The Logic of Stone Spaces

John Harding
Department of Mathematical Sciences
New Mexico State University
Las Cruces, NM 88003, USA
email jharding@nmsu.edu

An old result of McKinsey and Tarski from the 1940's characterized the modal logic of the reals as Lewis' system S4. To explain this, each topological space (X, τ) can be viewed as a Boolean algebra given by the power set $P(X)$ of X with a unary operation on this Boolean algebra given by the closure operator C of the topological space. The resulting structure is aptly named a closure algebra. One might ask whether there are any identities true of the closure algebra for the reals that are not true for the closure algebras of all topological spaces. Here, the term identity means a universally quantified equation such as $C(A \cup B) = CA \cup CB$. McKinsey and Tarski showed that this is not the case. In fact, they showed that for any dense in itself metrizable space, there are no such equations valid that are not valid for all topological spaces. This shows that the logic of the reals is S4, and similarly the logics of the rationals, the plane, three-space, and the Cantor space are all S4.

Guram Bezhanishvili and I wondered what one could say about some of the interesting non-metrizable spaces. In particular, we decided to investigate the logic of the Stone Cech compactification of the natural numbers with the discrete topology. In alternate form, this space is the Stone space of the power set of the natural numbers. After a nasty struggle, we were able to determine the logic of the closure algebra for this space as S4.1.2. From here we have extended our aims to characterizing the logical systems for Stone spaces of other Boolean algebras. In this we have definite avenues of progress, but the general situation remains elusive. We will discuss progress on these problems.