce to the desired form

$$((F_{11}(A_1) \,\&\, F_{12}(A_2)) \vee \ldots \vee (F_{p1}(A_1) \,\&\, F_{p2}(A_2))).$$

- Elementary formulas are already of the desired form – provided, of course, that we allow free variables.
- We will show that:
  - if we apply a propositional connective or a quantifier to a formula of this type,
  - then we can reduce the result again to the formula of this type.

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19. Applying Propositional Connectives

- We apply propositional connectives to formulas of the type
$$((F_{11}(A_1)\,\&\,F_{12}(A_2))\vee\ldots\vee(F_{p1}(A_1)\,\&\,F_{p2}(A_2))).$$
- We thus get a propositional combination of the formulas of the type $F_{ij}(A_j)$.
- An arbitrary propositional combination can be described as a disjunction of conjunctions (DNF form).
- Each conjunction combines properties related to A_1 and properties related to A_2, i.e., has the form
$$G_1(A_1)\,\&\,\ldots\,\&\,G_p(A_1)\,\&\,G_{p+1}(A_2)\,\&\ldots\,\&\,G_q(A_2).$$
- Thus, each conjunction has the from $G(A_1)\,\&\,G'(A_2)$, where $G(A_1)\Leftrightarrow(G_1(A_1)\,\&\,\ldots\,\&\,G_p(A_1))$.
- Thus, the disjunction of such properties has the desired form.

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20. Applying Existential Quantifiers

- When we apply $\exists a_1$, we get a formula

  $\exists a_1\,((F_{11}(A_1)\,\&\,F_{12}(A_2)) \vee \ldots \vee (F_{p1}(A_1)\,\&\,F_{p2}(A_2)))$.

- It is known that $\exists a\,(A \vee B)$ is equivalent to $\exists a\,A \vee \exists a\,B$.
- Thus, the above formula is equivalent to a disjunction

  $\exists a_1\,(F_{11}(A_1)\,\&\,F_{12}(A_2)) \vee \ldots \vee \exists a_1\,(F_{p1}(A_1)\,\&\,F_{p2}(A_2))$.

- Thus, it is sufficient to prove that each formula $\exists a_1\,(F_{i1}(A_1)\,\&\,F_{i2}(A_2))$ has the desired form.
- The term $F_{i2}(A_2)$ does not depend on a_1 at all, it is all about elements of A_2.
- Thus, the above formula is equivalent to

$$(\exists a_1\,F_{i1}(A_1))\,\&\,F_{i2}(A_2).$$

- So, it is equivalent to the formula $F'_{i1}(A_1)\,\&\,F_{i2}(A_2)$, where $F'_{i1} \Leftrightarrow \exists a_1\,F_{i1}(A_1)$.

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21. Applying Universal Quantifiers

- When we apply a universal quantifier, e.g., $\forall a_1$, then we can use the fact that $\forall a_1\, F$ is equivalent to $\neg\exists a_1\, \neg F$.
- We assumed that the formula F is of the desired type

$$(F_{11}(A_1)\,\&\,F_{12}(A_2)) \vee \ldots \vee (F_{p1}(A_1)\,\&\,F_{p2}(A_2)).$$

- By using the propositional part of this proof, we conclude that $\neg F$ can be reduced to the desired type.
- Now, by applying the $\exists$ part of this proof, we conclude that $\exists a_1\,(\neg F)$ can also be reduced to the desired type.
- By using the propositional part again, we conclude that $\neg(\exists a_1\,\neg F)$ can be reduced to the desired type.
- By induction, we can now conclude that the original formula can be reduced to the desired type.
- The main result is proven.

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22. Example of Applying the Algorithm

- Let us apply our algorithm to checking whether a Cartesian product is totally ordered.
- In this case, F has the form $\forall a \forall a' \,((a \le a') \vee (a' \le a))$.
- We first replace each variable $a, a' \in A_1 \times A_2$ with the corresponding pair:

$$\forall (a_1, a_2) \forall (a'_1, a'_2) \,(((a_1, a_2) \le (a'_1, a'_2)) \vee ((a'_1, a'_2) \le (a_1, a_2))).$$

- Replacing the ordering relation on the Cartesian product with its definition, we get

$$\forall (a_1, a_2) \forall (a'_1, a'_2) \,((a_1 \le a'_1 \,\&\, a_2 \le a'_2) \vee (a'_1 \le a_1 \,\&\, a'_2 \le a_2)).$$

- Replacing $\forall a$ over pairs with individual $\forall a_i$, we get:

$$\forall a_1 \forall a_2 \forall a'_1 \forall a'_2 \,((a_1 \le a'_1 \,\&\, a_2 \le a'_2)) \vee ((a'_1 \le a_1 \,\&\, a'_2 \le a_2))).$$

- By using the $\forall \Leftrightarrow \neg \exists \neg$, we get an equivalent form

$$\neg \exists a_1 \exists a_2 \exists a'_1 \exists a'_2 \,\neg((a_1 \le a'_1 \,\&\, a_2 \le a'_2) \vee (a'_1 \le a_1 \,\&\, a'_2 \le a_2))).$$

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23. Example (cont-d)

- So far, we got:

$$\neg\exists a_1 \exists a_2 \exists a_1' \exists a_2' \, \neg((a_1 \le a_1' \,\&\, a_2 \le a_2') \vee (a_1' \le a_1 \,\&\, a_2' \le a_2))).$$

- Moving $\neg$ inside the propositional formula, we get

$$\neg\exists a_1 \exists a_1 \exists a_1' \exists a_2' \, ((a_1 \not\le a_1' \vee a_2 \not\le a_2') \,\&\, (a_1' \not\le a_1 \vee a_2' \not\le a_2))).$$

- The formula $(a_1 \not\le a_1' \vee a_2 \not\le a_2')) \,\&\, (a_1' \not\le a_1 \vee a_2' \not\le a_2)$ must now be transformed into a DNF form.
- The result is $(a_1 \not\le a_1' \,\&\, a_1' \not\le a_1) \vee (a_1 \not\le a_1' \,\&\, a_2' \not\le a_2) \vee (a_2 \not\le a_2' \,\&\, a_1' \not\le a_1) \vee (a_2 \not\le a_2' \,\&\, a_2' \not\le a_2)$.
- Thus, our formula is $\Leftrightarrow \neg(F_1 \vee F_2 \vee F_3 \vee F_4)$, where

$$F_1 \Leftrightarrow \exists a_1 \exists a_2 \exists a_1' \exists a_2' \, (a_1 \not\le a_1' \,\&\, a_1' \not\le a_1),$$

$$F_2 \Leftrightarrow \exists a_1 \exists a_2 \exists a_1' \exists a_2' \, (a_1 \not\le a_1' \,\&\, a_2' \not\le a_2),$$

$$F_3 \Leftrightarrow \exists a_1 \exists a_2 \exists a_1' \exists a_2' \, (a_2 \not\le a_2' \,\&\, a_1' \not\le a_1),$$

$$F_4 \Leftrightarrow \exists a_1 \exists a_2 \exists a_1' \exists a_2' \, (a_2 \not\le a_2' \,\&\, a_2' \not\le a_2).$$

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24. Example (cont-d)

- So far, we got $\Leftrightarrow \neg(F_1 \vee F_2 \vee F_3 \vee F_4)$, where

$$F_1 \Leftrightarrow \exists a_1 \exists a_2 \exists a'_1 \exists a'_2\,(a_1 \not\leq a'_1 \,\&\, a'_1 \not\leq a_1),$$

$$F_2 \Leftrightarrow \exists a_1 \exists a_2 \exists a'_1 \exists a'_2\,(a_1 \not\leq a'_1 \,\&\, a'_2 \not\leq a_2),$$

$$F_3 \Leftrightarrow \exists a_1 \exists a_2 \exists a'_1 \exists a'_2\,(a_2 \not\leq a'_2 \,\&\, a'_1 \not\leq a_1),$$

$$F_4 \Leftrightarrow \exists a_1 \exists a_2 \exists a'_1 \exists a'_2\,(a_2 \not\leq a'_2 \,\&\, a'_2 \not\leq a_2).$$

- By applying the quantifiers to the corresponding parts of the formulas, we get

$$F_1 \Leftrightarrow \exists a_1 \exists a'_1\,(a_1 \not\leq a'_1 \,\&\, a'_1 \not\leq a_1),$$

$$F_2 \Leftrightarrow (\exists a_1 \exists a'_1\, a_1 \not\leq a'_1) \,\&\, (\exists a_2 \exists a'_2\, a'_2 \not\leq a_2),$$

$$F_3 \Leftrightarrow (\exists a_1 \exists a'_1\, a'_1 \not\leq a_1) \,\&\, (\exists a_2 \exists a'_2\, a_2 \not\leq a'_2),$$

$$F_4 \Leftrightarrow \exists a_2 \exists a'_1 \exists a'_2\,(a_2 \not\leq a'_2 \,\&\, a'_2 \not\leq a_2).$$

- Then, we again reduce $\neg(F_1 \vee F_2 \vee F_3 \vee F_4)$ to DNF.

25. Products of Ordered Sets: What Is Known

- At present, two product operations are known:
  - *Cartesian* product

$$(a_1, a_2) \le (a'_1, a'_2) \Leftrightarrow (a_1 \le_1 a'_1 \,\&\, a_2 \le_2 a'_2);$$

  and
  - *lexicographic* product

$$(a_1, a_2) \le (a'_1, a'_2) \Leftrightarrow$$

$$((a_1 \le_1 a'_1 \,\&\, a_1 \ne a'_1) \vee (a_1 = a'_1 \,\&\, a_2 \le_2 a'_2).$$

- *Question:* what other operations are possible?

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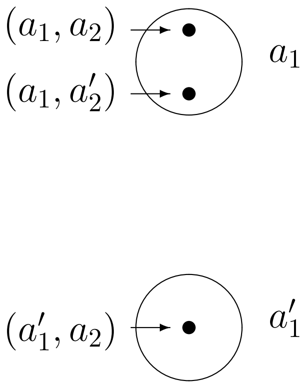
26. Possible Physical Meaning of Lexicographic Order

Idea:

- A_1 is *macroscopic* space-time,
- A_2 is *microscopic* space-time:



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27. Possible Logical Meaning of Different Orders

- *Reminder:* our certainty in S is described by a *pair* $(a_1, a_2) \in A_1 \times A_2$.
- We must therefore define a *partial order* on the set $A_1 \times A_2$ of all pairs.
- *Cartesian product:* our confidence in S is higher than in S' if and only if it is higher for both experts.
- *Meaning:* a *maximally cautious* approach.
- *Lexicographic product:* means that we have *absolute confidence* in the first expert.
- We only use the opinion of the 2nd expert when, to the 1st expert, the degrees of certainty are equivalent.

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28. Main Theorem

- By a *product operation*, we mean a Boolean function

$$P : \{T, F\}^4 \to \{T, F\}.$$

- For every two partially ordered sets A_1 and A_2, we define the following relation on $A_1 \times A_2$:

$$(a_1, a_2) \le (a_1', a_2') \stackrel{\text{def}}{=}$$

$$P(a_1 \le_1 a_1', a_1' \le_1 a_1, a_2 \le_2 a_2', a_2' \le_2 a_2).$$

- We say that a product operation is *consistent* if $\le$ is always a partial order, and

$$(a_1 \le_1 a_1' \,\&\, a_2 \le_2 a_2') \Rightarrow (a_1, a_2) \le (a_1', a_2').$$

- **Theorem:** *Every consistent product operation is the Cartesian or the lexicographic product.*

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29. Auxiliary Results: General Idea and First Example

- For each property of intervals in an ordered set A, we analyze:
  - which properties need to be satisfied for A_1 and A_2
  - so that the corresponding property is satisfies for intervals in $A_1 \times A_2$.
- *Connectedness property (CP):* for every two points $a, a' \in A$, there exists an interval that contains a and a':

$$\forall a\, \forall a'\, \exists a^-\, \exists a^+\, (a^- \le a, a' \le a^+).$$

- This property is equivalent to two properties:
  - A is *upward-directed*: $\forall a\, \forall a'\, \exists a^+\, (a, a' \le a^+)$;
  - A is *downward-directed*: $\forall a\, \forall a'\, \exists a^-\, (a^- \le a, a')$.
- *Cartesian product*: A is upward-(downward-) directed $\Leftrightarrow$ both A_1 and A_2 are upward-(downward-) directed.

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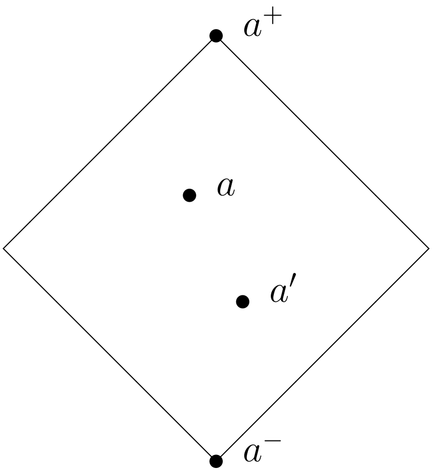
30. Connectedness Property Illustrated

Connectedness property (CP): for every two points $a, a' \in A$, there exists an interval that contains a and a':

$$\forall a\, \forall a'\, \exists a^-\, \exists a^+\, (a^- \le a, a' \le a^+).$$



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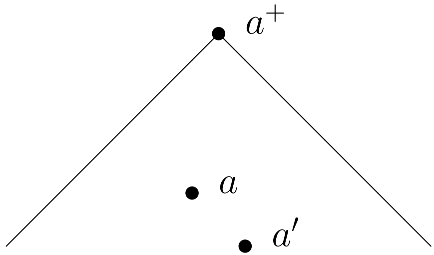
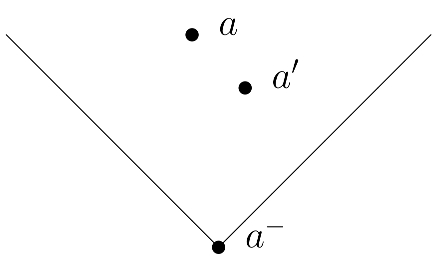
31. Upward and Downward Directed: Illustrated

Upward-directed: $\forall a\, \forall a'\, \exists a^+\, (a, a' \le a^+)$;



Downward-directed: $\forall a\, \forall a'\, \exists a^-\, (a^- \le a, a')$.



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32. First Example, Case of Cartesian Product: Proof

- *Part 1:*
  - Let us assume that $A_1 \times A_2$ is upward-directed.
  - We want to prove that A_1 is upward-directed.
  - For any $a_1, a_1' \in A_1$, take any $a_2 \in A_2$, then
  $$\exists a^+ = (a_1^+, a_2^+) \text{ such that } (a_1, a_2), (a_1', a_2) \le a^+.$$
  - Hence $a_1, a_1' \le_1 a_1^+$, i.e., A_1 is upward-directed.
- *Part 2:*
  - Assume that both A_i are upward-directed.
  - We want to prove that $A_1 \times A_2$ is upward-directed.
  - For any $a = (a_1, a_2)$ and $a' = (a_1', a_2')$, for $i = 1, 2$,
  $$\exists\, a_i^+ \, (a_i, a_i' \le_i a_i^+).$$
  - Hence $(a_1, a_2), (a_1', a_2') \le (a_1^+, a_2^+)$, i.e., $A_1 \times A_2$ is upward-directed.

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33. First Example: Case of Lexicographic Product

- $A_1 \times A_2$ is upward-directed $\Leftrightarrow$ the following two conditions hold:
  - the set A_1 is upward-directed, and
  - if A_1 has a maximal element $\overline{a}_1$ (= for which there are no a_1 with $\overline{a}_1 \prec_1 a_1$), then A_2 is upward-directed.
- $A_1 \times A_2$ is downward-directed $\Leftrightarrow$ the following two conditions hold:
  - the set A_1 is downward-directed, and
  - if A_1 has a minimal element $\underline{a}_1$ (= for which there are no a_1 for which $a_1 \prec_1 \underline{a}_1$), then A_2 is downward-directed.

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34. Case of Lexicographic Product: Proof

- Let us assume that $A_1 \times A_2$ is upward-directed.
- *Part 1:*
  - We want to prove that A_1 is upward-directed.
  - For any $a_1, a'_1 \in A_1$, take any $a_2 \in A_2$, then
    $$\exists a^+ = (a^+_1, a^+_2) \text{ for which } (a_1, a_2), (a'_1, a_2) \le a^+.$$
  - Hence $a_1, a'_1 \le_1 a^+_1$, i.e., A_1 is upward-directed.
- *Part 2:*
  - Let $\overline{a}_1$ be a maximal element of A_1.
  - For any $a_2, a'_2 \in A_2$, we have
    $$\exists a^+ = (a^+_1, a^+_2) \text{ for which } (\overline{a}_1, a_2), (\overline{a}_1, a'_2) \le a^+.$$
  - Here, $\overline{a}_1 \le_1 a^+_1$ and since $\overline{a}_1$ is maximal, $a^+_1 = \overline{a}_1$.
  - Hence $a_2, a'_2 \le_2 a^+_2$, i.e., A_2 is upward-directed.

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35. Proof (cont-d)

- Let us assume that A_1 is upward-directed.
- Let us assume that if A_1 has a maximal element, then A_2 is upward-directed.
- We want to prove that $A_1 \times A_2$ is upward-directed.
- Take any $a = (a_1, a_2)$ and $a' = (a'_1, a'_2)$ from $A_1 \times A_2$.
- Since A_1 is upward-directed, $\exists a_1^+ \, (a_1, a'_1 \leq_1 a_1^+)$.
- If $a_1 \prec_1 a_1^+$, then $(a_1, a_2), (a'_1, a'_2) \leq (a_1^+, a'_2)$.
- If $a'_1 \prec_1 a_1^+$, then $(a_1, a_2), (a'_1, a'_2) \leq (a_1^+, a_2)$.
- If $a_1 = a_1^+ = a'_1$, and a_1 is not a maximal element, then $\exists a''_1 \, (a_1 \prec_1 a''_1)$, hence $(a_1, a_2), (a'_1, a'_2) \leq (a''_1, s_2)$.
- If $a_1 = a_1^+ = a'_1$, and a_1 is a maximal element, then A_2 is upward-directed, hence $\exists a_2^+ \, (a_2, a'_2 \leq_2 a_2^+)$ and
$$(a_1, a_2), (a_1, a'_2) \leq (a_1, a_2^+).$$

36. Second Example: Intersection Property

- The intersection of every two intervals is an interval.
- *Comment:* this is true for intervals on the real line.
- This can be similarly reduced to two properties:
  - the intersection of every two future cones $Q^+_a \stackrel{\text{def}}{=} \{b : a \le b\}$ is a future cone;
  - the intersection of every two past cones $Q^-_a \stackrel{\text{def}}{=} \{b : b \le a\}$ is a past cone.
- If both properties hold, then the intersection of every two intervals $[a, b] = Q^+_a \cap Q^-_b$ is an interval.
- Ordered sets with Q^+ and Q^- properties are called *upper* and *lower* semi-lattices.
- *For Cartesian product:* $A_1 \times A_2$ is an upper (lower) semi-lattice $\Leftrightarrow$ both A_i are upper (lower) semi-lattices.

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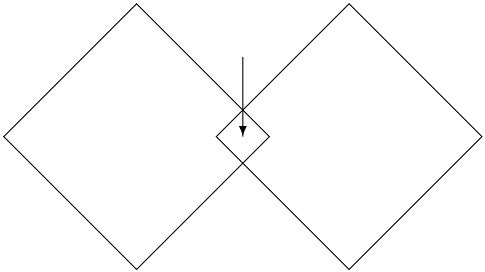
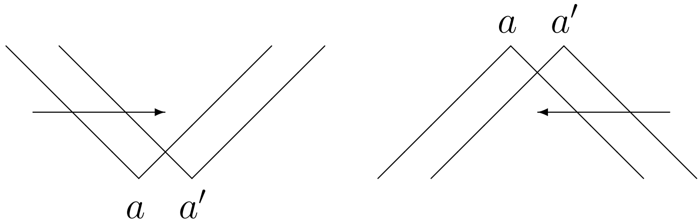
37. Intersection Property Illustrated

Intersection property for intervals:



Upper and lower semi-lattice properties:



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38. My Publications

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44. Acknowledgments

The author is greatly thankful:

- to members of my committee for their help:
  - to Dr. Martine Ceberio,
  - to Dr. Vladik Kreinovich,
  - to Dr. Luc Longpré, and
  - to Dr. Piotr Wojciechowski.
- to Dr. Eric Freudenthal for his mentorship;
- to CONACyT for their financial support;
- last but not the least, to my family for their love and support.

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