PARTIAL ORDERS FOR REPRESENTING UNCERTAINTY, CAUSALITY AND DECISION MAKING: GENERAL PROPERTIES, OPERATIONS, AND ALGORITHMS

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This dissertation is dedicated to my parents,

for their support and active engagement in

my journey through Graduate School.

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Abstract

One of the main objectives of science and engineering is to help people select the most beneficial decisions. To make these decisions, we must know people's preferences, we must have the information about different possible consequences of different decisions. Since information is never absolutely accurate and precise, we must also have information about the degree of certainty of different parts on information. All these types of information naturally lead to partial orders:

- For preferences, $a \preccurlyeq b$ means that b is preferable to a. This relation is used in decision theory.
- For events, $a \preccurlyeq b$ means that a can influence b. This causality relation is one of the fundamental notions of physics, especially of physics of space-time.
- For uncertain statements, $a \preccurlyeq b$ means that a is less certain than b. This relation is used in logics describing uncertainty, such as fuzzy logic.

In each of these areas, there is abundant research about studying the corresponding partial orders. This research has revealed that *some ideas are common in all three applications of partial orders*. In this dissertation, we analyze general properties, operations, and algorithms related to partial orders for representing uncertainty, causality, and decision making, with a special emphasis on uncertainty.

Under uncertainty, instead of a *single* partial order, we have a *class* C of possible partial orders. In such situations, it makes sense to ask when it is *possible* that a precedes b (i.e., when a precedes b according to *some* of these orders), and when it is *necessary* that a precedes b (i.e., when a precedes b according to *all* these orders). In Chapter 2, we give a general characterization of such "possible order" and "necessary order" relations.

In Chapter 3, we consider a special case of such a situation, when different partial orders result from measurements with different accuracy. In this case, we can distinguish between

the original ("closed") partial order $\preccurlyeq$ and the "open" partial order relation $a \prec b$ meaning that $a \preccurlyeq b$ and we can verify this based on measurements (i.e., $a \preccurlyeq \widetilde{b}$ for all $\widetilde{b}$ from some neighborhood of b). It has been proven that once we know the open order, then we can uniquely reconstruct the closed order. Whether it is possible, vice versa, to reconstruct the open order from the closed one was an open problem. In Chapter 3, we prove that, under reasonable conditions, such a reconstruction is indeed possible.

In Chapter 4, we move from *potentially* detectable (measurable) orders to orders which can be detected for a given accuracy. A typical example is when we only know the lower bound $\underline{a}$ and the upper bound $\overline{a}$ for an object a; in this case, we only know that a belongs to the *interval* $[\underline{a}, \overline{a}]$. In Chapter 4, we describe all possible relations between such intervals.

Once an order is defined, we are interested in its *properties*, e.g., whether the order is a *lattice*. For special-relativity-type partial orders, a new necessary and sufficient criterion for being a lattice is described in Chapter 5.

In many practical applications, we need to *combine* different partial orders. In Chapter 6, we describe all possible *combination operations*, and in Chapter 7, we provide a *general algorithm* that reduces the analysis of properties of such combined spaces to properties of individual partially ordered spaces.

Table of Contents

Page

Acknowledgements . . . v

Abstract . . . vi

Table of Contents . . . viii

List of Figures . . . xi

Chapter

1 Introduction and Prior Work . . . 1

2 Uncertainty Is Ubiquitous in Applications of Partial Orders . . . 12

- 2.1 Formulation of the Problem . . . 12
- 2.2 Results: Possible and Necessary Orders . . . 14
- 2.3 Results: Possible and Necessary Strict Orders . . . 15
- 2.4 Results: Possible and Necessary Linear Orders . . . 16
- 2.5 Auxiliary Results: Possible and Necessary Equivalence Relations . . . 16
- 2.6 Auxiliary Results: Possibly and Necessarily Reflexive Relations . . . 17
- 2.7 Auxiliary Results: Possible and Necessary Anti-Reflexive Relations . . . 18
- 2.8 Auxiliary Results: Possible and Necessary Reflexive-and-Symmetric Relations 19
- 2.9 Graphical Representation of the Results . . . 20
- 2.10 Proofs . . . 21

3 Original Order Relation and the Uncertainty-Motivated Experimentally Confirmable Relation . . . 32

- 3.1 Formulation of the Problem . . . 33
- 3.2 Definitions and Results . . . 37
- 3.3 Auxiliary Result: Relation to de Vries Algebras . . . 42
- 3.4 Proofs . . . 48

4 From Potentially Experimentally Confirmable Relation to Actually Experimentally Confirmable One: Extending Allen's Interval Algebra to General Partially Ordered Sets 57
4.1 Formulation of the Problem 58
4.2 Results: Possible i-Relations Between Intervals 62
4.3 Results: Possible Orders Between Intervals Generated By Orders Between Endpoints 69
4.4 First Auxiliary Result: Interval Relations Reformulated In Logical Terms . 75
4.5 Second Auxiliary Result: Extending Interval Graphs to Partially Ordered Sets 76
4.6 Proofs 77
5 Properties of Ordered Spaces: When Is the Resulting Ordered Space a Lattice . 89
5.1 Lattice Property Holds for Ordered Sets Coming from Considering All Possible Subsets 90
5.2 When a Special-Relativity-Type Ordered Space Is a Lattice 100
5.3 Modal Intervals as a New Logical Interpretation of the Usual Lattice Order Between Interval Truth Values 104
5.4 Proofs 119
6 How to Combine Ordered Sets 126
6.1 Formulation of the Problem 126
6.2 Products of Partially Ordered Sets: What Is Known 127
6.3 Result: a General Description of Product Operations on Partially Ordered Sets 129
6.4 Proofs 130
7 How to Tell When a Product of Two Partially Ordered Spaces Has a Certain Property 134
7.1 Formulation of the Main Problem 134
7.2 Definitions and the Main Algorithmic Result 139
7.3 Description of the Algorithm and Proof of Its Correctness 141

7.4 Example of Applying the Algorithm 144
7.5 Auxiliary Results 145
7.6 Proofs 151
8 Conclusions and Future Work 163
References 167
Curriculum Vitae 177

List of Figures

1.1 Causality in Special Relativity – an example of partial order 3

4.1 Partial order corresponding to special relativity 64
4.2 Order $\prec$ between i-relations 64
4.3 Eight points showing that all eight pairs of i-relations $p = (p_-, p_+)$ are possible 67
4.4 Order $\preceq$ between pairs p 68
4.5 Implications between truth values $t_{\pm\pm}$ 71
4.6 Combinations corresponding to $p_- = p_+$. 80
4.7 An example of a graph G 87
4.8 A poset for which the interval graph is the original graph G 88

5.1 Future cone of the event e 91
5.2 The intersection of two future cones is also a future cone: $C^+_e \cap C^+_{e'} = C^+_{e''}$. 94
5.3 The intersection of two past cones is also a past cone: $C^-_e \cap C^-_{e'} = C^-_{e''}$. . . 94
5.4 x_α is on the geodesic arc from x to y 120
5.5 x_α is on the geodesic arc from z to y 121

6.1 Physical meaning of lexicographic order 128

7.1 Connectedness property 146
7.2 Upward-directed partially ordered set 147
7.3 Downward-directed partially ordered set 147
7.4 Intersection property 149
7.5 Intersection property: analysis 149

Chapter 1

Introduction and Prior Work

Partial orders are important. One of the main objectives of science and engineering is to help people *select the most beneficial decisions.* To make these decisions,

- we must know people's *preferences*,
- we must have the information about different events – *possible consequences of different decisions*, and
- since information is never absolutely accurate and precise, we must also have information about the *degree of certainty.*

All these types of information naturally lead to partial orders:

- For preferences, $a \preccurlyeq b$ means that b is preferable to a. This relation is used in decision theory.
- For events, $a \preccurlyeq b$ means that a can influence b. This causality relation is used in space-time physics.
- For uncertain statements, $a \preccurlyeq b$ means that a is less certain than b. This relation is used in logics describing uncertainty such as fuzzy logic.

In each of these areas, there is abundant research about studying the corresponding partial orders. This research has revealed that *some ideas are common in all three applications of partial orders.* In our research, we *analyze general properties, operations*, and *algorithms related to partial orders* for representing uncertainty, causality, and decision making. In this analysis, we will be most interested in uncertainty.

Let us describe the corresponding partial orders one by one.

The use of partial orders in decision making. An important application of partial orders is decision making, when we need to describe human preferences [17, 37, 52, 67, 77].

Causality: the use of partial orders in space-time physics. In Newton's physics, signals can potentially travel with an arbitrarily large speed. To describe the corresponding causality relation between events, let us denote an event occurring at the spatial location x at time t by $a = (t, x)$. In these notations, Newton's causality relation is trivial: an event $a = (t, x)$ can causally (physically) influence an event $a' = (t', x')$ if and only if $t \le t'$:

$$(t, x) \preccurlyeq (t', x') \Leftrightarrow t \le t'.$$

The fundamental role of the non-trivial causality relation emerged with the Special Relativity; see, e.g., [21]. In Special Relativity, the speed of all the signals is limited by the speed of light c. As a result, $a = (t, x) \preccurlyeq a' = (t', x')$ if and only if $t' \ge t$ and in time $t' - t$, the speed needed to traverse the distance $d(x, x')$ does not exceed c, i.e., $\dfrac{d(x, x')}{t' - t} \le c$. The resulting causality relation has the form

$$(t, x) \preccurlyeq (t', x') \Leftrightarrow c \cdot (t' - t) \ge d(x, x'),$$

i.e.,

$$(t, x) \preccurlyeq (t', x') \Leftrightarrow c \cdot (t' - t) \ge \sqrt{(x_1 - x'_1)^2 + (x_2 - x'_2)^2 + (x_3 - x'_3)^2}.$$

This relation can be graphically described; see Figure 1.1.

In the original formulation of the Special Relativity Theory, causality was just one of the concepts. Its central role was revealed by A. D. Alexandrov [1, 2] who showed that in Special Relativity, causality implied Lorenz group. To be more precise, he proved that every order-preserving transforming of the corresponding partial ordered set is linear, and is a composition of:

- spatial rotations,
- Lorentz transformations (describing a transition to a moving reference frame), and

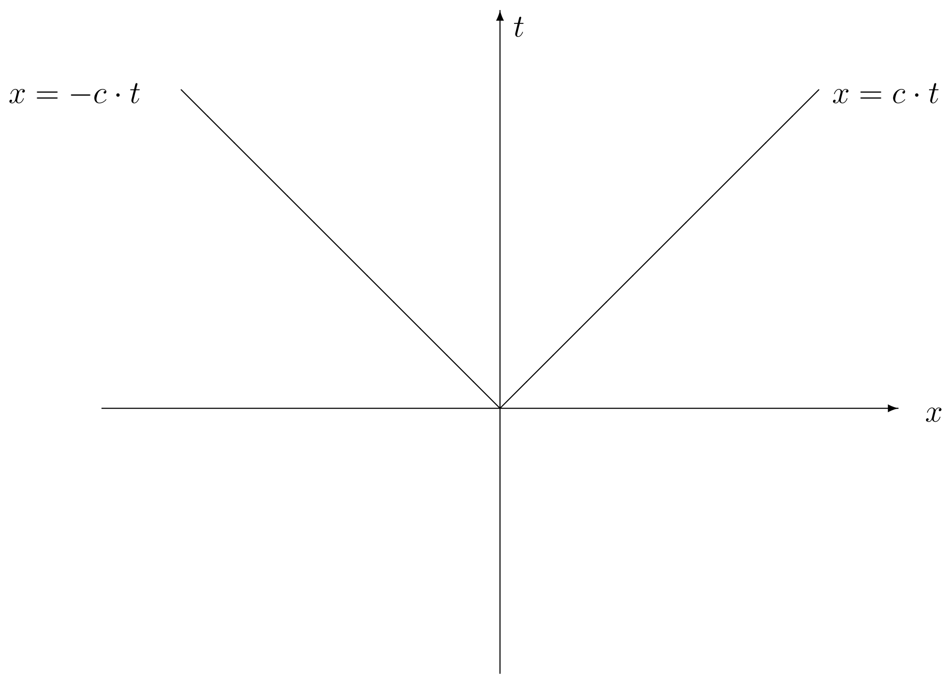

Figure 1.1: Causality in Special Relativity – an example of partial order

- re-scalings $x \to \lambda \cdot x$ (corresponding to a change of unit for measuring space and time).

This theorem was later generalized by E. Zeeman [90] and is therefore known as the *Alexandrov-Zeeman theorem*; see, e.g., [3, 4, 5, 7, 8, 26, 27, 28, 29, 30, 31, 35, 38, 41, 44, 46, 47, 48, 49, 50, 61, 69, 90].

Special relativity theory is an approximate description of space-time, a description that does not take into account that space-time is curved. To describe curved space-times, we need General Relativity Theory and its generalizations. Starting from general relativity, space-time models are usually formulated in terms of appropriate physical fields, e.g., a metric field; see, e.g., [57]. These fields assume that the space-time is smooth. However, there are important situations of non-smoothness:

- *singularities* like the Big Bang or a black hole, and

- *quantum fluctuations.*

According to modern physics (see, e.g., [57]), a proper description of the corresponding non-smooth space-time models means that we no longer have a metric field, we only have a *causality* relation $\preccurlyeq$ between events – a partial order. To describe such situations, in the 1960s, geometers H. Busemann and R. Pimenov and physicists E. Kronheimer and R. Penrose developed a theory of *kinematic spaces* [14, 43, 73]; see also [36, 39, 41, 44, 51, 79].

The use of partial orders in logic and expert reasoning. In logic, partial orders are used when we formalize common sense reasoning and expert reasoning. In this application, to each statement, we assign the expert's degree of certainty that this statement is true. A natural partial ordering relation $a \preccurlyeq b$ describes the fact that we are more certain in b than in a; see, e.g., [34, 54, 65, 66, 83, 84].

In this technique, to describe informal expert statements like "x is small", we assign, to each value x, a number $\mu(x)$ from the interval $[0,1]$ that describes the expert's confidence that this particular value x is small.

In the traditional approach to such fuzzy logic (see, e.g., [34, 66]), the degree of confidence ("truth value") of each statement is characterized by a number from the interval $[0,1]$:

- the value 1 means that the expert is absolutely confident in this statement;

- the value 0 means that the expert is absolutely confident that this statement is false; and

- values between 0 and 1 describe typical situations when the expert has some degree of confidence in the statement, but he or she is not absolutely sure that this statement is true.

These degrees of confidence are easy to compare: if our degree of confidence d in a statement S is larger than our confidence d' in a statement S', this means that we have more confidence in the statement S than in the statement S'.

Zadeh's original idea of using numerical values to describe degrees of confidence has led to many successful practical applications of fuzzy techniques [34, 66]. Many of these applications start with eliciting the corresponding degrees of confidence form the experts.

There are many different ways to elicit such degrees. For example, we can ask an expert to mark his or her confidence of a scale, e.g., on a scale from 0 to 5, 0 meaning no confidence at all, and 5 meaning absolute confidence. If an expert marks his or her confidence by 3, then we estimate the corresponding degree of confidence as $3/5 = 0.6$.

Another possibility is to poll several experts; if out of 10 experts, 7 believe that the statement is true, we take 7/10 as our degree of confidence in this statement. There are many other ways of eliciting the corresponding degrees.

At first glance, all these techniques provide a number that measures the expert's degree of confidence – the same way as the height in inches or centimeters measures the person's height. However, there is a big difference between these two types of measurements: if we measure the height of a person again and again, by using different rules, we get (more or less) the same value – describing the actual height of this person. In contrast, if we use slightly different versions of the same elicitation techniques, we get somewhat different values.

For example, if we ask a person to mark his or her confidence on a scale from 0 to 5, then possible marks are 0, 1, 2, 3, 4, and 5, and the resulting degrees of confidence are $0/5 = 0$, $1/5 = 0.2$, $2/5 = 0.4$, $3/5 = 0.6$, $4/5 = 0.8$, and $5/5 = 1.0$. To get a better estimate, we can use a more detailed scale, e.g., the scale from 0 to 6. However, with the new scale, we get numbers $0/6 = 0$, $1/6$, $2/6 = 1/3$, $3/6 = 0.5$, $4/6 = 2/3$, $5/6$, and $6/6 = 1.0$. With the exception of 0 and 1, none of the previous values can appear in this new scale. So if, e.g., a person selected 3 on a scale from 0 to 5, and we got 0.6 as the degree of confidence, on a new scale, we may get values $3/6 = 0.5$ or $4/6 = 2/3 = 0.66\ldots$, but never the exact same value 0.6.

To avoid this problem, we could ask the expert to make his or her degree of confidence on a scale, for example, from 0 to 100, but this runs into a different problem: that people are rarely able to meaningfully distinguish between, e.g., values of 70 and 71 on this scale.

Similarly, when we poll 10 experts, we can only get values 0, 0.1, 0.2, ..., 1.0. If we want to get a more accurate estimate, we can ask one more expert, but the resulting values 0, 1/11, 2/11, ..., 1 are all different from the previous values – with the exception, of course, of the values 0 and 1.

In other words, the numerical values depend not only on the actual expert's degree of confidence, they also depend on the technique that was used to elicit these degrees. For example, the same value 0.5 coming from an on-a-scale-from-0-to-something elicitation can mean different things.

- It can mean that we got 1 on a scale from 0 to 2. In this scale, we basically consider three different options: 0 if we are confident that the statement is false, 2 if we are confident that the statement is true, and 1 in cases when we are uncertain. Thus, the fact that the expert selected 1 simply means that the expert is not certain about this statement, and it does not tell us much about the degree of this uncertainty.

- On the other hand, this same value 0.5 could mean that the expert selected 5 on a scale from 0 to 10. This is a completely different story. Here, the expert had 9 values describing uncertainty to choose from: 1, 2, ..., 9, and the fact that the expert selected the midpoint 5 and not any other value means that this expert probably has as many reasons to believe in the original statement as in its negation.

When we make decisions based on the expert's degrees of confidence in different statements, it is definitely desirable to take into account the difference between the above two situations. Since in both situations, we have the exact same numerical value 0.5 of the expert's uncertainty, this means that we need to go beyond the numerical truth values.

A natural way to go beyond numerical truth values is to use *interval* truth values, when the expert's degree of confidence is described not by a number d from the interval [0,1], but rather by a subinterval $[\underline{d}, \overline{d}]$ of this interval; see, e.g., [34, 54, 55, 66].

Indeed, when a person select 3 on a scale from 0 to 5, this does not necessarily mean that his or her degree of confidence corresponds exactly to the value 3, it simply means that this degree is closer to 3 than to other marks (0, 1, 2, 4, and 5) on scale. Values which are closer to 3 than to all other integers are easy to describe: they form an interval $[2.5, 3.5]$. Based on our scale-from-0-to-5 request, we do not get the actual expert's degree of confidence, we only conclude that this actual (unknown) degree is between $2.5/5 = 0.5$ and $3.5/5 = 0.7$, i.e., that this degree is in the interval $[0.5, 0.7]$. It is therefore reasonable to return this interval as the available information about the expert's degree of confidence in a given statement.

The ultimate purpose of processing expert knowledge – and, in particular, processing degrees of belief in different statements – is to make decisions. Let us consider a simple example. Suppose that we want to achieve a certain objective. We know of two possible actions each of which can lead to this objective with some confidence, and we need to select the most promising action.

When the degree of confidence is described by a number, this problem is easy to solve: for each of the actions, we estimate the degree of confidence that this particular action will lead to the desired objective, and we select the action for which this degree is the largest possible.

However, when we use intervals to describe degrees of belief, it is not always clear which of the two actions is better. For example, suppose that for one of the actions, we have no information about its possible consequences. In this case, the interval-valued degree of belief is the whole interval $[0, 1]$. Suppose also that for the second action, we have some arguments for and against the success of this action, and we have exactly as many arguments for as we have arguments against. In this case, it is reasonable to take the midpoint 0.5 between 0 ("false") and 1 ("true") as the degree of belief in the second statement. In this situation, it is not clear which of these two alternatives we should prefer. In other words, in this case, it is also natural to consider a *partial* order.

Uncertainty is ubiquitous in applications of partial orders. While uncertainty is explicitly mentioned only in the computer-science example of partial orders, uncertainty is ubiquitous in describing our knowledge about all three types of partial orders.

How can we describe this uncertainty? We start with a general description of uncertainty with which we know a partial order. This general description is presented in Chapter 2.

Original order relation and the uncertainty-motivated experimentally confirmable relation. After the general description, we analyze how uncertainty affects specific partial orders. For example, in physics, we only observe an event with some accuracy. We may want to check what is happening exactly 1 second after a certain reaction. However, in practice, we cannot measure time exactly, so, we can only observe an event which is close to b – e.g., an event that occurs 1 ± 0.001 sec after the reaction. In general, we can only guarantee that the observed event is within a certain neighborhood U_b of the event b. In decision making, we similarly know the user's preferences only with some accuracy.

Because of this uncertainty, the only possibility to experimentally confirm that a precedes b (e.g., that a can causally influence b) is when for some neighborhood U_b of the event b, we have $a \preccurlyeq \widetilde{b}$ for all $\widetilde{b} \in U_b$. In topological terms, this "experimentally confirmable" relation $a \prec b$ means that b is contained in the future cone $C_a^+ = \{c : a \preccurlyeq c\}$ of the event a together with some neighborhood, i.e., that b belongs to the *interior* K_a^+ of the closed cone C_a^+. Such relations, in which future cones are open, are called *open.*

In usual space-time models, once we know the open cone K_a^+, we can reconstruct the original cone C_a^+ as the closure of K_a^+: $C_a^+ = \overline{K_a^+}$. A natural question is: vice versa, *can we uniquely reconstruct an open order if we know the corresponding closed order?* In Chapter 3, we show that under reasonable conditions, this reconstruction is possible; this result provides a partial solution to a known open problem.

It is worth mentioning that the corresponding pair of relations turns out to be connected to a mathematical notion of *de Vries algebras*. This connection is also described in Chapter 3.

From potentially experimentally confirmable relation to actually experimentally confirmable one. In addition to checking what is *potentially*, eventually deducible (when the accuracy increases), it is also important to check what can be confirmed at present, when we only have a finite number of observations with a given accuracy. For example, instead of knowing the exact time location of an event a, we only know an event $\underline{a}$ that precedes a and an event $\overline{a}$ that follows a. In this case, the only information that we have about the actual (unknown) event a is that it belongs to the interval $[\underline{a}, \overline{a}] \stackrel{\text{def}}{=} \{a : \underline{a} \preccurlyeq a \preccurlyeq \overline{a}\}$. It is desirable to describe possible relations between such intervals.

Such a description has already been done for intervals on the real line; the resulting description is known as Allen's algebra; in these terms, what we want is to generalize Allen's algebra to intervals over an arbitrary partially ordered set. This generalization is described in Chapter 4.

Properties of ordered spaces. Once a new ordered set is defined, we may be interested in properties of these spaces. For example, we may be interested in knowing when such an order is a lattice, i.e., when for every two elements, there is the greatest lower bound and the least upper bound. If this product is not a lattice, we may want to know, e.g., when the order is a *lower semi-lattice*, i.e., when every two elements have the least upper bound, etc. These questions are analyzed in Chapter 5.

For ordered sets coming from considering all possible subsets, we prove the corresponding lattice property. Ordered spaces coming from the special relativity-type situations are not always lattices; we describe necessary and sufficient conditions for such an order to be a lattice. Finally, for interval truth values, we provide a natural logical interpretation of the lattice order.

Need to combine several ordered sets. In physics, a system often consists of several subsystems; we need to combine the information related to each subsystem into a single description. In mathematical terms, we need to combine the corresponding ordered sets.

Similarly, in describing uncertainty, we may have different experts who provide different descriptions of what is more certain and what is less certain. It is desirable to combine the descriptions of several experts into a single description.

In all these cases, we need to combine several partial orders on different sets X_1 and X_2 into a single partial order on the set $X_1 \times X_2$ of all the pairs (x_1, x_2), where $x_i \in X_i$. This set of pairs is called a *product* of the sets X_i; in these terms, our question becomes a question of describing possible orders on the product of two ordered sets. Such operations are described in Chapter 6. It turns out that under reasonable conditions, there are only two possible product orders:

- component-wise order, when $(x_1, x_2) \preccurlyeq (y_1, y_2)$ if and only if $x_1 \preccurlyeq y_1$ and $x_2 \preccurlyeq y_2$;

- lexicographic order, when $(x_1, x_2) \preccurlyeq (y_1, y_2)$ if and only if either $x_1 \prec y_1$ or

$$(x_1 = y_1 \text{ and } x_2 \prec y_2).$$

Both orders make perfect sense when we combine expert opinions:

- the component-wise order means that we consider one statement to be more certain than the other if both experts agree on this;

- the lexicographic order means that we fully accept the opinion of the first expert, and we only consider the opinion of the second (auxiliary) expert if the first expert is undecided.

Properties of combined ordered sets. Once the product order operation is fixed, a next natural question is to analyze the properties of the resulting order. For example, we may be interested in knowing when such an order is a lattice or a semi-lattice.

In Chapter 7, we provide a general algorithm that enables us to reduce each such question about the product $X_1 \times X_2$ of two ordered sets into similar questions about the sets X_1 and X_2. We also answer specific questions about lattices and semi-lattices (and several other relevant properties).

Chapter 2

Uncertainty Is Ubiquitous in Applications of Partial Orders

In Chapter 1, we explained that ordering relations are important in different application areas. In practice, we usually do not have the complete information about the corresponding relation; as a result, we have *uncertainty*, i.e., we have several different relations consistent with our knowledge. In such situations, it is desirable to know which elements a and b are *possibly* connected by the relation and which are *necessarily* connected by this relation. In this chapter, we provide a full description of all such possible and necessary order relations. For example, possible orders are exactly reflexive relations, while necessary orders are exactly order relations.

As a side result, we also describe possible and necessary *equivalence* relations. The results from this chapter will appear in [82].

2.1 Formulation of the Problem

Relations are ubiquitous. In many practical situations, we are interested in a relation $r \subseteq U \times U$ on a given set U:

- As we have mentioned in Chapter 1, in many practical situations, we are interested in an *order* relation that describes preferences.
- In many other practical situations, we are interested in an *equivalence relation* that describes clustering of objects into groups of similar ones.

Need to consider possible and necessary relations. Often, we do not have the full information about the desired relation, so several different relations are consistent with our knowledge. In other words, the class C of all the relations which are consistent with our knowledge has at least two different elements.

If we knew the exact relation r, then, for every two elements $a, b \in U$, we could be able to check whether $a\,r\,b$, i.e., whether these elements are in relation r. For example, we could be able to check whether a is preferable to b, whether a is equivalent to b, etc. Since we do not have the full information about r, we cannot always check whether $a\,r\,b$. Instead:

- we can check whether it is *possible* that $a\,r\,b$, i.e., whether $a\,r\,b$ for some $r \in C$, and
- we can check whether it is *necessary* that $a\,r\,b$, i.e., whether $a\,r\,b$ for all $r \in C$.

In modal logic (see, e.g., [13, 23, 56]), *possible* is denoted by $\Diamond$, and necessary by $\Box$. Thus, the corresponding possible and necessary relations can be described as $\Diamond(a\,r\,b)$ and $\Box(a\,r\,b)$

Comment. Possible orders also appear in *cooperative game theory*; see, e.g., [60]. In this theory, we consider games between n players. For each possible coalition $S \subseteq \{1, 2, \ldots, n\}$, we can form an ordering relation $a \preccurlyeq_S b$ meaning that this coalition can force the outcome to go from the original a to b, increasing their incomes, the outcome b is preferable to the outcome a. Then we define a *dominance* relation: a is dominated by b if $a \preccurlyeq_S b$ for some coalition S. The *von Neumann-Morgenstern solution* is then defined as a set $\mathcal{C}$ of outcomes for which the following two properties are satisfied:

- no two outcomes from the set $\mathcal{C}$ dominate each other, and
- every outcomes which does not belong to the class $\mathcal{C}$ is dominated by an outcome from this set $\mathcal{C}$.

The meaning of the set $\mathcal{S}$ is that it represents a stable set of socially acceptable outcomes:

- for every outcome which is not in this set, at least one of the coalitions can force this outcome into a socially acceptable one;

- once we agreed to a socially acceptable outcome, no coalition can force us to move to another socially acceptable outcome.

Formulation of the problem. For a given class of relations – e.g., orders, equivalent relations, etc. – how can we describe the corresponding possible and ordinary relations?

What was known. Possible and necessary relations are described, e.g., [62]; see also references therein.

A similar idea of possible and necessary relations was considered in [78]. In that paper, however, a different situation was considered, when the relation r is fixed, but the elements a and b are known with uncertainty.

What we do in this chapter. This chapter provides a description of possible and necessary orders and equivalent relations.

2.2 Results: Possible and Necessary Orders

Reminder. A relation r is called *reflexive* if $a\,r\,a$ for all a, *antisymmetric* if $a\,r\,b$ and $b\,r\,a$ imply $a = b$, *transitive* if $a\,r\,b$ and $b\,r\,c$ imply $a\,r\,c$, and *order* (or *partial order*) if it is reflexive, antisymmetric, and transitive.

Definition 2.1. *Let U be a set. We say that a relation $R \subseteq U \times U$ is a* possible order *if there exists a non-empty class C of ordering relations on U for which $a\,R\,b$ if and only if $a\,r\,b$ for some $r \in C$:*

$$a\,R\,b \Leftrightarrow \exists r \in C\,(a\,r\,b).$$

Definition 2.2. *Let U be a set. We say that a relation $R \subseteq U \times U$ is a* necessary order *if there exists a non-empty class C of ordering relations on U for which $a\,R\,b$ if and only if $a\,r\,b$ for all $r \in C$:*

$$a\,R\,b \Leftrightarrow \forall r \in C\,(a\,r\,b).$$

Proposition 2.1. *A relation R is a possible order if and only if it is reflexive.*

Proposition 2.2. *A relation R is a necessary order if and only if it is an order.*

Comment. For reader's convenience, in this dissertation, all the proofs are placed in a special section at the end of the corresponding chapter.

2.3 Results: Possible and Necessary Strict Orders

Case of strict orders. Sometimes, it makes sense to consider *strict orders*, i.e., transitive relations which are *strictly antisymmetric*, i.e., for which $a \prec b$ implies that $b \not\prec a$. All such relations are *anti-reflexive*: $a \not\prec a$. For strict order relations, similar proofs lead to the following similar results:

Definition 2.3. *Let U be a set. We say that a relation $R \subseteq U \times U$ is a* possible strict order *if there exists a non-empty class C of strict orders on U for which $a\,R\,b$ if and only if $a\,r\,b$ for some $r \in C$:*

$$a\,R\,b \Leftrightarrow \exists r \in C\,(a\,r\,b).$$

Definition 2.4. *Let U be a set. We say that a relation $R \subseteq U \times U$ is a* necessary strict order *if there exists a non-empty class C of strict orders on U for which $a\,R\,b$ if and only if $a\,r\,b$ for all $r \in C$:*

$$a\,R\,b \Leftrightarrow \forall r \in C\,(a\,r\,b).$$

Proposition 2.3. *A relation R is a possible strict order if and only if it is anti-reflexive.*

Proposition 2.4. *A relation R is a necessary strict order if and only if it is a strict order.*

2.4 Results: Possible and Necessary Linear Orders

Idea. If instead of general (partial) orders, we can consider *linear* (*total*) orders, for which for every a and b, we have $a \preccurlyeq b$ or $b \preccurlyeq a$.

Definition 2.5. *Let U be a set. We say that a relation $R \subseteq U \times U$ is a* possible linear order *if there exists a non-empty class C of linear orders on U for which $a\,R\,b$ if and only if $a\,r\,b$ for some $r \in C$:*

$$a\,R\,b \Leftrightarrow \exists r \in C\,(a\,r\,b).$$

Definition 2.6. *Let U be a set. We say that a relation $R \subseteq U \times U$ is a* necessary linear order *if there exists a non-empty class C of orders on U for which $a\,R\,b$ if and only if $a\,r\,b$ for all $r \in C$:*

$$a\,R\,b \Leftrightarrow \forall r \in C\,(a\,r\,b).$$

Proposition 2.5. *A relation R is a necessary linear order if and only if it is an order.*

Comment. It is not clear how to easily describe *possible* linear orders.

2.5 Auxiliary Results: Possible and Necessary Equivalence Relations

Reminder. A relation r is called *symmetric* if $a\,r\,b$ implies $b\,r\,a$, and an *equivalence relation* if it is reflexive, symmetric, and transitive.

Definition 2.7. *Let U be a set. We say that a relation $R \subseteq U \times U$ is a* possible equivalence relation *if there exists a non-empty class C of equivalence relations on U for which $a\,R\,b$ if and only if $a\,r\,b$ for some $r \in C$:*

$$a\,R\,b \Leftrightarrow \exists r \in C\,(a\,r\,b).$$

Definition 2.8. *Let U be a set. We say that a relation $R \subseteq U \times U$ is a* necessary equivalence relation *if there exists a non-empty class C of equivalence relations on U for which $a\,R\,b$ if and only if $a\,r\,b$ for all $r \in C$:*

$$a\,R\,b \Leftrightarrow \forall r \in C\,(a\,r\,b).$$

Proposition 2.6. *A relation R is a possible equivalence relation if and only if it is reflexive and symmetric.*

Proposition 2.7. *A relation R is a necessary equivalence relation if and only if it is an equivalence relation.*

2.6 Auxiliary Results: Possibly and Necessarily Reflexive Relations

Formulation of the question. We have shown that possible orders are exactly reflexive relations, and necessary order relations are orders. It is natural to ask: what are possibly and necessarily reflexive relations?

Definition 2.9. *Let U be a set. We say that a relation $R \subseteq U \times U$ is a* possibly reflexive relation *if there exists a non-empty class C of reflexive relations on U for which $a\,R\,b$ if and only if $a\,r\,b$ for some $r \in C$:*

$$a\,R\,b \Leftrightarrow \exists r \in C\,(a\,r\,b).$$

Definition 2.10. *Let U be a set. We say that a relation $R \subseteq U \times U$ is a* necessarily reflexive relation *if there exists a non-empty class C of reflexive relations on U for which*

$a\,R\,b$ if and only if $a\,r\,b$ for all $r \in C$:

$$a\,R\,b \Leftrightarrow \forall r \in C\,(a\,r\,b).$$

Proposition 2.8. *A relation R is a possibly reflexive relation if and only if it is reflexive.*

Proposition 2.9. *A relation R is a necessarily reflexive relation if and only if it is a reflexive relation.*

2.7 Auxiliary Results: Possible and Necessary Anti-Reflexive Relations

Formulation of the question. We have shown that possible strict orders are exactly anti-reflexive relations, and necessary strict orders are strict orders. It is natural to ask: what are possibly and necessarily anti-reflexive relations?

Definition 2.11. *Let U be a set. We say that a relation $R \subseteq U \times U$ is a* possibly anti-reflexive relation *if there exists a non-empty class C of anti-reflexive relations on U for which $a\,R\,b$ if and only if $a\,r\,b$ for some $r \in C$:*

$$a\,R\,b \Leftrightarrow \exists r \in C\,(a\,r\,b).$$

Definition 2.12. *Let U be a set. We say that a relation $R \subseteq U \times U$ is a* necessarily anti-reflexive relation *if there exists a non-empty class C of anti-reflexive relations on U for which $a\,R\,b$ if and only if $a\,r\,b$ for all $r \in C$:*

$$a\,R\,b \Leftrightarrow \forall r \in C\,(a\,r\,b).$$

Proposition 2.10. *A relation R is possibly anti-reflexive if and only if it is anti-reflexive.*

Proposition 2.11. *A relation R is necessarily anti-reflexive if and only if it is anti-reflexive.*

2.8 Auxiliary Results: Possible and Necessary Reflexive-and-Symmetric Relations

Formulation of the question. We have shown that possible equivalence relations are exactly reflexive and symmetric relations, and necessary equivalent relations are equivalent relations. It is natural to ask: what are possibly and necessarily reflexive-and-symmetric relations?

Definition 2.13. *Let U be a set. We say that a relation $R \subseteq U \times U$ is a* possibly reflexive-and-symmetric relation *if there exists a non-empty class C of reflexive-and-symmetric relations on U for which $a\,R\,b$ if and only if $a\,r\,b$ for some $r \in C$:*

$$a\,R\,b \Leftrightarrow \exists r \in C\,(a\,r\,b).$$

Definition 2.14. *Let U be a set. We say that a relation $R \subseteq U \times U$ is a* necessarily reflexive-and-symmetric relation *if there exists a non-empty class C of reflexive-and-symmetric relations on U for which $a\,R\,b$ if and only if $a\,r\,b$ for all $r \in C$:*

$$a\,R\,b \Leftrightarrow \forall r \in C\,(a\,r\,b).$$

Proposition 2.12. *A relation R is a possibly reflexive-and-symmetric relation if and only if it is reflexive and symmetric.*

Proposition 2.13. *A relation R is a necessarily reflexive-and-symmetric relation if and only if it is a reflexive-and-symmetric relation.*

2.9 Graphical Representation of the Results

Graphical description. By using the symbols $\Diamond$ and $\Box$, we can describe our results in the following graphical form:

$$\begin{array}{ccccc} \preccurlyeq\overset{\Box}{\Leftrightarrow} & ; & \prec\overset{\Box}{\Leftrightarrow} & ; & \equiv\overset{\Box}{\Leftrightarrow} \\ \Downarrow\Diamond & ; & \Downarrow\Diamond & ; & \Downarrow\Diamond \\ \text{refl.}\overset{\Diamond,\Box}{\Leftrightarrow} & ; & \text{anti-refl.}\overset{\Diamond,\Box}{\Leftrightarrow} & ; & \text{refl.-sym.}\overset{\Diamond,\Box}{\Leftrightarrow} \end{array}$$

Left diagram. The left diagram means that if we start with a (partial) order $\preccurlyeq$, then:

- the necessary modality $\Box$ leads again to an order – we denoted this by $\overset{\Box}{\Leftrightarrow}$, while
- the possible modality $\Diamond$ leads to reflexive relations – which we denoted by refl.

For reflexive relations, both modalities $\Box$ and $\Diamond$ lead again to reflexive relations – we denoted this by $\overset{\Diamond,\Box}{\Leftrightarrow}$.

Middle diagram. The middle diagram means that if we start with a strict order $\prec$, then:

- the necessary modality $\Box$ leads again to a strict order – we denoted this by $\overset{\Box}{\Leftrightarrow}$, while
- the possible modality $\Diamond$ leads to anti-reflexive relations – which we denoted by anti-refl.

For anti-reflexive relations, both modalities $\Box$ and $\Diamond$ lead again to anti-reflexive relations – we denoted this by $\overset{\Diamond,\Box}{\Leftrightarrow}$.

Right diagram. The right diagram means that if we start with an equivalence relation $\equiv$, then:

- the necessary modality $\Box$ leads again to an equivalence relation – we denoted this by $\overset{\Box}{\Leftrightarrow}$, while

- the possible modality $\Diamond$ leads to reflexive and symmetric relation – which we denoted by refl.-sym.

For reflexive and symmetric relations, both modalities $\Box$ and $\Diamond$ lead again to reflexive and symmetric relations – we denoted this by $\overset{\Diamond,\Box}{\Leftrightarrow}$.

2.10 Proofs

Proof of Proposition 2.1.

1°. Let us first prove that every possible order is reflexive.

Indeed, let R be a possible order corresponding to a class C of orders. By definition of an order relation, we have $a\,r\,a$ for all $r \in C$. Thus, $a\,r\,a$ for some $r \in C$ and therefore, $a\,R\,a$.

2°. Vice versa, let us assume that R is a reflexive relation. Let us prove that R is a possible order.

Indeed, for each pair $(x, y) \in R$, we can consider the ordering relation $\preccurlyeq_{u,v}$ that consists of this pair and all pairs (u, u) $(u \in U)$. In this relation, $a \preccurlyeq b$ if and only if either $a = x$ and $b = y$, or $a = b$. One can easily see that this relation $\preccurlyeq_{u,v}$ is indeed an order. So, if we take the class of all such relation as C, then $a\,r\,b$ for some $r \in C$ if and only if $a \preccurlyeq_{u,v} b$ for some $(u, v) \in R$, i.e., if and only if either $a = u$ and $b = v$ for some pair $(u, v) \in R$ (i.e., equivalently, if $(u, v) \in R$) or $a = b$ – in which case also $(a, b) = (a, a) \in R$. Thus, the corresponding possible order is indeed the original relation R.

The proposition is proven.

Proof of Proposition 2.2.

1°. One can easily prove that each order $\preccurlyeq$ is a necessary order: it is sufficient to consider the class $C = \{\preccurlyeq\}$ that consists of only this order relation.

2°. Vice versa, let us assume that a relation R is a necessary order, i.e., $a\,R\,b$ if and only if $a\,r\,b$ for all orders r from some class C. Let us prove that this relation R is an order, i.e., that it is reflexive, antisymmetric, and transitive.

2.1°. Let us first prove that the relation R is reflexive.

Indeed, for every a, and for every $r \in C$, we have $a\,r\,a$, so we conclude that $a\,R\,a$. Thus, R is indeed reflexive.

2.2°. Let us now prove that the relation R is antisymmetric.

Indeed, if $a\,R\,b$ and $b\,R\,a$, this means that $a\,r\,b$ and $b\,r\,a$ for all $r \in C$. Since each relation r is antisymmetric, this implies $a = b$. We have thus proved that $a\,R\,b$ and $b\,R\,a$ imply $a = b$, i.e., that the necessary order relation is also antisymmetric.

2.3°. Finally, let us prove that the relation R is transitive.

Indeed, if $a\,R\,b$ and $b\,R\,c$, this means that for every order $r \in R$, we have $a\,r\,b$ and $b\,r\,c$. Since each r is an order and hence, transitive, we conclude that $a\,r\,c$ for all c. By definition of a necessary order, this means that $a\,R\,c$. Thus, the relation R is indeed transitive.

The proposition is proven.

Proof of Proposition 2.3.

1°. Let us first prove that every possible strict order is anti-reflexive.

Indeed, let R be a possible order corresponding to a class C of strict orders. By definition of a strict order relation, we have $\neg(a\,r\,a)$ for all $r \in C$. Thus, $\neg(a\,R\,a)$.

2°. Vice versa, let us assume that R is an anti-reflexive relation. Let us prove that R is a possible strict order.

Indeed, for each pair $(x, y) \in R$, we can consider the strict ordering relation $\prec_{u,v}$ that consists only of this pair. In this relation, $a \prec b$ if and only if $a = x$ and $b = y$. One can easily see that this relation $\prec_{u,v}$ is indeed an order. So, if we take the class of all such relation as C, then $a\,r\,b$ for some $r \in C$ if and only if $a \prec_{u,v} b$ for some $(u, v) \in R$, i.e., if and only if $a = u$ and $b = y$ for some pair $(u, v) \in R$ (i.e., equivalently, if $(u, v) \in R$). Thus, the corresponding possible strict order is indeed the original relation R.

The proposition is proven.

Proof of Proposition 2.4.

1°. One can easily prove that each strict order $\prec$ is a necessary strict order: it is sufficient to consider the class $C = \{\prec\}$ that consists of only this strict order relation.

2°. Vice versa, let us assume that a relation R is a necessary strict order, i.e., $a\,R\,b$ if and only if $a\,r\,b$ for all strict orders r from some class C. Let us prove that this relation R is asymmetric, and transitive.

2.1°. Let us first prove that the relation R is strictly antisymmetric.

Indeed, let us assume that $a\,R\,b$. This means that $a\,r\,b$ for all $r \in C$. Since all relations $r \in C$ are strict orders, we thus conclude that $\neg(b\,r\,a)$ for all $r \in C$. Thus, $\neg(b\,R\,a)$. Thus, the relation R is indeed strictly antisymmetric.

2.2°. Let us now prove that the relation R is transitive.

Indeed, if $a\,R\,b$ and $b\,R\,c$, this means that for every order $r \in R$, we have $a\,r\,b$ and $b\,r\,c$. Since each r is an order and hence, transitive, we conclude that $a\,r\,c$ for all c. By definition of a necessary strict order, this means that $a\,R\,c$. Thus, the relation R is indeed transitive.

The proposition is proven.

Proof of Proposition 2.5.

1°. Let us prove that each order $\preccurlyeq$ is a necessary linear order, i.e., that for each (partial) order, $\preccurlyeq$, there exists a family C of linear orders for which, for every a and b, $a \preccurlyeq b \Leftrightarrow \forall r \in C\,(a\,r\,b)$.

This proof is based on the known result that every linear order can be extended to a linear order. This result, in its turn, is based on a following auxiliary result:

1.1°. Let U be an ordered set with an order $\preccurlyeq$ which is not a linear order. The fact that $\preccurlyeq$ is not a linear order means that there exists elements a and b for which $a \not\preccurlyeq b$ and $b \not\preccurlyeq a$. Let us pick two such elements a and b. Then, we can extend the original order $\preccurlyeq$ to a new order $\preccurlyeq^*$ in which $b \preccurlyeq^* a$.

Indeed, we can define the new relation $\preccurlyeq^*$ as follows:

$$p \preccurlyeq^* q \Leftrightarrow (p \preccurlyeq q \vee (p \preccurlyeq b \,\&\, a \preccurlyeq q)).$$

This relation clearly extends the original order $\preccurlyeq$. To prove the above statement, we thus need to prove that this new relation is an order, i.e., that it reflexive, antisymmetric, and transitive.

1.1.1°. Let us first prove that the relation $\preccurlyeq^*$ is reflexive.

Indeed, since $\le$ is an order, we have $p \preccurlyeq p$ for every p and thus, $p \preccurlyeq^* p$ for every p.

1.1.2°. Let us now prove that the relation $\preccurlyeq^*$ is antisymmetric, i.e., that $p \preccurlyeq^* q$ and $q \preccurlyeq^* p$ imply that $p = q$.

The condition $p \preccurlyeq^* q$ means that either $p \preccurlyeq q$ or $(p \preccurlyeq b \,\&\, a \preccurlyeq q)$. Similarly, the condition $q \preccurlyeq^* p$ means that either $q \preccurlyeq p$ or $(q \preccurlyeq b \,\&\, a \preccurlyeq p)$. To complete our proof, let us consider all $2 \cdot 2 = 4$ combinations of these conditions.

If $p \preccurlyeq q$ and $q \preccurlyeq p$, then $p = q$ since $\preccurlyeq$ is an order and is, thus, antisymmetric.

If $p \preccurlyeq b$, $a \preccurlyeq q$, and $q \preccurlyeq p$, then, by transitivity, we get $a \preccurlyeq b$, which contradicts to our original assumption that $a \not\preccurlyeq b$ and $b \not\preccurlyeq a$.

Similarly, if $p \preccurlyeq q$, $q \preccurlyeq b$, and $a \preccurlyeq p$, then we also get $a \preccurlyeq b$ and thus, a contradiction.

Finally, if $p \preccurlyeq b$, $a \preccurlyeq q$, $q \preccurlyeq b$, $a \preccurlyeq p$, then we get $a \preccurlyeq q \le b$ and $a \preccurlyeq b$, which is also impossible.

Antisymmetry is proven.

1.1.3°. Finally, let us prove that the relation $\preccurlyeq^*$ is transitive, i.e., that $p \preccurlyeq^* q$ and $q \preccurlyeq^* r$ imply that $p \preccurlyeq^* r$.

The condition $p \preccurlyeq^* q$ means that either $p \preccurlyeq q$ or $(p \preccurlyeq b \,\&\, a \preccurlyeq q)$. Similarly, the condition $q \preccurlyeq^* r$ means that either $q \preccurlyeq r$ or $(q \preccurlyeq b \,\&\, a \preccurlyeq r)$. To complete our proof, let us consider all $2 \cdot 2 = 4$ combinations of these conditions.

If $p \preccurlyeq q$ and $q \preccurlyeq r$, then $p \preccurlyeq r$ since $\preccurlyeq$ is an order and is, thus, transitive.

If $p \preccurlyeq q$, $q \preccurlyeq b$, and $a \preccurlyeq r$, then, by transitivity, we get $p \preccurlyeq b$ and $a \preccurlyeq r$, i.e., $p \preccurlyeq^* r$.

Similarly, if $p \preccurlyeq b$, $a \preccurlyeq q$, and $q \preccurlyeq r$, then by transitivity, we get $p \preccurlyeq b$ and $a \preccurlyeq r$, i.e., also $p \preccurlyeq^* r$.

Finally, if $p \preccurlyeq b$, $a \preccurlyeq q$, $q \preccurlyeq b$, $a \preccurlyeq r$, then by transitivity, from $a \preccurlyeq q \preccurlyeq b$, we conclude that $a \preccurlyeq b$, which contradicts to our assumption that $a \not\preccurlyeq b$ and $b \not\preccurlyeq a$.

Transitivity is proven, so $\preccurlyeq^*$ is indeed an order.

1.2°. For a finite set U, we can consistently add pairs and thus, eventually get a linear order that extends our original order $\preccurlyeq$. For an infinite set, we can do the same by using transfinite induction or, equivalently, Zorn's Lemma.

1.3°. We can now prove that a given partial order $\preccurlyeq$ is a necessary linear order, i.e., that there exists a family C of linear orders for which $a \preccurlyeq b \Leftrightarrow \forall r \in C\,(a\,r\,b)$.

As this family C, we take all linear orders that extend the original order $\preccurlyeq$. Let us prove that this family has the desired property, by considering three possible cases: the case when $a \preccurlyeq b$, the case when $b \preccurlyeq a$ and $b \neq a$, and the case when $a \not\preccurlyeq b$ and $b \not\preccurlyeq a$. We will prove that in all these three case:

- when the condition $a \preccurlyeq b$ is true, then the condition $\forall r \in C\,(a\,r\,b)$ is also true, and
- when the condition $a \preccurlyeq b$ is false, then the condition $\forall r \in C\,(a\,r\,b)$ is also false.

1.3.1°. If $a \preccurlyeq b$, then, since all orders $r \in C$ extend $\preccurlyeq$, we have $a\,r\,b$ for all $r \in C$. So, the condition $\forall r \in C\,(a\,r\,b)$ is also true.

1.3.2°. If $b \preccurlyeq a$ and $b \neq a$, then we have $a \not\preccurlyeq b$, so the condition $a \preccurlyeq b$ is false. Since all orders $r \in C$ extend $\preccurlyeq$, we have $b\,r\,a$ for all $r \in C$. Since $b \neq a$ and each r is an order, this implies that $\neg(a\,r\,b)$ for all $r \in C$. Thus, the condition $\forall r \in C\,(a\,r\,b)$ is also false.

1.3.3°. Finally, let us consider the case when $a \not\preccurlyeq b$ and $b \not\preccurlyeq a$. Here, the condition $a \preccurlyeq b$ is false. In this case, as we have proven earlier, we can:

- extend the original order $\preccurlyeq$ to a new order $\preccurlyeq^*$ in which $b \preccurlyeq^* a$, and then
- extend this new order $\preccurlyeq^*$ to a linear order r.

In this linear order $r \in C$, we have $b\,r\,a$ hence $\neg(a\,r\,b)$, so the condition $\forall r \in C\,(a\,r\,b)$ is also false.

The statement is proven.

2°. Vice versa, let us assume that a relation R is a necessary order linear order, i.e., $a\,R\,b$ if and only if $a\,r\,b$ for all linear orders r from some class C. Let us prove that this relation R is an order, i.e., that is is reflexive, antisymmetric, and transitive.

2.1°. Let us first prove that the relation R is reflexive.

Indeed, for every a, and for every $r \in C$, we have $a\,r\,a$, so we conclude that $a\,R\,a$. Thus, R is indeed reflexive.

2.2°. Let us now prove that the relation R is antisymmetric.

Indeed, if $a\,R\,b$ and $b\,R\,a$, this means that $a\,r\,b$ and $b\,r\,a$ for all $r \in C$. Since each relation r is antisymmetric, this implies $a = b$. We have thus proved that $a\,R\,b$ and $b\,R\,a$ imply $a = b$, i.e., that the necessary order relation is also antisymmetric.

2.3°. Finally, let us prove that the relation R is transitive.

Indeed, if $a\,R\,b$ and $b\,R\,c$, this means that for every order $r \in R$, we have $a\,r\,b$ and $b\,r\,c$. Since each r is an order and hence, transitive, we conclude that $a\,r\,c$ for all c. By definition of a necessary order, this means that $a\,R\,c$. Thus, the relation R is indeed transitive.

The proposition is proven.

Proof of Proposition 2.6.

1°. Let us first assume that R is a possible equivalence relation corresponding to a class C of equivalent relations. Let us prove that this relation R is reflexive and symmetric.

1.1°. Let us first prove that the relation R is reflexive.

Indeed, by definition of an equivalence relation, we have $a\,r\,a$ for all $r \in C$. Thus, $a\,r\,a$ for some $r \in C$ and therefore, $a\,R\,a$, i.e., R is indeed reflexive.

1.2°. Let us now prove that the relation R is symmetric.

Indeed, if aRb, this means that $a\,r\,b$ for some $r \in C$. This relation r is an equivalence relation, so we have $b\,r\,a$ for this $r \in C$. Thus, we conclude that $b\,R\,a$, i.e., the relation R is indeed symmetric.

2°. Vice versa, let us assume that R is a reflexive and symmetric relation. Let us prove that R is a possible equivalence relation.

Indeed, for each pair $(x, y) \in R$, we can consider the equivalence relation $\equiv_{u,v}$ that consists of this pair (u, v), the "dual" pair (v, u), and all pairs (u, u) $(u \in U)$. In this relation, $a \equiv_{u,v} b$ if and only if either the pair (a, b) coincides with (u, v) or (v, u), or $a = b$. One can easily see that this relation $\equiv_{u,v}$ is indeed an equivalence relation: in this relation, elements u and v are grouped in one equivalence class, while all other equivalence classes consists of a single element.

So, if we take the class of all such relation as C, then $a\,r\,b$ for some $r \in C$ if and only if $a \equiv_{u,v} b$ for some $(u, v) \in R$, i.e., if and only if either $(a, b) = (u, v)$ or $(a, b) = (v, u)$ for some pair $(u, v) \in R$ (since R is symmetric, this is equivalent to $(u, v) \in R$) or $a = b$ – in which case also $(a, b) = (a, a) \in R$. Thus, the corresponding possible equivalence relation is indeed the original relation R.

The proposition is proven.

Proof of Proposition 2.7.

1°. One can easily prove that each equivalence relation $\equiv$ is a necessary equivalence relation: it is sufficient to consider the class $C = \{\equiv\}$ that consists of only this equivalence relation.

2°. Vice versa, let us assume that R is a necessary equivalence relation, i.e., $a\,R\,b$ if and only if $a\,r\,b$ for all equivalence relations r from some class C. Let us prove that R is an equivalence relation, i.e., that it is reflexive, symmetric, and transitive.

2.1°. Let us first prove that the relation R is reflexive.

Indeed, for every a, and for every $r \in C$, we have ara, so we conclude that $a\,R\,a$. Thus, R is reflexive.

2.2°. Let us now prove that the relation R is symmetric.

Indeed, if $a\,R\,b$, this means that $a\,r\,b$ for all $r \in C$. Since each relation r is symmetric, this implies $b\,r\,a$. Since this is true for all $r \in R$, we thus have $b\,R\,a$. We have thus proved that $a\,R\,b$ implies $b\,R\,a$, i.e., that the necessary equivalence relation is also symmetric.

2.3°. Finally, let us prove that the relation R is transitive.

Indeed, if $a\,R\,b$ and $b\,R\,c$, this means that for every equivalence relation $r \in R$, we have $a\,r\,b$ and $b\,r\,c$. Since each r is an equivalence relation and hence, transitive, we conclude that $a\,r\,c$ for all c. By definition of a necessary equivalence relation, this means that $a\,R\,c$. Thus, the relation R is indeed transitive.

The proposition is proven.

Proof of Proposition 2.8.

1°. Let us first prove that every possibly reflexive relation is reflexive.

Indeed, let R be a possible reflexive relation corresponding to a class C of reflexive relations. For every element a, by definition of a reflexive relation, we have ara for all $r \in C$. Thus, $a\,r\,a$ holds at least for *some* relations $r \in C$ and therefore, $a\,R\,a$. So, the relation R is indeed reflexive.

2°. Vice versa, it is easy to show that every reflexive relation R is possibly reflexive: indeed, we can take a set $S = \{R\}$ consisting of only this relation.

The proposition is proven.

Proof of Proposition 2.9.

1°. One can easily prove that each reflexive relation r is a necessarily reflexive relation: it is sufficient to consider the class $C = \{r\}$ that consists of only this relation.

2°. Vice versa, let us assume that R is a necessarily reflexive relation, i.e., $a\,R\,b$ if and only if $a\,r\,b$ for all reflexive relations r from some class C. Let us prove that this relation R is reflexive.

Indeed, for every a, and for every $r \in C$, we have $a\,r\,a$, so we conclude that $a\,R\,a$. Thus, R is reflexive.

The proposition is proven.

Proof of Proposition 2.10.

1°. Let R be a possibly anti-reflexive relation corresponding to a class C of anti-reflexive relations. Let us prove that this relation R is anti-reflexive.

Indeed, since all the relations $r \in C$ are anti-reflexive, we have $\neg(a\,r\,a)$ for all $r \in C$. Thus, we cannot have $a\,r\,a$ for any $r \in C$ and therefore, we have $\neg(a\,R\,a)$.

2°. Vice versa, it is easy to show that every anti-reflexive relation R is possibly anti-reflexive: indeed, we can take a set $S = \{R\}$ consisting of only this relation.

The proposition is proven.

Proof of Proposition 2.11.

1°. One can easily prove that each anti-reflexive relation r is necessarily anti-reflexive: it is sufficient to consider the class $C = \{r\}$ that consists of only this relation.

2°. Vice versa, let us assume that R is a necessarily anti-reflexive relation, i.e., $a\,R\,b$ if and only if $a\,r\,b$ for all anti-reflexive relations r from some class C. Let us prove that this relation R is anti-reflexive.

Indeed, for every a, and for every $r \in C$, we have $\neg(a\,r\,a)$, so we conclude that $\neg(a\,R\,a)$. Thus, R is indeed anti-reflexive.

The proposition is proven.

Proof of Proposition 2.12.

1°. Let R be a possibly reflexive-and-symmetric relation corresponding to a class C of reflexive-and-symmetric relations. Let us prove that this relation R is reflexive and symmetric.

1.1°. Let us prove that the relation R is reflexive.

Indeed, since all the relations $r \in C$ are reflexive, we have $a\,r\,a$ for all $r \in C$. Thus, $a\,r\,a$ at least for some $r \in C$ and therefore, $a\,R\,a$.

1.2°. Let us now prove that the relation R is symmetric, Indeed, suppose that $a\,R\,b$. By definition, this means that $a\,r\,b$ for some $r \in C$. Since every relation $r \in C$ is symmetric, we conclude that $b\,r\,a$, which implies that $b\,R\,a$. Thus, the relation R is indeed symmetric.

2°. Vice versa, it is easy to show that every reflexive-and-symmetric relation R is possibly reflexive-and-symmetric: indeed, we can take a set $S = \{R\}$ consisting of only this relation.

The proposition is proven.

Proof of Proposition 2.13.

1°. One can easily prove that each reflexive-and-symmetric relation r is necessarily reflexive-and-symmetric: it is sufficient to consider the class $C = \{r\}$ that consists of only this relation.

2°. Vice versa, let us assume that R is a necessarily reflexive-and-symmetric relation, i.e., $a\,R\,b$ if and only if $a\,r\,b$ for all reflexive-and-symmetric relations r from some class C. Let us prove that this relation R is reflexive and symmetric.

2.1°. Let us first prove that the relation R is reflexive.

Indeed, for every a, and for every $r \in C$, we have $a\,r\,a$, so we conclude that $a\,R\,a$. Thus, R is reflexive.

2.2°. Let us now prove that the relation R is symmetric.

Indeed, if $a\,R\,b$, this means that $a\,r\,b$ for all $r \in C$. Since each relation r is symmetric, this implies $b\,r\,a$. So, we have $b\,r\,a$ for all $r \in C$, and thus, we have $b\,R\,a$. We have thus proved that $a\,R\,b$ implies $b\,R\,a$, i.e., that the necessary reflexive-and-symmetric relation is indeed symmetric.

The proposition is proven.

Chapter 3

Original Order Relation and the Uncertainty-Motivated Experimentally Confirmable Relation

After the general description, we analyze how uncertainty affects specific partial orders. For example, in physics, we only observe an event with some accuracy. We may want to check what is happening exactly 1 second after a certain reaction. However, in practice, we cannot measure time exactly, so, we can only observe an event which is close to b – e.g., an event that occurs 1 ± 0.001 sec after the reaction. In general, we can only guarantee that the observed event is within a certain neighborhood U_b of the event b. In decision making, we similarly know the user's preferences only with some accuracy.

Because of this uncertainty, the only possibility to experimentally confirm that a precedes b (e.g., that a can causally influence b) is when for some neighborhood U_b of the event b, we have $a \preccurlyeq \widetilde{b}$ for all $\widetilde{b} \in U_b$. In topological terms, this "experimentally confirmable" relation $a \prec b$ means that b is contained in the future cone $C_a^+ = \{c : a \preccurlyeq c\}$ of the event a together with some neighborhood, i.e., that b belongs to the *interior* K_a^+ of the closed cone C_a^+. Such relations, in which future cones are open, are called *open*.

In usual space-time models, once we know the open cone K_a^+, we can reconstruct the original cone C_a^+ as the closure of K_a^+: $C_a^+ = \overline{K_a^+}$. A natural question is: vice versa, *can we uniquely reconstruct an open order if we know the corresponding closed order?* In this chapter, we show that under reasonable conditions, this reconstruction is possible; this result provides a partial solution to a known open problem.

It is worth mentioning that the corresponding pair of relations turns out to be connected to a mathematical notion of *de Vries algebras*. This connection is also described in this chapter.

The results of this chapter first appeared in [82] and [87].

3.1 Formulation of the Problem

Order-preserving mappings of topological spaces in logic and in physics: general reminder. As we have mentioned in Chapter 1, ordered spaces and order-preserving mappings naturally appear in space-time physics and in other areas of logic.

In physics, a natural ordering relation is the *causality* relation between events, when $a \preccurlyeq b$ means that an event a can influence the event b.

In physical models of space-time, it is also assumed that there is a topology that corresponds to closeness of events. Intuitively, convergence $b_n \to b$ in this topology means that for any fixed measurement accuracy, when n is sufficiently large, we cannot distinguish events b_n and b by measurements of this accuracy.

It is usually assumed that the relation $\preccurlyeq$ is "closed" in the sense that for every element a, its *future cone* $C_a^+ \stackrel{\text{def}}{=} \{b : a \preccurlyeq b\}$ is closed. In other words, if a sequence b_n converges to the element b in the sense of the underlying topology, and $a \preccurlyeq b_n$ for all n, then $a \preccurlyeq b$.

This closeness makes intuitive sense. Indeed, if $b_n \to b$, this means, as we have mentioned, that for every given measurement accuracy, when n is sufficiently large, we cannot distinguish events b_n and b. So,

- if $a \preccurlyeq b_n$, i.e., if physical evidence shows a can influence all events b_n, and
- $b_n \to b$, meaning that b is indistinguishable from b_n,

then this same evidence shows that a can influence b ($a \preccurlyeq b$).

In logic, partial orders are used when we formalize commonsense and expert reasoning. In this application, to each statement, we assign the expert's degree of certainty that this statement is true. A natural partial ordering relation $a \preccurlyeq b$ describes the fact that we are more certain in b than in a.

In this application, topology represents the closeness of the corresponding degrees of certainty. In this case, it is also reasonable to require that the relation $\preccurlyeq$ is closed.

As we have mentioned in Chapter 1, another important application of partial orders is decision making, when we need to describe human preferences.

Need for open partial orders. In many applications, we only observe an event b with some accuracy. For example, in physics, we may want to check what is happening exactly 1 second after a certain reaction. However, in practice, we cannot measure time exactly, so, we can only observe an event which is close to b – e.g., an event that occurs 1 ± 0.001 sec after the reaction. In general, we can only guarantee that the observed event is within a certain neighborhood U_b of the event b.

Because of this uncertainty, the only possibility to experimentally confirm that a can influence b is when for some neighborhood U_b of the event b, we have $a \preccurlyeq \widetilde{b}$ for all $\widetilde{b} \in U_b$. In topological terms, this "experimentally confirmable" relation $a \prec b$ means that b is contained in the future cone $C_a^+ = \{c : a \preccurlyeq c\}$ of the event a together with some neighborhood, i.e., that b belongs to the *interior* K_a^+ of the closed cone C_a^+. Such relations, in which future cones are open, will be called *open*.

In usual space-time models, once we know the open cone K_a^+, we can reconstruct the original cone C_a^+ as the closure of K_a^+: $C_a^+ = \overline{K_a^+}$.

Comment. To avoid confusion, please note that here $a \prec b$ *does not* mean $a \preccurlyeq b$ and $a \neq b$.

Similar arguments justify the need to consider open cones also in case of uncertainty.

In physics, there is another motivation for open cones: open cones correspond to influences with speeds smaller than the speed of light. This is important because, according to modern physics, there are two types of objects (see, e.g., [21]):

- objects with non-zero rest mass that can travel with any possible speed which is smaller than the speed of light – but not with the speed of light, and

- objects with zero rest mass (like photons), that can travel only with the speed of light, but not with any smaller speed.

Thus, open cones correspond to causality by traditional (kinematic) objects. Because of this, the open relation $a \prec b$ is also known as *kinematic* causality, and spaces with this open relation $\prec$ are known as *kinematic spaces* [73].

Comment. For example, in special relativity, $a \prec b$ means that $b - a$ is a future-oriented timelike vector.

Natural questions: what is the relation between open and closed partial orders? In all the above applications, on the same space, we have three things:

- topology;

- the original (closed) partial order $\preccurlyeq$; and

- the (open) partial order $\prec$.

It is reasonable to ask to what extent knowing only *some* of these things enables us to reconstruct the others.

Relation between open and closed partial orders: what was known. It has been known that under some physically (and logically) reasonable assumptions, the open relation uniquely determines both the topology and the closed relation.

The corresponding topology was first introduced by A. D. Alexandrov and is thus known as *Alexandrov topology*. It is a topology whose base are *open intervals*

$$(a, b) \stackrel{\text{def}}{=} \{c : a \prec c \prec b\}.$$

For this definition to be valid, we need to make sure that intervals do form a base of a topology, i.e., when a point x belongs to the intersection of two open intervals, there a whole open interval containing x is contained in this intersection.

Once this topology is defined, we can define $a \preccurlyeq b$ as b belonging to the closure $\overline{K_a^+}$ of the open cone $K_a^+ = \{c : a \prec c\}$. Of course, we need to make sure that a dual definition $a \in \overline{K_b^-}$, where $K_b^- = \{c : c \prec b\}$ leads to the exact same ordering.

It is also usually assumed that for every element a, there are elements larger than a and smaller than a, and that if $a \prec b$, then there is a point in between a and b.

Under these conditions, the above description determines the topology and the closed order in terms of the open order $\prec$. Thus, the open order uniquely determines both the topology and the closed order.

In the case of special relativity, the inverse is also true: if we know the closed partial order, then we can uniquely reconstruct the open order as well – and so, the topology. Hence, every 1-1 transformation preserving a closed order also preserves the open order and the topology. This conclusion is used in many proofs that every order-preserving transformation is linear. The proof of this conclusion is based on the easy-to-check observation that when $a \preccurlyeq b$, we have $a \prec b$ if and only if the relation $\preccurlyeq$ restricted to the closed interval $[a, b] = \{c : a \preccurlyeq c \preccurlyeq b\}$ is *not* a total (linear) order, i.e., if and only if there exist c and c' for which $a \preccurlyeq c \preccurlyeq b$, $a \preccurlyeq c' \preccurlyeq b$, $c \not\preccurlyeq c'$, and $c' \not\preccurlyeq c$.

It is known, however, that this observation does not hold in general. For example, on the 3-D space $\mathbb{R}^3$ with a standard topology, we can define a component-wise partial order as follows: $a = (a_1, a_2, a_3) \preccurlyeq b = (b_1, b_2, b_3)$ if and only if $a_1 \le b_1$, $a_2 \le b_2$, and $a_3 \le b_3$. In this space, the corresponding open order is also easy to describe: $a = (a_1, a_2, a_3) \prec b = (b_1, b_2, b_3)$ if and only if $a_1 < b_1$, $a_2 < b_2$, and $a_3 \le b_3$. Here, however, we can have $a = (0, 0, 0) \preccurlyeq b = (0, 1, 1)$, $a \not\prec b$, but for $c = (0, 0, 1)$ and $c' = (0, 1, 0)$, we have $a \preccurlyeq c \preccurlyeq b$, $a \preccurlyeq c' \preccurlyeq b$, $c \not\preccurlyeq c'$, and $c' \not\preccurlyeq c$.

Remaining problem – that we solve in this chapter. A natural question is: what happens in the general case? Can we uniquely reconstruct an open order if we know the corresponding closed order? In this chapter, we show that under reasonable assumptions, such a reconstruction is indeed possible.

3.2 Definitions and Results

Definition 3.1. [73] *A set X with a partial order $\prec$ is called a* kinematic space *if is satisfies the following conditions:*

$$\forall a\,\exists a_-, a_+\,(a_- \prec a \prec a_+);$$

$$\forall a, b\,(a \prec b \rightarrow \exists c\,(a \prec c \prec b));$$

$$\forall a, b, c\,(a \prec b, c \rightarrow \exists d\,(a \prec d \prec b, c));$$

$$\forall a, b, c\,(b, c \prec a \rightarrow \exists d\,(b, c \prec d \prec a)).$$

Definition 3.2. *For every partial ordered set, and every $a \prec b$, by an* interval *(or* open interval*), we mean the set $(a, b) \stackrel{\text{def}}{=} \{c : a \prec c \prec b\}$.*

Definition 3.3. *A kinematic space is called* separable *if there exists a countable set $\{x_n\}$ such that every open interval contains one of the elements x_i.*

Definition 3.4. [73] *For every separable kinematic space, we define* convergence $s_n \rightarrow a$ *as follows:*

$$s_n \rightarrow a \Leftrightarrow \forall a_-, a_+\,(a_- \prec a \prec a_+ \Rightarrow \exists N\,\forall n\,(n \geq N \Rightarrow a_- \prec s_n \prec a_+))).$$

For each set S, its closure $\overline{S}$ *is defined as the set of all the points a for which $s_n \rightarrow a$ for some $\{s_n\} \subseteq S$.*

Comment. In other words, $s_n \to a$ if and only if every interval (a_-, a_+) containing a also contains almost all elements of the sequence s_n – i.e., in other words, that it contains all but finitely many of these elements.

Comment. From the physical viewpoint, the fact that we consider separable spaces is not really restrictive, because all space-time models considered in mainstream physics are separable:

- The original Minkowski space is separable.
- Modern physics describes space-time as a manifold equipped with pseudo-Riemannian metric. All such manifolds are separable. To describe singularities, physicists consider manifolds with borders; these spaces are also separable; see, e.g., a classical textbook [57].

Mathematical comment. In this chapter, we consider separable kinematic spaces, i.e., spaces in which there is a countable set $\{x_n\}$ which is everywhere dense in X. In such spaces, to describe topology, it is sufficient to consider convergence of sequences. Our result, however, can be easily extended to general (not necessarily separable) kinematic spaces if, instead of sequences $\{s_n\}$, we consider *nets* $\{s_\alpha\}_{\alpha \in A}$ corresponding to directed sets A; see, e.g., [33].

Definition 3.5. [73] *A kinematic space is called* normal *if*

$$b \in \overline{\{c : a \prec a\}} \Leftrightarrow a \in \overline{\{c : c \prec b\}}.$$

Notation. For a normal kinematic space, we denote $b \in \overline{\{c : a \prec c\}}$ by $a \preccurlyeq b$. For every $a \preccurlyeq b$, the set $[a, b] \stackrel{\text{def}}{=} \{c : a \preccurlyeq c \preccurlyeq b\}$ is called a *closed interval.*

The following transitivity and closure properties hold for this relation:

Proposition 3.1. [73] *For every separable normal kinematic space and for every elements a, b, and c, the following holds:*

- $a \preccurlyeq a$;
- *if* $a \prec b$, *then* $a \preccurlyeq b$;
- *if* $a \preccurlyeq b$ *and* $b \prec c$, *then* $a \prec c$;
- *if* $a \prec b$ *and* $b \preccurlyeq c$, *then* $a \prec c$.

The proof of the first part of Proposition 3.1 is based on the following lemma:

Definition 3.6. *We say that a sequence* $\{s_n\}$ *is* $\prec$-decreasing *if* $s_{n+1} \prec s_n$ *for all* n.

Lemma 3.1. *For every separable kinematic space, if* $a \prec b$, *then there exists a* $\prec$-*decreasing sequence* $\{s_n\}$ *of elements* $a \prec s_n$ *for which* $s_1 = b$ *and* $s_n \to a$.

Comment. For readers' convenience, all the proofs are placed in the special (final) Proofs section.

A dual lemma also holds:

Definition 3.7. *We say that a sequence* $\{s_n\}$ *is* $\prec$-increasing *if* $s_n \prec s_{n+1}$ *for all* n.

Lemma 3.2. *For every separable kinematic space, for every element* a, *there exists a* $\prec$-*increasing sequence* s_n *of element* $s_n \prec a$ *for which* $s_n \to a$.

Proposition 3.2. [73] *For every separable normal kinematic space:*

- *if* $b \preccurlyeq s_n$ *for all* n *and* $s_n \to a$, *then* $b \preccurlyeq a$;
- *if* $s_n \preccurlyeq b$ *for all* n *and* $s_n \to a$, *then* $a \preccurlyeq b$;
- *if* $a \preccurlyeq b$ *and* $b \preccurlyeq c$, *then* $a \preccurlyeq c$.

Comment. In the causality relations corresponding to Einstein's space-time models, the causality relation is an order relation. However, from the mathematical viewpoint, it is possible to define kinematic spaces in which the limit relation $\preccurlyeq$ is not an order, i.e., in which, for some elements a and b, we have $a \preccurlyeq b$ and $b \preccurlyeq a$, but $a \neq b$.

For example, in Newtonian space-time, $e = (t, x) \preccurlyeq e' = (t', x') \Leftrightarrow t \leq t'$. In this order, $(0, (0, 0, 0)) \preccurlyeq (0, (1, 0, 0))$ and $(0, (1, 0, 0)) \preccurlyeq (0, (0, 0, 0))$, but $(0, (0, 0, 0)) \neq (0, (1, 0, 0))$.

In such non-order cases, we have a non-trivial equivalence relation

$$a \equiv b \Leftrightarrow (a \preccurlyeq b \,\&\, b \preccurlyeq a).$$

For each element a, its equivalence class $\{b : b \equiv a\}$ is equal to $[a, a]$.

To formulate our result, we need to introduce an additional completeness property.

Definition 3.8. *We say that a sequence $\{s_n\}$ is* $\preccurlyeq$-decreasing *if $s_{n+1} \preccurlyeq s_n$ for all n.*

Definition 3.9. *We say that a sequence $\{s_n\}$ is* bounded from below *if there exists an element b for which $b \preccurlyeq s_n$ for all n.*

Definition 3.10. *We say that a separable kinematic space is* complete *if every $\preccurlyeq$-decreasing bounded sequence has a limit.*

Physical comment. From the application viewpoint, this requirement does not change much. Indeed, the events are only approximately known anyway, so explicitly adding a limit event $a = \lim s_n$ does not affect the physical picture: for all practical purposes, the limit a is indistinguishable from s_n for large n.

The following result shows that completeness holds in most physically interesting cases, for example, in the space-time corresponding to special relativity and in many models corresponding to General Relativity.

Definition 3.11. *A kinematic space is called* intervally compact *if in this space, every closed interval is compact.*

Proposition 3.3. *Every intervally-compact separable normal kinematic space is complete.*

Definition 3.12. *For every element $e \in X$, let S_e denote the set of all $\preccurlyeq$-monotonically decreasing sequences $s = \{s_n\}$ for which $e \preccurlyeq s_n$ for all n and $\bigcap\limits_{n=1}^{\infty} [e, s_n] = [e, e]$. On this set of sequences, we can define a new pre-ordering $s \geq s' \Leftrightarrow \forall n\, \exists m\, (s'_m \preccurlyeq s_n)$.*

Definition 3.13. *We say that a sequence $s \in S_e$ is S_e-*maximum *if $s' \preccurlyeq s$ for every $s' \in S_e$.*

Comment. For every e, we may have several different S_e-maximum sequences.

The following result states that a complete (open) kinematic order can indeed be uniquely reconstructed from the corresponding closed order.

Theorem 3.1. *For every complete separable normal kinematic space, $b \prec a$ if and only if there exist an element e for which $b \preccurlyeq e$ and a sequence $\{s_n\}$ which is S_e-maximum and for which $s_1 = a$.*

Comment. This theorem describes $\prec$ in terms of $\preccurlyeq$. Thus, the relation $\prec$ is preserved by an arbitrary $\preccurlyeq$-isomorphism between kinematic spaces. In other words, we arrive at the following corollary.

Corollary. *If two complete separable normal kinematic orders $\prec$ and $\prec'$ on the same set X lead to the same closed order $\preccurlyeq = \preccurlyeq'$, then $\prec = \prec'$.*

Physical meaning of this result. One of the main objectives of physics is to reduce observed phenomena to fundamental theories and explanations. Because of this activity, it is important to try our best to describe seemingly less fundamental properties in terms of more fundamental ones. For example, wherever an empirical law is discovered, like the Ohm law that describes the relation between current and voltage, physicists try to reduce this empirical law to more fundamental laws of nature.

Another example of such a reduction was given earlier: Lorentz transformations – describing coordinates transformations during a transition to a moving reference frame – can be explained as transformations that preserve causality. This description reduces such less fundamental properties as coordinates and coordinate transformations to causality – one of the most fundamental notions of physics; see, e.g., [21].

The above result can be viewed as another example of such a reduction. Indeed, as we have mentioned, from the physical viewpoint,

- the relation $\preccurlyeq$ is the causality relation, while
- the relation $a \prec b$ means that a can influence b by using particle with positive rest mass – i.e., particles that travel with speeds smaller than the speed of light.

In physics:

- causality is one of the most fundamental properties, while
- the properties like the rest mass are less fundamental.

Thus, the fact that we can express $\prec$ in terms of $\preccurlyeq$ means that we are reducing the less fundamental relation – of being influences by particles of non-zero rest mass – to causality, one of the most fundamental notions of physics.

3.3 Auxiliary Result: Relation to de Vries Algebras

In the previous sections, we analyzed the situations in which we have *two* ordering relations: a closed one (corresponding to causality) and an open one (corresponding to experimentally confirmable causality). It turns out that the corresponding pair of relations turns out to be connected to a mathematical notion of *de Vries algebras*. This connection is described in the present section.

Order relations are needed in describing space-time. As we have mentioned in Chapter 1, traditionally, in physics, space-times are described by (pseudo-)Riemann spaces, i.e., by smooth manifolds with a tensor metric field. However, in several physically interesting situations smoothness is violated and metric is undefined:

- near the singularity (Big Bang),
- at the black holes, and
- on the microlevel, when we take into account quantum effects.

In all these situations, what remains is causality $\preccurlyeq$ – an ordering relation. To describe such situations, in the 1960s, geometers H. Busemann and R. Pimenov and physicists E. Kronheimer and R. Penrose developed a theory of *kinematic spaces* [14, 43, 73].

When is causality experimentally confirmable: reminder. Since causality is important, it is desirable to analyze how it can be experimentally detected whether an event a can influence event b.

As we have mentioned in Section 3.1, in many applications, we only observe an event b with some accuracy. For example, in physics, we may want to check what is happening exactly 1 second after a certain reaction. However, in practice, we cannot measure time exactly, so, we observe an event occurring 1 ± 0.001 sec after a.

In general, we can only guarantee that the observed event is within a certain neighborhood U_b of the event b. Because of this uncertainty, the only possibility to experimentally confirm that a can influence b is when a can influence *all* the events from a neighborhood. i.e., when

$$\exists U_b \, \forall \widetilde{b} \in U_b \, \left(a \preccurlyeq \widetilde{b}\right).$$

Let us denote this "experimentally conformable" causality relation by $a \prec b$. In topological terms, $a \prec b$ means that b is in the interior K_a^+ of the *future cone* $C_a^+ \stackrel{\text{def}}{=} \{c : a \preccurlyeq c\}$, i.e., of the set of all the events that can be influenced by the event a.

Kinematic orders. In special relativity, the relation $a \prec b$ corresponds to influences with speeds smaller than the speed of light. This relation has a natural physical interpretation if we take into account that in special relativity, there are two types of objects:

- objects with non-zero rest mass can travel with any possible speed $v < c$ but not with the speed c; and
- objects with zero rest mass (e.g., photons) can travel only with the speed c, but not with $v < c$.

In these terms, the relation $\prec$ corresponds to causality by traditional (kinematic) objects. Because of this fact, the relation $\prec$ is called *kinematic causality*, and spaces with this relation $\prec$ are called *kinematic spaces*.

Kinematic spaces: towards a description. To describe space-time, we thus need a (pre-)ordering relation $\preccurlyeq$ (causality) and topology (= closeness). There are some natural relations between them.

For example, a natural continuity idea implies that in every neighborhood of an event a, there are events causally following a and causally preceding a. In other words, for every event a and for every neighborhood U_a, there exist a^- and a^+ for which $a^- \prec a$ and $a \prec a^+$.

It is reasonable to assume that if the events $a' \prec a''$ are close to the event a, then every event in between a' and a'' should also be close to a. In precise terms, every neighborhood U_a should contains an entire open interval

$$(a', a'') \stackrel{\text{def}}{=} \{b : a' \prec b \prec a''\}.$$

Another reasonable requirement comes from the fact a motion with speed c is a limit of motions with speeds $v < c$ when $v \to c$. It is therefore reasonable to require that the future cone $C_a^+ = \{b : a \preccurlyeq b\}$ in terms of the original causality relation $\preccurlyeq$ is a closure of the future cone $K_a^+ = \{b : a \prec b\}$ in terms of the kinematic causality: $C_a^+ = \overline{K_a^+}$. A similar property holds for the past cones

$$C_a^- \stackrel{\text{def}}{=} \{b : b \preccurlyeq a\} \text{ and } K_a^- \stackrel{\text{def}}{=} \{b : b \prec a\} :$$

$C_a^- = \overline{K_a^-}$. In other words,

$$a \preccurlyeq b \Leftrightarrow \forall U_b \, \exists \widetilde{b} \left(\widetilde{b} \in U_b \,\&\, a \prec \widetilde{b}\right).$$

In particular, for the neighborhood $U_b = (b', b'')$, for which $b \prec b''$, we get $a \prec \widetilde{b} \prec b''$ and hence $a \prec b''$. Thus,

$$a \preccurlyeq b \Leftrightarrow \forall c \, (b \prec c \Rightarrow a \prec c).$$

These requirements lead to the following definition of a kinematic space.

Definition 3.14.

- *A set X with a partial order $\prec$ is called a* kinematic space *if is satisfies the following conditions:*

$$\forall a \, \exists a_-, a_+ \, (a_- \prec a \prec a_+);$$

$$\forall a, b \, (a \prec b \rightarrow \exists c \, (a \prec c \prec b));$$

$$\forall a, b, c \, (a \prec b, c \rightarrow \exists d \, (a \prec d \prec b, c));$$

$$\forall a, b, c \, (b, c \prec a \rightarrow \exists d \, (b, c \prec d \prec a)).$$

- *On a kinematic space, we take a topology generated by* intervals

$$(a, b) \stackrel{\text{def}}{=} \{c : a \prec c \prec b\}.$$

- *A kinematic space is called* normal *if*

$$b \in \overline{\{c : a \prec c\}} \Leftrightarrow a \in \overline{\{c : c \prec b\}}.$$

- *For a normal kinematic space, we denote $b \in \overline{\{c : a \prec c\}}$ by $a \preccurlyeq b$.*

Remark. It has been proven that $a \prec b \preccurlyeq c$ or $a \preccurlyeq b \prec c$ imply $a \prec c$ [73].

Symmetry: a fundamental property of the physical world. One of the main objectives of science is prediction. The main basis for prediction is that we have observed similar situations in the past, and thus we expect similar outcomes as in those past situations.

In mathematical terms, similarity corresponds to symmetry, and similarity of outcomes – to invariance. For example, suppose that I dropped the ball, then it fall down. I can then shift my position, and drop the ball again. It is reasonable to expect that the ball will fall, and that its trajectory will be the same as before – to be more precise, obtained by shift from the previous one. Similarly, if I rotate myself 90 degrees and drop the ball again, I will get the same trajectory but rotated by 90 degrees.

The notion of symmetry is very important in modern physics, to the extent that, starting with the quarks, physical theories are usually formulated in terms of symmetries – and not in terms of differential equations as in the past; see, e.g., [21].

In particular, an important symmetry is *T-transformation*, a symmetry with respect to reversal of time $t \to -t$. One important property of this transformation is that if we apply it twice, we get the same point back. Another property is that T-transformation reverses the order of causality. Thus, we arrive at the following definition.

Definition 3.15. *A 1-1 mapping $t : X \to X$ of a kinematic space onto itself is called a* T-transformation *if $t(t(a)) = a$ for all a and $a \preccurlyeq b \Leftrightarrow t(b) \preccurlyeq t(a)$.*

de Vries algebras. In mathematics, there is another case when we have a set with two orders: the case of so-called *de Vries algebras*. The original example of such an object is the class $\mathcal{R}X$ of all regular open subsets of a compact Hausdorff space X, i.e., open subsets A for which the interior $\text{Int}\left(\overline{A}\right)$ of the closure $\overline{A}$ coincides with the original set A.

On $\mathcal{R}X$, we can define $A \preccurlyeq B \Leftrightarrow A \subseteq B$ and $A \prec B \Leftrightarrow \overline{A} \subseteq B$. One can check that the class $\mathcal{R}X$ with the relation $\preccurlyeq$ is a complete Boolean algebra, with negation $\neg A \stackrel{\text{def}}{=} \text{Int}(X - A)$. In general, a de Vries algebra is defined as a Boolean algebra with an additional relation $\prec$ that satisfies the same properties as the algebra $\mathcal{R}X$. This idea leads to the following definition [10, 11, 12]:

Definition 3.16. *A* de Vries algebra *is a pair consisting of a complete Boolean algebra* $(B, \preccurlyeq)$ *with the relation* $\preccurlyeq$ *(precedes) and a binary relation* $\prec$ *(strictly precedes) for which:*

- $1 \prec 1$;
- $a \prec b$ *implies* $a \preccurlyeq b$;
- $a \preccurlyeq b \prec c \preccurlyeq d$ *implies* $a \prec d$;
- $a \prec b, c$ *implies* $a \prec b \wedge c$;
- $a \prec b$ *implies* $\neg b \prec \neg a$;
- $a \prec b$ *implies that there exists a* c *such that* $a \prec c \prec b$;
- $a \neq 0$ *implies that there exists a* $b \neq 0$ *such that* $b \prec a$.

Possible relations with kinematic spaces: discussion. In a kinematic space, if we associate with every element a an open cone K_a^+, then we get $a \preccurlyeq b \Leftrightarrow K_b^+ \subseteq K_a^+$ and $a \prec b \Leftrightarrow \overline{K_b^+} \subseteq K_a^+$ [73].

Please note that we need to be cautious about this observation, since standard examples of de Vries algebras come from a compact space X, while a kinematic space is never compact [73]. However, this observation can indeed be transformed into a formal relation between kinematic spaces and de Vries algebras. To describe this formal relation, we need to introduce the following auxiliary definitions.

Definition 3.17. *We say that a de Vries algebra is* connected *if* $a \prec a$ *implies that* $a = 0$ *or* $a = 1$.

Remark. The name comes from the fact that, as it is easy to check, for an algebra $\mathcal{R}X$, this is indeed equivalent to connectedness of the topological space X, i.e., to the fact that the space X cannot be represented as a union of two disjoint open sets (which are different from X and $\emptyset$).

Theorem 3.2. *For every connected de Vries algebra B:*

- *the set $B - \{0, 1\}$ with a proximity relation $\prec$ is a normal kinematic space, and*
- *the original relation $\preccurlyeq$ coincides with the closure of $\prec$ in the sense of kinematic spaces.*

3.4 Proofs

Proof of Lemma 3.1. Since the kinematic space is separable, there exists a sequence x_n that has elements in every open interval. We will construct a sequence s_n with the following additional property: for every $n \geq 2$, if $a \prec x_n$, then $s_n \prec x_n$.

Let us take $s_1 = b$; then, $a \prec b$ implies that $a \prec s_1$.

Let us now assume that the values $a \prec s_{n-1} \prec \ldots \prec s_1$ have already been constructed. The construction of the next element s_n will depend on whether $a \prec x_n$ or not. If $a \prec x_n$, then we have $a \prec x_n, s_{n-1}$. So, by definition of a kinematic space, there exists an element c for which $a \prec c \prec x_n, s_{n-1}$. We will take one of these elements c as s_n.

If $a \not\prec x_n$, then we have $a \prec s_{n-1}$. So, by definition of a kinematic space, there exists an element c for which $a \prec c \prec s_{n-1}$. We will take one of these elements c as s_n.

We have constructed a $\prec$-decreasing sequence. Let us prove that this sequence converges to a, i.e., that for every $a_- \prec a \prec a_+$, there exists an N such that for all $n \geq N$, we have $a_- \prec s_n \prec a_+$. Indeed, since $a \prec a_+$, the sequence x_n has an element x_N in an open interval (a, a_+): $a \prec x_N \prec a_+$. By our construction, $a \prec s_N$ implies that $a \prec s_N \prec x_N$. By transitivity, we conclude that $a_- \prec s_N \prec a_+$. Since the sequence s_n is $\prec$-decreasing, we conclude that for $n > N$, we have $a \prec s_n \prec s_N$, so, by transitivity, $a_- \prec s_n \prec a_+$. Convergence is proven.

Proof of Lemma 3.2. This proof is similar to the proof of Lemma 3.1.

Proof of Proposition 3.1. Let us first prove that $a \preccurlyeq a$. Indeed, by Lemma 3.1, there exists a sequence s_n for which $a \prec s_n$ and $s_n \to a$. Thus, $a \preccurlyeq a$.

Let us prove that if $a \prec b$, then $a \preccurlyeq b$. Indeed, we can take $s_n = b$. Each open interval neighborhood of b contains b and thus, contains all elements of the sequence s_n. Thus, $s_n \to b$ and hence, $a \preccurlyeq b$.

Let us now prove that if $a \preccurlyeq b$ and $b \prec c$, then $a \prec c$. Indeed, by definition, $a \preccurlyeq b$ means that there is a sequence $s_n \to b$ for which $a \prec s_n$ for all n. By definition of convergence, $s_n \to b$ means that for every two elements $b_- \prec b \prec b_+$, there exists N for which, for all $n \geq N$, $b_- \prec s_n \prec b_+$. By definition of a kinematic space, there is an element $b_- \prec b$. As b_+, we take $b_+ = c$. In this case, for sufficiently large n, we have $s_n \prec c$, so $a \prec s_n$ and transitivity imply that $a \prec c$.

Finally, let us prove that if $a \prec b$ and $b \preccurlyeq c$, then $a \prec c$. Indeed, since the kinematic space is normal, $b \preccurlyeq c$ means that there exists a sequence $s_n \to b$ for which $s_n \prec c$ for all n. By definition of convergence, $s_n \to b$ means that for every two elements $b_- \prec b \prec b_+$, there exists N for which, for all $n \geq N$, $b_- \prec s_n \prec b_+$. By definition of a kinematic space, there is an element $b \prec b_+$. As b_-, we take $b_- = a$. In this case, for sufficiently large n, we have $a \prec s_n$, so $s_n \prec c$ and transitivity imply that $a \prec c$.

Finally, let us prove that the relation $\preccurlyeq$ is transitive. Let $b \preccurlyeq a$ and $c \preccurlyeq b$. By definition, $b \preccurlyeq a$ means that there exists a sequence $b \prec s_n$ for which $s_n \to a$. As we have shown, from $b \prec s_n$ and $c \preccurlyeq b$, we conclude that $c \prec s_n$. Thus, $c \prec s_n$ for some sequence $s_n \to a$. This is exactly what is means to have $c \preccurlyeq a$. The statement is proven.

The proposition is proven.

Proof of Proposition 3.2. Without losing generality, let us prove the first statement, i.e., let us assume that $s_n \to a$ and $b \preccurlyeq s_n$, and let us prove that $b \preccurlyeq a$. For that, we will need to prove that there exists a sequence $b \prec s'_n$ for which $s'_n \to a$. As such a sequence, we will take a $\prec$-decreasing sequence s'_n for which $a \prec s'_n$ and $s'_n \to a$, a sequence whose existence was proved in Lemma 1. Since $s'_n \to a$, to complete our proof, it is sufficient to prove that $b \prec s'_n$ for all n.

Indeed, let n be an arbitrary natural number. By definition of a kinematic space, there exists an element $a_- \prec a$, so we have $a_- \prec a \prec s'_n$. Since the element a is contained in the open interval (a_-, s'_n) and $s_n \to a$, by definition of convergence, there exists an N for which $a_- \prec a_N \prec s'_n$. By definition, $s_N \geq b$ means that there exists a sequence of elements $s_{N,1}, s_{N,2}, \ldots$ for which $s_{N,k} > b$ and $s_{N,k} \to s_N$. Since $s_N \in (a_-, a'_n)$, by definition of convergence, this implies that for some K, we have $s_{N,K} \in (a_-, s'_n)$. From $s_{N,K} \prec s'_n$ and $b \prec s_{N,K}$, we conclude that $b \prec s'_n$. The proposition is proven.

Proof of Proposition 3.3. If $\{s_n\}$ is a $\preccurlyeq$-decreasing bounded sequence, with a bound b, then all its elements belong to the interval $[b, s_1]$. Since the kinematic space is intervally-compact, this interval is compact. Thus, by known properties of compactness, the sequence $\{s_n\}$ has a convergent subsequence $s_{n_k} \to a$, where $n_k \to \infty$. By definition of the Alexandrov topology on a kinematic space, this means that for every $a_- \prec a \prec a_+$, there exists a K for which, for all $k \geq K$, we have $a_- \prec s_{n_k} \prec a_+$. Let us show that $s_n \to a$, i.e., that for every a_- and a_+, there exists an N for which, for all $n \geq N$, we have $a_- \prec s_n \prec a_+$. Indeed, let K be the value corresponding to these a_- and a_+, and let us take $N = n_K$. In this case, $s_N = s_{n_K} \prec a_+$.

When $n \geq N$, then, due to the fact that the sequence is $\preccurlyeq$-decreasing, we have $s_N \preccurlyeq s_n$, so due to $s_N \prec a_+$, we have $s_n \prec a_+$.

Since $n_k \to \infty$, there exists a value $k_0 \geq K$ for which $n_{k_0} \geq n$ and hence, $s_{n_{k_0}} \preccurlyeq s_n$. Thus, from $a_- \prec s_{n_{k_0}}$ and $s_{n_{k_0}} \preccurlyeq s_n$, we conclude that $s_- \prec s_n$. Convergence is proven, and so it the proposition.

Proof of Theorem 3.1. Our proof is based on the following three lemmas:

Lemma 3.3. *For every complete separable normal kinematic space, if $s \in S_e$, then*

$$s_n \to e.$$

Lemma 3.4. *For every complete separable normal kinematic space, if $s_n \to e$, and $\{s_n\}$ is $\prec$-decreasing, then the sequence $\{s_n\}$ is in S_e.*

Lemma 3.5. *For every complete separable normal kinematic space, if a sequence $s \in S_e$ is $\prec$-decreasing, then it is S_e-maximum.*

Proof of Lemma 3.3. If $s \in S_e$, then s_n is a $\preccurlyeq$-decreasing sequence which is bounded by e. Since the kinematic space is complete, this sequence has a limit. Let us denote this limit by b.

From $e \preccurlyeq s_n$ and $s_n \to b$, in the limit, we get $e \preccurlyeq b$; see Proposition 2. From the fact that $s_N \preccurlyeq s_n$ for all $N \geq n$, in the limit, we get $b \preccurlyeq s_n$ for all n. Thus, $e \preccurlyeq b \preccurlyeq s_n$ for all n, i.e., b belongs to all the closed intervals $[e, s_n]$ and so, b belongs to the intersection $[e, e]$ of all these closed intervals.

The fact that $b \in [e, e]$ means that $b \preccurlyeq e$ and $e \preccurlyeq b$. Now, for every element x, if $x \prec b$ then from $x \prec b$ and $b \prec e$, we conclude that $x \prec e$. Vice versa, if $x \prec e$, then from $x \prec e$ and $e \preccurlyeq b$, we conclude that $x \prec b$. Thus, $x \prec b$ if and only if $x \prec e$. Similarly, for every element x, we have $b \prec x$ if and only if $e \prec x$. So, in terms of the open relation $\prec$, the elements e and b are interchangeable. Since the limit is defined in terms of the open relation $\prec$, the fact that $s_n \to b$ implies that $s_n \to e$. The lemma is proven.

Proof of Lemma 3.4. By definition of the class S_e, to prove the lemma, we must prove that $e \preccurlyeq s_n$ for all n and that $\bigcap\limits_{n=1}^{\infty} [e, s_n] = [e, e]$.

Let us first prove that for every n, we have $e \preccurlyeq s_n$. Indeed, since the sequence $\{s_n\}$ is $\prec$-decreasing, for every $m > 0$, we have $s_{n+m} \prec s_n$ and thus, $s_{n+m} \preccurlyeq s_n$. From $s_n \to e$, we conclude that $s_{n+m} \to e$. Due to Proposition 2, $s_{n+m} \preccurlyeq s_n$ and $s_{n+m} \to e$ imply that $e \preccurlyeq s_n$.

Let us now prove that $\bigcap_{n=1}^{\infty} [e, s_n] = [e, e]$. For that, we prove that every element from the intersection belongs to the interval $[e, e]$, and that every element from the interval $[e, e]$ belongs to the intersection. Indeed, let t be an element from the intersection $\bigcap_{n=1}^{\infty} [e, s_n]$. This means that for every n, $t \in [e, s_n]$, i.e., $e \preccurlyeq t$ and $t \preccurlyeq s_n$. Due to Proposition 2, $t \preccurlyeq s_n$ and $s_n \to e$ imply that $t \preccurlyeq e$. Thus, $e \preccurlyeq t$ and $t \preccurlyeq e$, i.e., indeed, $t \in [e, e]$.

Vice versa, let $t \in [e, e]$, i.e., let $e \preccurlyeq t$ and $t \preccurlyeq e$. Let us prove that t belongs to the intersection $\bigcap_{n=1}^{\infty} [e, s_n]$, i.e., that for every n, we have $t \in [e, s_n]$. Indeed, we know that $e \preccurlyeq t$. Due to Proposition 3.2, from $t \preccurlyeq e$ and $e \preccurlyeq s_n$, we conclude that $t \preccurlyeq s_n$. Thus, $e \preccurlyeq t \preccurlyeq s_n$, i.e., indeed, $t \in [e, s_n]$. The lemma is proven.

Proof of Lemma 3.5. Let us show that if s is an $\prec$-decreasing element of S_e and $s' \in S_e$, then $s' \preccurlyeq$, i.e., that for every n, there exist an m for which $s'_m \preccurlyeq s_n$. Indeed, by Lemma 3.3, $s' \in S_e$ implies that $s'_n \to e$. By definition of convergence, this means that for every $a_- \prec e \prec a_+$, there exists an m_0 for which, for all $m \geq m_0$, we have $a_- \prec s'_m \prec a_+$.

By definition of a kinematic space, there exists an element $a_- \prec e$. Since $s_{n-1} \prec s_n$ and $e \preccurlyeq s_{n-1}$, we conclude, by Proposition 2, that $e \prec s_n$. So, we can take $a_+ = s_n$. Then, there exists an m for which $s'_m \prec a_+ = s_n$ and thus, $s'_m \preccurlyeq s_n$. The Lemma is proven.

Now, we can complete the proof of Theorem 3.1. Let $b \prec a$. Then, according to Lemma 1, we can construct a $\prec$-decreasing sequence s_n for which $s_1 = a$, $b \prec s_n$, and $s_n \to b$. Due to Lemma 4, we can thus conclude that $s \in S_b$. So, due to Lemma 5, we conclude that the sequence s is S_b-maximum.

Vice versa, let us assume that for some $b \preccurlyeq e$, there exists an S_e-maximum sequence s for which $a = s_1$. This means that for every other sequence $s' \in S_e$, we have $s \geq s'$. In particular, as s', we can take a $\prec$-decreasing sequence s'_n for which $s'_n \to e$. For this sequence, $e \prec s'_n$ for all n. From $s' \preccurlyeq s$, we conclude, in particular, that there exists an m

for which $s'_m \preccurlyeq a = s_1$. From $b \preccurlyeq e \prec s'_m \preccurlyeq a$, we now conclude – via Proposition 3.1 – that $b \prec a$.

The Theorem is proven.

Proof of Theorem 3.2.

1°. Let us first prove that for every $a \in B - \{0,1\}$, there exists a point $a^- \in B - \{0,1\}$ for which $a^- \prec a$.

Indeed, by the definition of de Vries algebra, since $a \neq 0$, there exists a $b \neq 0$ for which $b \prec a$. We will show that this b is the desired a^-. We already know that $b \prec a$ and that $b \neq 0$. So, to complete the proof, it is sufficient to show that $b \neq 1$.

We can prove the inequality $b \neq 1$ by contradiction. Indeed, if $b = 1$, then from $b \prec a$, we would be able to conclude that $1 \prec a$. Since $a \preceq 1$, from $a \preceq 1 \prec a$, we would then conclude that $a \prec a$, which contradicts to our assumption that the de Vries algebra B is connected.

2°. Let us now prove that for every $a \in B - \{0,1\}$, there exists a point $a^+ \in B - \{0,1\}$ for which $a \prec a^+$.

Let us use an auxiliary element $b \stackrel{\text{def}}{=} \neg a$. Since $\neg$ is a 1-1 mapping and it maps 0 to 1 and vice versa, we conclude that b is different from 0 and 1, i.e., that $b \in B - \{0,1\}$. Thus, due to Part 1 of this proof, there exists a value $b^- \in B - \{0,1\}$ for which $b^- \prec b$. By definition of the de Vries algebra, this implies $\neg b \prec \neg b^-$, i.e., $a \prec a^+ \stackrel{\text{def}}{=} \neg b^-$. Due to $b^- \in B - \{0,1\}$, we get $a^+ = \neg b^- \in B - \{0,1\}$. The statement is proven.

3°. If $a \prec b$, then the existence of a c for which $a \prec c \prec b$ follows directly from the definition of a de Vries algebra.

4°. Let us prove that if $a \prec b$ and $a \prec c$, then there exists a d for which $a \prec d \prec b, c$.

Indeed, due to the properties of a de Vries algebra, $a \prec b, c$ implies that $a \prec b \wedge c$, where $b \wedge c \preccurlyeq b, c$. Due to Part 3 of this proof, there exists a d for which $a \prec d \prec b \wedge c$. From $d \prec b \wedge c \preccurlyeq b, c$, we conclude that $d \prec b, c$. The statement is proven.

5°. The dual statement, that if $b \prec b$ and $c \prec a$, then there exists a d for which $c, d \prec d \prec a$, follows from Part 4 of this proof by considering the values $\neg a$, $\neg b$, and $\neg c$ (just like we reduced Part 2 of this proof to Part 1).

6°. So, the set $B - \{0, 1\}$ is indeed a kinematic space. Thus, intervals form a basis of a topology. Let us prove that $\preccurlyeq$ is indeed a closure of $\prec$ in this topology, and that this kinematic space is normal.

7°. Let us prove that in $B - \{0, 1\}$, we have

$$a \preccurlyeq b \Leftrightarrow \forall c\,(b \prec c \rightarrow a \prec c).$$

Indeed, if $a \preccurlyeq b$ and $b \prec c$, then, due to the definition of a de Vries algebra, we have $a \prec c$. Vice versa, let us assume that $\forall c\,(b \prec c \rightarrow a \prec c)$, i.e., that a strictly precedes ($\prec$) all the elements that b strictly precedes. It is known that in a de Vries algebra, we have $b = \vee\{c : b \prec c\}$ [10, 11, 12]. Since a strictly precedes all elements of the set $\{c : b \prec c\}$, it thus precedes ($\preccurlyeq$) all these elements and thus, precedes their infimum b: $a \prec b$.

8°. By using duality, we can now prove that

$$a \preccurlyeq b \Leftrightarrow \forall c\,(c \prec a \rightarrow c \prec b).$$

9°. From Parts 7 and 8 of this proof, we can now conclude, by using known results about kinematic spaces [73], that for every a:

- the cone $C_a^+ = \{b : a \preccurlyeq b\}$ is equal to the closure of the cone

$$K_a^+ = \{b : a \prec b\}, \text{ and}$$

- the cone $C_a^- = \{b : b \preccurlyeq a\}$ is equal to the closure of the cone

$$K_a^- = \{b : b \prec a\}.$$

The fact that $b \in K_a^+ \Leftrightarrow a \preccurlyeq b \Leftrightarrow a \in K_b^-$ means that the kinematic space $(B - \{0,1\}, \prec)$ is normal.

The theorem is proven.

The inverse is also true, in the following sense.

Theorem 3.3 *Let S be a normal kinematic space, let t be a T-transformation, and let us assume that when we add the smallest element 0 and the largest element 1 to the corresponding set $(S, \preccurlyeq)$, we get a complete Boolean algebra, with t as negation. In this case, if we extent $\prec$ to $S \cup \{0,1\}$ by taking and that $0 \prec a \prec 1$ for all a, then $(S \cup \{0,1\}, \preccurlyeq, \prec)$ becomes a connected de Vries algebra.*

Proof of Theorem 3.3. Let us prove the properties of de Vries algebra one by one.

1°. The property $1 \prec 1$ follows from our definition of the order $\prec$.

2°. For $a, b \in S$, the desired property – that $a \prec b$ implies $a \preccurlyeq b$ – comes from the known properties of a kinematic space. When in the pair (a, b), at least one of the elements a or b is equal to 0 or 1, this implication follows from our definitions of $\prec$ and $\preccurlyeq$ for such pairs.

3°. For $a, b, c \in S$, the desired property – that $a \preccurlyeq b \prec c \preccurlyeq d$ implies $a \prec d$ – follows from the above-mentioned results about kinematic spaces. When at least one of the elements a, b, c, or d is equal to 0 or 1, this implication follows from the above results and from our definitions of $\prec$ and $\preccurlyeq$ for the pairs containing 0 or 1.

4°. When $a \prec b, c$ for $a, b, c \in S$, then, according to the definition of a kinematic space, there exists a d for which $a \prec d \prec b, c$. Since $\prec$ implies $\preccurlyeq$, we have $d \preccurlyeq b, c$ and thus, $a \preccurlyeq b \wedge c$. From $a \prec d \preccurlyeq b \wedge c$, we conclude that $a \prec b \wedge c$. This inequality is also easy to prove when one of the elements a, b, and c coincides with 0 or 1.

5°. For $a, b \in S$, the desired property – that $a \prec b$ implies $\neg b \prec \neg a$ – follows from the fact that t is a T-transformation. When in the pair (a, b), at least one of the elements a or b is equal to 0 or 1, this implication follows from our definition of $\prec$ for such pairs.

6°. When $a \prec b$ for $a, b \in S$, then, according to the definition of a kinematic space, there exists a c for which $a \prec c \prec b$. This inequality is also easy to prove when one of the elements a or b coincides with 0 or 1: e.g., if $0 \prec a$, then $0 \prec 0 \prec a$, so we can take $c = 0$. If $a \prec 1$, then $a \prec 1 \prec 1$, so we can take $c = 1$.

7°. Let us assume that $a \neq 0$, We need to prove that there exists $b \neq 0$ for which $b \prec a$. To prove this property, let us consider two cases: $a = 1$ and $a \neq 1$.

7.1°. If $a = 1$, we can take $b = 1$. In this case, $1 \neq 0$ and $1 \prec 1$ (due to Part 1 of this proof).

7.2°. If $a \neq 1$, then, since a is also different from 0, the element a belongs to the original set S. Thus, due to the definition of a kinematic space, there exists an element $a^- \in S$ for which $a^- \prec a$. This element is different from 0, so we can take it as the desired element b.

The theorem is proven.

Chapter 4

From Potentially Experimentally Confirmable Relation to Actually Experimentally Confirmable One: Extending Allen's Interval Algebra to General Partially Ordered Sets

In addition to checking what is *potentially*, eventually deducible (when the accuracy increases), it is also important to check what can be confirmed at present, when we only have a finite number of observations with a given accuracy. For example, instead of knowing the exact time location of an event a, we only know an event $\underline{a}$ that preceded a and an event $\overline{a}$ that follows a. In this case, the only information that we have about the actual (unknown) event a is that it belongs to the interval $[\underline{a}, \overline{a}] \stackrel{\text{def}}{=} \{a : \underline{a} \preccurlyeq a \preccurlyeq \overline{a}\}$. In such situations, we need to compare intervals.

Such a description has already been done for intervals on the real line; the resulting description is known as Allen's algebra; in these terms, what we want is to generalize Allen's algebra to intervals over an arbitrary partially ordered set. Such a generalization is described in this chapter. As auxiliary results, we provide a logical interpretation of the relation between intervals, and extend the results about interval graphs to intervals over posets.

The results of this chapter will appear in [88].

4.1 Formulation of the Problem

Need to compare values. In order to compare different objects, we need to compare the values of their corresponding quantities. For example, one object is heavier than the other if its weight is larger than the weight of the other object, it is faster than the other if its velocity is larger, etc.

The result of comparing two values is often called a *relation* between the two values v and v':

- if $v < v'$, we say that v and v' are in relation $<$;
- if $v = v'$, we say that v and v' are in relation $=$; etc.

Comment. In this chapter, we use $\leq$ to describe a general partial order.

Important terminological comment. It should be mentioned that this usual use of the word "relation" can lead to confusion, since in mathematics, a *relation* is defined as a *set* of pairs: e.g., the relation $<$ between real numbers is defined as the set of all the pairs (a, b) for which $a < b$. To avoid confusion, in this chapter, we will call the relation symbol between the two values an *individual relation*, or *i-relation*, for short.

Need to take into account interval uncertainty. In the ideal situation, when we represent the value in question as a real number $x \in \mathbb{R}$, and we know the exact values of the quantities for both objects $x, y \in \mathbb{R}$, we can compare these values and conclude either that the first value is smaller $x < y$, or that the first value is larger $y > x$, or that these values are equal $x = y$. In practice, we rarely know the exact values of the corresponding quantity: the values usually come from measurements, and measurements are never absolutely accurate – the measurement result $\widetilde{x}$ is, in general, different from the actual (unknown) value x of this quantity. In many practical situations, we only know the upper bound Δ on the absolute value $|\Delta x|$ of the measurement error $\Delta x \stackrel{\text{def}}{=} \widetilde{x} - x$. In this case, once we know $\widetilde{x}$ and Δ, the only information that we have about the value x is that x belongs to the real interval $[\underline{x}, \overline{x}] \subseteq \mathbb{R}$, where $\underline{x} = \widetilde{x} - \Delta$ and $\overline{x} = \widetilde{x} + \Delta$.

i-Relations between values under interval uncertainty. If we know the two values with interval uncertainty, we may not be able to tell whether the first value is smaller or larger than the second value. For example, if the first value x is in the interval $[0.9, 1.1]$ and the second value y is in the interval $[1.0, 1.2]$, then it may be that $x = 0.9 < y = 1.2$, or it may be that $x = 1.1 > y = 1.0$.

Interval i-relations: what is known. Methods using intervals on the real line are prominent in quantitative analysis; see, e.g., [59]. Let $\mathbf{x} = [\underline{x}, \overline{x}] \subseteq \mathbb{R}$ be a generic real interval and let $\hat{\mathbb{R}}$ be the set of all real intervals. The possible i-relations between real intervals $\mathbf{x}, \mathbf{y} \in \hat{\mathbb{R}}$ generated by i-relations between endpoints were explicated in the early 1980s in [6] (see also [63]); the class of such i-relations is known as *Allen's algebra.*

Specifically, if we are given two numbers x and y, then we have three possible i-relations between them: $<$, $=$, and $>$. In the interval case, instead of each number x (or y), we have *two* numbers: $\underline{x}$ and $\overline{x}$ (or $\underline{y}$ and $\overline{y}$). So, to fully describe the i-relation between the intervals, we need to describe all possible combinations of i-relations between these numbers. For non-degenerate intervals, we know the i-relation between x-bounds and the i-relation between the y-bounds: $\underline{x} < \overline{x}$ and $\underline{y} < \overline{y}$. So, to describe possible i-relations between (non-degenerate) intervals, it is sufficient to describe the i-relations between the x-bounds and the y-bounds, i.e., the 4-tuple $(r_{--}, r_{-+}, r_{+-}, r_{++})$, where:

- r_{--} is the i-relation between $\underline{x}$ and $\underline{y}$;
- r_{-+} is the i-relation between $\underline{x}$ and $\overline{y}$;
- r_{+-} is the i-relation between $\overline{x}$ and $\underline{y}$;
- r_{++} is the i-relation between $\overline{x}$ and $\overline{y}$.

Each i-relation between numbers can have three possible values $<$, $=$, and $>$, so in principle, we can have 3^4 possible 4-tuples of i-relations. In our case, however, due to the properties of order, not all such 4-tuples are possible: e.g., if $\overline{x} < \underline{y}$, then, due to transitivity, we also have $\underline{x} < \overline{y}$, $\underline{x} < \overline{y}$, and $\underline{x} < \overline{y}$. What Allen did was described all possible 4-tuples of i-relations between the endpoints $\underline{x}$, $\overline{x}$, $\underline{y}$, $\overline{y}$ of two non-degenerate intervals $[\underline{x}, \overline{x}]$ and $[\underline{y}, \overline{y}]$.

Possible orders between intervals: what is known. Allen's algebra defines many different relations between intervals. An important class of relations is the class of orders. It is therefore natural to ask: which relations of Allen's algebra define orders?

At first glance, this question may seem easy to answer: we already have a full description of all possible i-relations between intervals, so we can simply check which of these i-relations define an order. However, the situation is not as simple: e.g., for real numbers, none of the three i-relations $<$, $=$, and $>$ define an order. To get an order, we need to consider a *propositional combination* of these relations: e.g., the usual order $x \le y$ means that $(x < y) \vee (x = y)$, and the order $x \ge y$ means that $(x > y) \vee (x = y)$.

Similarly, to describe orders between intervals, Allen considers propositional combinations. Specifically, since we know that the orders between numbers are $\le$ and $\ge$, Allen considers propositional combinations of the corresponding relations $\le$ and $\ge$. We want to describe orders that, in the degenerate case when $[\underline{x}, \overline{x}] = [x, x]$ and $[\underline{y}, \overline{y}] = [y, y]$, reduce to the usual numerical order $x \le y$. Thus, it is reasonable to consider propositional combinations of the truth value of the following four relations: $\underline{x} \le \underline{y}$, $\underline{x} \le \overline{y}$, $\overline{x} \le \underline{y}$, and $\overline{x} \le \overline{y}$. It turns out that only two such combinations lead to orders that extend $x \le y$:

- In the *strong order*, relation $\mathbf{x} \le \mathbf{y}$ means that $\overline{x} \le \underline{y}$, so that every value from the interval $[\underline{x}, \overline{x}]$ is smaller than or equal to every value from the interval $\left[\underline{y}, \overline{y}\right]$. This is the common and by far most prominent sense of "interval order", as advocated e.g., in [22].

- In the *weak order*, relation $\mathbf{x} \le \mathbf{y}$ means that $\underline{x} \le \underline{y}$ and $\overline{x} \le \overline{y}$, so that the respective endpoints satisfy $\le$ on the reals. This is a very natural sense of an interval order, for example saying that one event extended in time can be prior to another even if it is still underway when the subsequent event initiates.

If we do not require that the combination reduces to $x \leq y$ in the degenerate case, then we can additional orders, e.g., the *containment order* $\mathbf{x} \subseteq \mathbf{y}$ (see, e.g., [76]), in which $\mathbf{x} \leq \mathbf{y}$ means that $\underline{x} \geq \underline{y}$ and $\overline{x} \leq \overline{y}$.

Relation between different interval orders. It is worth mentioning that the strong order implies the weak order.

Also, the weak order and the containment order are generally conjugate, in that pairs of real intervals $\mathbf{x}, \mathbf{y} \in \hat{\mathbb{R}}$ are comparable in exactly one or the other[1]. In fact, the weak order is actually just the Cartesian product $\leq \times \leq$ of the natural order $\leq$ on $\mathbb{R}$, whereas the containment order is defined as $\geq \times \leq$ ([70]).

Need to consider partially ordered sets. The set of all the real numbers is totally (linearly) ordered: for every two numbers x and y, either $x < y$ or $y < x$, or $x = y$. In many practical situations, however, we are interested in the quantities that are only partially ordered.

For example, in space-time geometry, we do not have the exact location of an event in space-time, we usually only know the events $\underline{x}$ that can causally affect the given event x ($\underline{x} \leq x$) and the events $\overline{x}$ that can causally be affected by x ($x \leq \overline{x}$). In this case, the only information that we have about the event x is that it belongs to the interval $[\underline{x}, \overline{x}] = \{x : \underline{x} \leq x \leq \overline{x}\}$. This description looks similar to the above interval case, but the important difference is that the causality relation is space-time is only a partial order: there exist events x and y for which $x \not\leq y$ and $y \not\leq x$; such events are called *incompatible* and denoted by $x \parallel y$; see, e.g., [21, 43, 57, 87] and references therein.

[1] Note that this is *almost* always true, in that endpoint equality also has to be taken into account, yielding intervals that are equal at *one* endpoint comparable in both orders.

There are other cases when we have intervals in partially ordered spaces: e.g., preferences often form only a partial order; see, e.g., [37, 77, 80]. So, if we know the lower and upper bounds, we end up with an interval in a partially ordered space.

In order theory, i.e., in the mathematics of lattices and partially ordered sets (see, e.g., [16]), intervals are readily available. Recall that for two elements x and y in a partially ordered set, we have the following possible relations: $x < y$, $x = y$, $x > y$, and $x \parallel y$ (meaning that x and y are incompatible, i.e., that $x \neq y$, $x \not< y$, and $x \not> y$). For two elements x, y where $x \leq y$, then we simply define the interval $\mathbf{x} = [x, y]$ as the set $\mathbf{x} = \{z : x \leq z \leq y\}$.

Need to extend interval orders to partially ordered sets. Since in practice, we encounter intervals in partially ordered spaces, it is desirable to describe possible relations between such intervals – i.e., to extend interval orders and Allen's algebra to partially ordered sets. In particular, we would like to list all possible ordering relations between two intervals in a partially ordered set.

Remaining open problem. In this chapter, we consider the case when we know a preceding event $\underline{x}$ and a following event $\overline{x}$. In this case, the only information that we have about the event of interest x is that it belongs to the *interval* $[\underline{x}, \overline{x}]$. In principle, we may have *several* lower bounds and several upper bounds. In this more general case, the set of possible values of x is an intersection of several intervals. In other cases, we may have an even more general set. It is desirable to further generalize the results of this chapter by extending these results from intervals to intersections of intervals – and to more general sets[2].

4.2 Results: Possible i-Relations Between Intervals

Comparison between points x and y: reminder.

[2] The authors are thankful to an anonymous referee for this interesting suggestion.

Definition 4.1 *Let X be a partially ordered set. By an* i-relation *between elements $x, x' \in X$, we mean:*

- *a symbol $<$ if $x < x'$;*
- *a symbol $=$ if $x = x'$;*
- *a symbol $>$ if $x > x'$;*
- *a symbol $\|$ if $x \| x'$.*

i-relations between points can be illustrated on the example of a 2-D analog of the causality relation of special relativity. In special relativity, it is assumed that all the speeds are limited by the speed of light c. Thus, an event (x_0, x_1) occurring at moment x_0 at a spatial point x_1 can influence an event (y_0, y_1) if and only if $y_0 > x_0$ and during the time $y_0 - x_0$, the signal traveling with speed of light c can cover the distance $|x_1 - y_1|$ between the corresponding spatial points, i.e., if

$$x = (x_0, x_1) \le y = (y_0, y_1) \Leftrightarrow c \cdot (y_0 - x_0) \ge |x_1 - y_1|.$$

This relation is illustrated by Figure 4.1, in which:

- the symbol $>$ marks all the points y for which $x > y$,
- the symbol $<$ marks all the points y for which $x < y$,
- etc.

Comparison between a point x and an interval $[\underline{y}, \overline{y}]$. We have already described possible i-relations between points. Each point x can be viewed as a "degenerate" interval $[\underline{x}, \overline{x}]$. Thus, we have covered the case when both intervals are degenerate.

Before we consider the general case of comparing intervals, let us first consider the case when the first interval is still degenerate (i.e., is a point), but the second interval $[\underline{y}, \overline{y}]$ is non-degenerate (i.e., $\underline{y} < \overline{y}$). In this case, instead of a *single* i-relation r ($>$, $<$, $=$, or $\|$) between x and y, we have *two* i-relations:

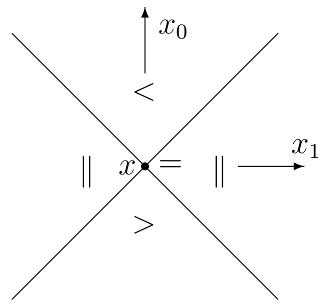

Figure 4.1: Partial order corresponding to special relativity

- the i-relation r_- between x and $\underline{y}$ (for which $xr_-\underline{y}$), and
- the i-relation r_+ between x and $\overline{y}$ (for which $xr_+\overline{y}$).

Our objective is to describe possible pairs $p = (r_-, r_+)$ of such i-relations. To come up with such a description, let us introduce the following order $\preceq$ between four possible relations:

$$> \prec =, \quad > \prec \|, \quad > \prec <, \quad = \prec <, \quad \| \prec <$$

This order is illustrated in Figure 4.2. The order $\prec$ means that $>$ precedes all other i-relations, and $=$ and $\|$ precede $<$. Alternatively, we can say that $<$ follows all other relations, and $=$ and $\|$ follow $>$.

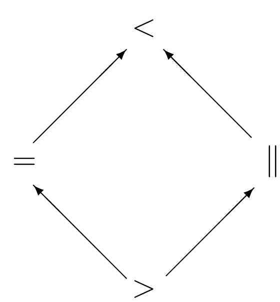

Figure 4.2: Order $\prec$ between i-relations

Proposition 4.1. *Let X be a partially ordered set, and let x, $\underline{y}$, and $\overline{y}$, be elements of X for which $\underline{y} \le \overline{y}$. Then $r_- \preceq r_+$, where:*

- *r_- is the i-relation between x and $\underline{y}$, and*
- *r_+ is the i-relation between x and $\overline{y}$.*

Proposition 4.2. *For a pair $p = (r_-, r_+)$ of i-relations, the following two conditions are equivalent to each other:*

- *there exists a partially ordered set and values $\underline{x} < \overline{x}$ and y from this set for which:*
  - *the i-relation r_- is the i-relation between x and $\underline{y}$, and*
  - *the i-relation r_+ is the i-relation between x and $\overline{y}$.*
- *the pair $p = (r_-, r_+)$ is equal to one of the following pairs:*

$$(<,<),\quad (=,<),\quad (\|,<),\quad (>,<),\quad (\|,\|),\quad (>,=),\quad (>,\|),\quad (>,>).$$

The possibility of all eight pairs can be illustrated on the example of the following points from the above-described 2-D analog of special relativity relation; see Figure 4.3. Here, we take $\underline{y} = (-1, 0)$, $\overline{y} = (1, 0)$, and we show eight points x that have eight possible pairs of i-relations (r_-, r_+) between x and $\underline{y}$ and between x and $\overline{y}$. Specifically, we have two points $x = \underline{y}$ and $x = \overline{y}$ and six additional points:

- the point $x = \underline{y}$ that corresponds to $(=,<)$;
- the point $x = \overline{y}$ that corresponds to $(>,=)$;
- the point $x = (2, 0)$ that corresponds to $(>,>)$;
- the point $x = (1, c)$ that corresponds to $(>,\|)$;
- the point $x = (0, 0)$ that corresponds to $(>,<)$;

- the point $x = (0, 2c)$ that corresponds to $(\|, \|)$;
- the point $x = (-1, c)$ that corresponds to $(\|, <)$;
- the point $x = (-2, 0)$ that corresponds to $(<, <)$.

Dashed lines describe ordering between the six additional points x.

One can see that the order presented on Figure 4.3 is inverse to the order presented on Figure 4.2. The reason for this inversion is that Figure 4.2 and Figure 4.3 describe *different* types of orders:

- Figure 4.2 describes orders between *i-relations*, while
- Figure 4.3 describes orders between *elements* of the original partially ordered set.

As a result:

- In Figure 4.2, the relation $<$ is on top, because once $x < \underline{y}$ (i.e., once the i-relation between x and $\underline{y}$ is $<$), then, due to transitivity, we also have $x < \overline{y}$, meaning that the i-relation between x and $\overline{y}$ is also $<$.
- In Figure 4.3, the pair $(>, >)$ is on top, because for the largest elements x, we have $x > \underline{y}$ and $x > \overline{y}$ and thus, the corresponding pair of i-relations is $p = (r_-, r_+) = (>, >)$.

Comparison between two non-degenerate intervals. We would like to describe all possible i-relations between intervals generated by i-relation between endpoints. In the previous text, we have described all such i-relations for the situations in which at least one of the intervals is degenerate. So, to complete our description, it is sufficient to describe all possible i-relations between two non-degenerate intervals.

Specifically, we will use the above result about i-relations between a number and an interval to describe possible i-relations between two non-degenerate intervals $[\underline{x}, \overline{x}]$ and $[\underline{y}, \overline{y}]$. In this case, instead of two i-relations r_- and r_+, we have four i-relations:

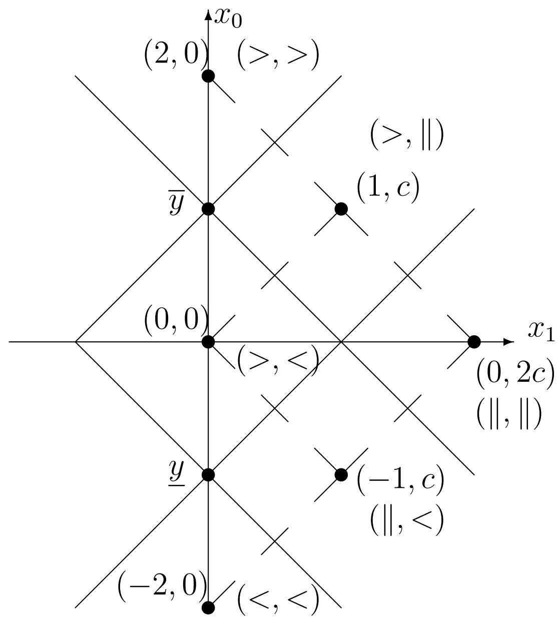

Figure 4.3: Eight points showing that all eight pairs of i-relations $p = (p_-, p_+)$ are possible

- the i-relation r_{--} between $\underline{x}$ and $\underline{y}$,
- the i-relation r_{-+} between $\underline{x}$ and $\overline{y}$,
- the i-relation r_{+-} between $\overline{x}$ and $\underline{y}$, and
- the i-relation r_{++} between $\overline{x}$ and $\overline{y}$.

Our objective is to describe possible combinations $(r_{--}, r_{-+}, r_{+-}, r_{++})$ of such relations.

Each such combination can be represented as a pair (p_-, p_+) of pairs $p_- \stackrel{\text{def}}{=} (r_{--}, r_{-+})$ and $p_+ \stackrel{\text{def}}{=} (r_{+-}, r_{++})$:

- the pair p_- describes the i-relations between the point $\underline{x}$ and the (endpoints of the) interval $\left[\underline{y}, \overline{y}\right]$, and

- the pair p_+ describes the i-relations between the point $\overline{x}$ and the (endpoints of the) interval $\left[\underline{y}, \overline{y}\right]$.

To come up with the desired description, let us introduce the order $\preceq$ between possible pairs as in Figure 4.4. This means that the pair $(<, <)$ precedes all other pairs, etc.

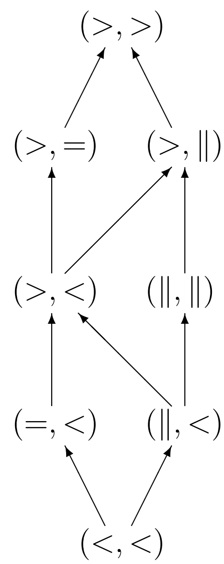

Figure 4.4: Order $\preceq$ between pairs p

Proposition 4.3. *Let X be a partially ordered set, and let $\underline{x} < \overline{x}$, and $\underline{y} < \overline{y}$ be elements of X. Then $p_- \preceq p_+$, where:*

- *p_- is a pair of i-relations between $\underline{x}$ and $\underline{y}$ and between $\underline{x}$ and $\overline{y}$, and*

- *p_+ is a pair of i-relations between $\overline{x}$ and $\underline{y}$ and between $\overline{x}$ and $\overline{y}$.*

Proposition 4.4. *For a combination of i-relations $(r_{--}, r_{-+}, r_{+-}, r_{++})$, the following two conditions are equivalent to each other:*

- *there exists a partially ordered set and values $\underline{x} < \overline{x}$ and $\underline{y} < \overline{y}$ from this set for which:*

    - *r_{--} is the i-relation between $\underline{x}$ and $\overline{y}$,*

– r_{-+} *is the i-relation between* $\underline{x}$ *and* $\overline{y}$*,*

– r_{+-} *is the i-relation between* $\overline{x}$ *and* $\underline{y}$*, and*

– r_{++} *is the i-relation between* $\overline{x}$ *and* $\overline{y}$*.*

- *the combination* $(r_{--}, r_{-+}, r_{+-}, r_{++})$ *is equal to one of the following combinations:*

$$(<,<,<,<),\ \ (<,<,=,<),\ \ (<,<,\|,<),\ \ (<,<,>,<),\ \ (<,<,\|,\|),\ \ ,(<,<,>,=),$$

$$(<,<,>,\|),\ \ (<,<,>,>),\ \ (=,<,>,<),\ \ (=,<,>,=),\ \ (=,<,>,\|),\ \ (=,<,>,>),$$

$$(\|,<,\|,<),\ \ (\|,<,>,<),\ \ (\|,<,\|,\|),\ \ (\|,<,>,=),\ \ (\|,<,>,\|),\ \ (\|,<,>,>),$$

$$(>,<,>,<),\ \ (>,<,>,=),\ \ (>,<,>,\|),\ \ (>,<,>,>),\ \ (\|,\|,\|,\|),\ \ (\|,\|,>,\|),$$

$$(\|,\|,>,>),\ \ (>,=,>,>),\ \ (>,\|,>,\|),\ \ (>,\|,>,>),\ \ (>,>,>,>).$$

4.3 Results: Possible Orders Between Intervals Generated By Orders Between Endpoints

It is desirable to describe all possible orders between intervals generated by orders between endpoints. In addition to describing all possible i-relations between intervals, we may also want to describe possible *orders* between intervals generated by orders between endpoints – in a general partially ordered case. Specifically, we would like to describe all i-relations that, in the degenerate case, when each interval consists of a single element, reduce to the order $x \le y$ between the elements.

Why this problem is non-trivial. At first glance, this problem is simple, since we already have a full description of all possible i-relations, so we can simply check which of these i-relations describe order.

However, the situation is not as simple, because in addition to the original "basic" i-relations, we can have *propositional combinations* of these relations.

For example, the usual order $x \le y$ means

$$(x < y) \vee (x = y).$$

Similarly, the strong order $\overline{x} \le \underline{y}$ means that we have one of the following tuples $(r_{--}, r_{-+}, r_{+-}, r_{++})$:

$$(<,<,<,<),\quad (<,<,=,<),\quad (=,<,=,<),\quad (<,<,=,=),\ \text{or}\ (=,=,=,=).$$

While the number of possible combinations is finite, it is huge, and simply checking all these combinations is not simple. Thus, instead of using the above classification, we start "from scratch", and use a different approach.

Towards describing all possible orders between intervals generated by orders between endpoints. In the interval case:

- instead of a single element x, we have two endpoints $\underline{x}$ and $\overline{x}$, and
- instead of a single element y, we have two endpoints $\underline{y}$ and $\overline{y}$.

Thus, instead of a single i-relation $x \le y$, we have $2 \times 2 = 4$ possible i-relations: $\overline{x} \le \underline{y}$, $\underline{x} \le \underline{y}$, $\overline{x} \le \overline{y}$, and $\underline{x} \le \overline{y}$.

In addition to these relations, we can also have propositional combinations of these i-relations, i.e., i-relations of the type

$$[\underline{x}, \overline{x}] \le \left[\underline{y}, \overline{y}\right] \Leftrightarrow P\left(\underline{x} \le \underline{y}, \underline{x} \le \overline{y}, \overline{x} \le \underline{y}, \overline{x} \le \overline{y}\right) \tag{4.3.1}$$

for some propositional function $P : \{T, F\}^4 \to \{T, F\}$ that transforms four truth values of the four i-relations into a single truth value describing whether the intervals $[\underline{x}, \overline{x}]$ and $\left[\underline{y}, \overline{y}\right]$ are related.

Let us denote the truth value of the i-relation $\overline{x} \le \underline{y}$ between:

- the upper endpoint $\overline{x}$ of the first interval and

- the lower endpoint $\underline{y}$ of the second interval

by t_{+-}. Here:

- the first subscript $+$ means that we take the upper endpoint of the first interval, and

- the second subscript $-$ means that we take the lower endpoint of the second interval.

Similarly:

- the i-relation $\underline{x} \leq \underline{y}$ between the lower endpoints will be denoted by t_{--};

- the i-relation $\underline{x} \leq \overline{y}$ between the lower endpoint $\underline{x}$ of the first interval and the upper endpoint $\overline{y}$ of the second interval will be denoted by t_{-+}, and

- the i-relation $\overline{x} \leq \overline{y}$ between the upper endpoints will be denoted by t_{++}.

In these terms, the strong order relation $\overline{x} \leq \underline{y}$ means that $P(t_{--}, t_{-+}, t_{+-}, t_{++}) = t_{+-}$, i.e., that $[\underline{x}, \overline{x}] \leq \left[\underline{y}, \overline{y}\right]$ if and only if $\overline{x} \leq \underline{y}$. Similarly, the weak order relation $\underline{x} \leq \underline{y} \,\&\, \overline{x} \leq \overline{y}$ corresponds to $P(t_{--}, t_{-+}, t_{+-}, t_{++}) = t_{--} \,\&\, t_{++}$.

It is important to mention that not all combinations of truth values t_{--}, t_{-+}, t_{+-}, and t_{++} are possible: since the endpoints of each interval are related by the order, i.e., since $\underline{x} \leq \overline{x}$ and $\underline{y} \leq \overline{y}$, some of the four i-relations between endpoints imply each other. For example, by transitivity, $\underline{x} \leq \overline{x}$ and $\overline{x} \leq \underline{y}$ imply that $\underline{x} \leq \underline{y}$. In general, we have the implications pictured in Figure 4.5.

$$
\begin{array}{ccccc}
 & & \underline{x} \leq \underline{y} & & \\
 & \nearrow & & \searrow & \\
\overline{x} \leq \underline{y} & & & & \underline{x} \leq \overline{y} \\
 & \searrow & & \nearrow & \\
 & & \overline{x} \leq \overline{y} & &
\end{array}
$$

Figure 4.5: Implications between truth values $t_{\pm\pm}$

Let us enumerate all possible combinations.

Proposition 4.5. *For a combination*

$$t = (t_{--}, t_{-+}, t_{+-}, t_{++})$$

of four truth values, the following two conditions are equivalent to each other:

- *there exists a partial ordered set and value* $\underline{x} \le \overline{x}$ *and* $\underline{y} \le \overline{y}$ *from this set for which:*
  - t_{--} *is the truth value of the relation* $\underline{x} \le \underline{y}$*,*
  - t_{-+} *is the truth value of the relation* $\underline{x} \le \overline{y}$*,*
  - t_{+-} *is the truth value of the relation* $\overline{x} \le \underline{y}$*, and*
  - t_{++} *is the truth value of the relation* $\overline{x} \le \overline{y}$*;*
- *the combination* t *is equal to one of the following combinations:*

$$(T,T,T,T), \quad (T,T,F,T), \quad (T,T,F,F), \quad (F,T,F,T), \quad (F,T,F,F), \quad (F,F,F,F).$$

In the following text, the set of all possible combination will be denoted by

$$\mathcal{S} = \{(T,T,T,T),(T,T,F,T),(T,T,F,F), \quad (F,T,F,T),(F,T,F,F),(F,F,F,F)\}.$$

Let us describe the general situation in precise terms.

Definition 4.2.

- *By a* propositional formula*, we mean a function* $P : \mathcal{S} \to \{T, F\}$.
- *Let* X *be a partially ordered set, and let* P *be a propositional formula. By a* relation corresponding to P *(or* P-relation*, for short), we mean the following relation between intervals* $[\underline{x}, \overline{x}]$ *and* $[\underline{y}, \overline{y}]$*:*

$$[\underline{x}, \overline{x}] \le_P [\underline{y}, \overline{y}] \Leftrightarrow P\left(\underline{x} \le \underline{y}, \underline{x} \le \overline{y}, \overline{x} \le \underline{y}, \overline{x} \le \overline{y}\right).$$

We want the resulting i-relation between intervals to generalize the i-relation $x \le y$ between the elements: in the degenerate case when $\underline{x} = \overline{x} = x$ and $\underline{y} = \overline{y} = y$, the new i-relation should transform into the i-relation $x \le y$. In other words:

- If $x \le y$, then, in the degenerate case, all four i-relations $\underline{x} \le \underline{y}$, $\underline{x} \le \overline{y}$, $\overline{x} \le \underline{y}$, and $\overline{x} \le \overline{y}$ coincide with $x \le y$ and are thus true. So, in this case, we should have $P(T, T, T, T) = T$.

- Similarly, if $x \not\le y$, then, in the degenerate case, all four i-relations $\underline{x} \le \underline{y}$, $\underline{x} \le \overline{y}$, $\overline{x} \le \underline{y}$, and $\overline{x} \le \overline{y}$ coincide with $x \le y$ and are thus false. So, in this case, we should have $P(F, F, F, F) = F$.

Definition 4.3 *We say that a P-relation $\le_P$* extends the original order *if the corresponding propositional formula satisfies the condition*

$$P(T, T, T, T) = T \text{ and } P(F, F, F, F) = F.$$

According to Definition 4.3, the ideal case is when all four i-relations $\underline{x} \le \underline{y}$, $\underline{x} \le \overline{y}$, $\overline{x} \le \underline{y}$, and $\overline{x} \le \overline{y}$ are true. It may be possible, however, that the two intervals are related by a new interval i-relation $\le$ even when some of these relations are false. It is reasonable to require the following.

- Suppose that we have $[\underline{x}, \overline{x}] \le [\underline{y}, \overline{y}]$ for some case when some of the four i-relations are true and some are false.

- Then, if we keep true i-relations true and make some false i-relations true, we should have even fewer reasons not to conclude that that $[\underline{x}, \overline{x}] \le [\underline{y}, \overline{y}]$.

- Thus, we should be able to conclude that in the new situation, intervals are related.

In other words, if the formula $P(t_{--}, t_{-+}, t_{+-}, t_{++})$ is true for some values t_{ij}, and we keep all the values $t_{ij} = T$ unchanged, but change some false values $t_{ij} = F$ to T, then, for the changed values t'_{ij}, the formula P should still be true.

Definition 4.4. *We say that a P-relation $\leq_P$ is* reasonable *if for every two sequences of truth values $t_{--}, t_{-+}, t_{+-}, t_{++}$ and $t'_{--}, t'_{-+}, t'_{+-}, t'_{++}$ for which $P(t_{--}, t_{-+}, t_{+-}, t_{++}) = T$ and $t_{ij} = T$ implies $t'_{ij} = T$ for every i, j, we have*

$$P(t'_{--}, t'_{-+}, t'_{+-}, t'_{++}) = T.$$

This definition can be reformulated in more traditional mathematical terms:

Definition 4.5. *Let $F \leq T$ be an ordering on the set of truth values. We say that a P-relation $\leq_P$ is* monotonic *if $t_{ij} \leq t'_{ij}$ for all i, j imply that*

$$P(t_{--}, t_{-+}, t_{+-}, t_{++}) \leq P(t'_{--}, t'_{-+}, t'_{+-}, t'_{++}).$$

Proposition 4.6. *A P-relation $\leq_P$ is reasonable (in the sense of Definition 4.4) if and only if it is monotonic.*

Finally, since we want to define an order, we want to make sure that the relation (4.3.1) is transitive. The following result describes all possible monotonic transitive p-relations that extend the original order.

Proposition 4.7 *A P-relation $\leq_P$ is monotonic, transitive, and extends the original order if and only if the corresponding propositional formula P has one of the following forms:*

1. *$P(T, T, T, T) = T$ and $P(t_{--}, t_{-+}, t_{+-}, t_{++}) = F$ for all other tuples $(t_{+-}, t_{--}, t_{++}, t_{-+})$;*
2. *$P(T, t_{-+}, t_{+-}, t_{++}) = T$ and $P(F, t_{-+}, t_{+-}, t_{++}) = F$ for all t_{-+}, t_{+-}, and t_{++};*
3. *$P(t_{--}, t_{-+}, t_{+-}, T) = T$ and $P(t_{--}, t_{-+}, t_{-+}, F) = F$ for all t_{--}, t_{-+}, and t_{+-};*

4. *$P(T, t_{-+}, t_{+-}, T) = T$ for all t_{+-} and t_{-+} and $P(t_{--}, t_{-+}, t_{+-}, t_{++}) = F$ for all other tuples.*

As a result, we arrive at the following corollary:

Corollary 4.1 *There are four and only four monotonic transitive P-relations $\le_P$ that extends the original order:*

1. *$\overline{x} \le \underline{y}$ (strong order);*
2. *$\underline{x} \le \underline{y}$ (ordering of lower endpoints);*
3. *$\overline{x} \le \overline{y}$ (ordering of upper endpoints);*
4. *$\underline{x} \le \underline{y}$ and $\overline{x} \le \overline{y}$ (weak order).*

Remaining open problem. In this section, we only considered orders between intervals generated by orders between endpoints, i.e., generated by the truth values of the four ordering i-relations $\underline{x} \le \underline{y}$, $\underline{x} \le \overline{y}$, $\overline{x} \le \underline{y}$, and $\overline{x} \le \overline{y}$. In principle, we can add equalities to this list of i-relations, in which case we can have additional orders, such as $\underline{x} < \underline{y} \vee (\underline{x} = \underline{y} \,\&\, \overline{x} = \overline{y})$. It would be nice to describe all such possible orders.

4.4 First Auxiliary Result: Interval Relations Reformulated In Logical Terms

It is worth mentioning that the four i-relations t_{ij} correspond to different selection of quantifiers:

Proposition 4.8.

$$\overline{x} \le \underline{y} \Leftrightarrow \forall x_{\in[\underline{x},\overline{x}]} \, \forall y_{\in[\underline{y},\overline{y}]} \, (x \le y);$$

$$\underline{x} \le \underline{y} \Leftrightarrow \exists x_{\in[\underline{x},\overline{x}]} \, \forall y_{\in[\underline{y},\overline{y}]} \, (x \le y);$$

$$\overline{x} \le \overline{y} \Leftrightarrow \exists y_{\in[\underline{y},\overline{y}]} \, \forall x_{\in[\underline{x},\overline{x}]} \, (x \le y);$$

$$\underline{x} \le \overline{y} \Leftrightarrow \exists x_{\in[\underline{x},\overline{x}]} \, \exists y_{\in[\underline{y},\overline{y}]} \, (x \le y).$$

4.5 Second Auxiliary Result: Extending Interval Graphs to Partially Ordered Sets

What is an interval graph. In many practical applications – e.g., in scheduling, in bioinformatics – it is useful to consider *interval graphs*, i.e., undirected graphs in which vertices are real-line intervals, and two vertices are connected by an edge if and only if the corresponding intervals intersect; see, e.g., [15, 22, 53].

In precise terms, an undirected graph is defined as a pair (V, E), where V is a set whose elements are called *vertices*, and E is a set of unordered pairs of vertices (v, v'); such pairs are called *edges*. A graph (V, E) is called an *interval graph* if it is possible to put into correspondence, to every vertex $v \in V$, an interval $I(v)$ so that the vertices v and v' are connected by an edge $(v, v') \in E$ if and only if the corresponding intervals have a non-empty intersection: $I(v) \cap I(v') = \emptyset$.

In view of the fact that the notion of an interval graph is practically important, efficient algorithms have been developed for checking whether a given graph can be represented as such an interval graph.

Natural question. A natural question is: what if instead of real-valued intervals, we allow intervals in a general partially ordered set? It turns out that in this case, any undirected graph can be represented as an intersection graph of intervals:

Proposition 4.9. *For every undirected graph (V, E), there exists a poset $(X, \le)$ and a mapping I that maps $v \in V$ into intervals $I(v) \subseteq X$ so that vertices v and v' are* *connected by an edge if and only if corresponding intervals have a non-empty intersection:* $I(v) \cap I(v') = \emptyset$.

It is worth mentioning that this result holds for infinite graphs as well.

4.6 Proofs

Proof of Proposition 4.1. To prove this proposition, let us consider all possible values of the i-relation r_-: $<$, $=$, $\|$, and $>$.

1°. If r_- is $<$, i.e., if $x < \underline{y}$, then, since $\underline{y} \le \overline{y}$, by transitivity, we get $x < \overline{y}$, i.e., r_+ is $<$. Thus, we have $r_i \preceq r_+$.

2°. If the i-relation r_- is equality $=$, i.e., if $x = \underline{y}$, then, since $\underline{y} \le \overline{y}$, we have $x \le \overline{y}$, i.e., $x < \overline{y}$ or $x = \overline{y}$. In this case, the i-relation r_+ is either $<$ or $=$. In both cases, $r_- \preceq r_+$.

3°. If r_- is $\|$, i.e., $x \parallel \underline{y}$, then it is impossible to have $x \ge \overline{y}$. Indeed, in this case, we would have $x \ge \underline{y}$, while we have $x \parallel \underline{y}$. Thus, the i-relation r_+ between x and $\overline{y}$ can only be $\|$ or $<$. In both cases, we have $r_- \preceq r_+$.

4°. Finally, if r_- is $>$, then $r_- \preceq r_+$ for all possible i-relations r^+.

The proposition is proven.

Proof of Proposition 4.2.

1°. An example presented in the main text shows for each of the eight pairs of i-relations (r_-, r_+) from the formulation of this proposition, there exists a partially ordered set and values x and $\underline{y} < \overline{y}$ for which

$$x r_- \underline{y} \text{ and } x r_+ \overline{y}.$$

2°. So, to complete the proof, it is sufficient to prove that for every partially ordered set and for all values x and $\underline{y} < \overline{y}$ from this set, the corresponding pair of i-relations (r_-, r_+) is equal to one of the pairs listed in the formulation of the Proposition.

To prove this, we will consider two possible cases: when x is equal to one of the points $\underline{y}$ and $\overline{y}$, and when x is different from both these points.

2.1°. When x is equal to one of the points $\underline{y}$ or $\overline{y}$, then, due to $\underline{y} < \overline{y}$, we get pairs $(=, <)$ and $(>, =)$.

2.2°. When x is different from both points $\underline{y}$ and $\overline{y}$, then for each of these points, we have three possible i-relations with x: $<$, $>$, and $\|$. In principle, there are $3 \times 3 = 9$ possible pairs, but the pairs

$$(\|, >), \quad (<, >), \text{ and } (<, \|)$$

are impossible due to Proposition 4.1. Thus, we get exactly one of the six remaining pairs – which are listed in the formulation of the Proposition.

Proof of Proposition 4.3. We will prove that $p_- \preceq p_+$ by considering all possible pairs p_-.

1°. Let us first consider the case when $p_- = (>, >)$, i.e., when $\underline{x} > \underline{y}$ and $\underline{x} > \overline{y}$. Then, due to $\overline{x} > \underline{x}$, we have $\overline{x} > \underline{y}$ and $\overline{x} > \overline{y}$, i.e., $p_+ = (>, >)$. Thus, in this case, $p_- \preceq p_+$.

2°. For $p_- = (>, =)$, we have $\underline{x} > \underline{y}$ and $\underline{x} = \overline{y}$. In this case, from $\overline{x} > \underline{x}$, we conclude that $\overline{x} > \underline{y}$ and $\overline{x} > \overline{y}$, i.e., that $p_+ = (>, >)$. Thus, $p_- \preceq p_+$.

3°. For $p_- = (>, \|)$, we have $\underline{x} > \underline{y}$ and $\underline{x} \| \overline{y}$. In this case, from $\overline{x} > \underline{x}$, we conclude that $\overline{x} > \underline{y}$. We cannot have $\overline{x} \le \overline{y}$, because this would imply $\underline{x} < \overline{y}$ while we have $\underline{x} \| \overline{y}$. Thus, we can have either $\overline{x} > \overline{y}$ or $\overline{x} \| \overline{y}$, i.e., $p_+ = (>, >)$ or $p_+ = (>, \|)$. In both cases, $p_- \preceq p_+$.

4°. For $p_- = (>, <)$, we have $\underline{x} > \underline{y}$ and $\underline{x} < \overline{y}$. In this case, from $\overline{x} > \underline{x}$, we conclude that $\overline{x} > \underline{y}$. Thus, p_+ is equal to one of the pairs $(>, r_{++})$: $(>, >)$, $(>, =)$, $(>, <)$, and $(>, \|)$. In all four cases, $p_- \preceq p_+$.

5°. For $p_- = (\|, \|)$, we have $\underline{x} \parallel \underline{y}$ and $\underline{x} \parallel \overline{y}$. In this case, similarly to Part 3 of this proof, each of the i-relations r_{+-} and r_{++} is equal to either $>$ or to $\|$. If r_{++} is $>$, i.e., if $\overline{x} > \overline{y}$, then we have $\overline{x} > \underline{y}$, and $p_+ = (>, >)$. If $r_{++} =\|$, then we can have $p_+ = (\|, \|)$ and $p_+ = (>, \|)$. In all three cases, we have $p_- \preceq p_+$.

6°. For $p_- = (=, <)$, we have $\underline{x} = \underline{y}$ and $\underline{x} < \overline{y}$. In this case, $\underline{x} < \overline{x}$ implies that $\overline{x} > \underline{y}$. Thus, p_+ is equal to one of the pairs $(>, r_{++})$: $(>, >)$, $(>, =)$, $(>, <)$, and $(>, \|)$. In all four cases, $p_- \preceq p_+$.

7°. For $p_- = (\|, <)$, we have $\underline{x} \parallel \underline{y}$ and $\underline{x} < \overline{y}$. In this case, we cannot have $\overline{x} \le \underline{y}$, since then, due to $\underline{x} < \overline{x}$, we will have $\underline{x} < \underline{y}$, while we have $\underline{x} \parallel \underline{y}$. Thus, the first component r_{+-} of the pair $p_+ = (r_{+-}, r_{++})$ is either $>$ or $\|$. For all such pairs p_+, we have $p_- = (\|, <) \preceq p_+$.

8°. Finally, if $p_- = (<, <)$, then $p_- \preceq p_+$ for all pairs p_+.

The proposition is proven.

Proof of Proposition 4.4.

1°. Let us first prove that if there exists a partially ordered set and values $\underline{x} < \overline{x}$ and $\underline{y} < \overline{y}$, then the corresponding combination of i-relations $(r_{--}, r_{-+}, r_{+-}, r_{++})$ coincides with one of the combinations listed in the formulation of the Proposition.

Indeed, due to Proposition 4.3, we must have $p_- \preceq p_+$. In the formulation of the Proposition, we listed, for each pair p_-, all possible pairs $p_- \preceq p_+$, with two exceptions: combinations (p_-, p_+) corresponding to $p_- = p_+ = (=, <)$ and $p_- = p_+ = (>, =)$.

So, to prove the first implication, it is sufficient to prove that these two combinations are impossible. Let us do it case by case.

1.1°. If $p_- = (=, <)$, this means that $\underline{x} = \underline{y}$. Since we consider non-degenerate intervals, for which $\underline{x} < \overline{x}$, we cannot have $\overline{x} = \underline{y}$ and thus, we cannot have

$$p_+ = (=, <).$$

1.2°. Similarly, if $p_- = (>, =)$, this means that $\underline{x} = \overline{x}$. Since we consider non-degenerate intervals, for which $\underline{x} < \overline{x}$, we cannot have $\overline{x} = \overline{x}$ and thus, we cannot have

$$p_+ = (>, =).$$

The first implication is proven.

2°. To complete the proof of the Proposition, we must prove that for every combination (p_-, p_+) listed in the formulation, there exists a partially ordered set and values $\underline{x} < \overline{x}$ and $\underline{y} < \overline{y}$ that lead to this very combination.

For combinations for which $p_- \neq p_+$, we can have, as examples, points $\underline{y} = (-1, 0) < \overline{y} = (1, 0)$ described after the formulation of Proposition 4.2, and as the points $\underline{x} < \overline{x}$, points from this description corresponding to pairs p_- and p_+ (recall that in that example, we have one point y for each of the six pairs $p = (r_-, r_+)$).

For combinations for which $p_- = p_+$, we can take nearby points $\underline{x} < \overline{x}$ from the zone of all points x corresponding to this pair $p_- = p_+$; see Figure 4.6.

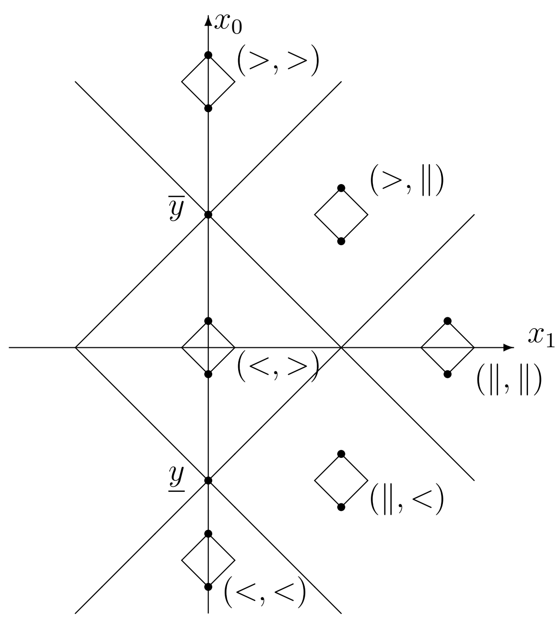

Figure 4.6: Combinations corresponding to $p_- = p_+$.

The statement is proven.

Proof of Proposition 4.5.

1°. Let us first prove that for every ordered set, the corresponding combination of truth values coincides with one of the six combinations listed in the formulation of the proposition.

2°. Let us start our analysis with the truth value of the third variable t_{+-}. This value can take either the value T or the value F. Let us consider these two values one by one.

3°. Let us first consider the case when $t_{+-} = T$, i.e., the case of combinations $(t_{--}, t_{-+}, T, t_{++})$. In this case, $\overline{x} \le \underline{y}$, and so, due to the above implications, all three other i-relations t_{--}, t_{-+}, and t_{++} are also true. Thus, we get the combination (T, T, T, T).

4°. Let us now consider the case when $t_{+-} = F$, i.e., the case of combinations of the type $(t_{--}, t_{-+}, F, t_{++})$.

Let us consider possible truth values of the first variable t_{--}, first the value T and then the value F.

5°. Let us consider combinations of the type (T, t_{-+}, F, t_{++}), in which $t_{--} = T$. In such situations, $t_{--} = T$ implies that $t_{-+} = T$, so the value of the second variable t_{-+} is always true.

The fourth variable t_{++} can be either true or false. Thus, in this situation, we have two possible combinations: (T, T, F, T) and (T, T, F, F).

6°. Let us now consider situations of the type (F, t_{-+}, F, t_{++}) in which not only $t_{+-} = F$, but also $t_{--} = F$. In such situations, the fourth variable t_{++} can be either true or false. Let us consider these two cases one by one.

6.1°. If $t_{++} = T$, then, by the above implications, we get $t_{-+} = T$. Thus, we get a combination (F, T, F, T).

6.2°. If $t_{++} = F$, then we can have two possible values of t_{-+}: true and false. Thus, we get two possible combinations: (F, T, F, F) and (F, F, F, F).

7°. We have proven for all partially ordered sets, the combination of truth values coincides with one of the six given combinations. To complete the proof, it is sufficient to prove that all six combinations are indeed possible. Indeed:

7.1°. The combination (T, T, T, T) occurs, e.g., for $[\underline{x}, \overline{x}] = [0, 1]$ and $\left[\underline{y}, \overline{y}\right] = [2, 3]$.

7.2°. The combination (T, T, F, T) occurs, e.g., for $[\underline{x}, \overline{x}] = [0, 2]$ and $\left[\underline{y}, \overline{y}\right] = [1, 3]$.

7.3°. The combination (T, T, F, F) occurs, e.g., for $[\underline{x}, \overline{x}] = [0, 3]$ and $\left[\underline{y}, \overline{y}\right] = [1, 2]$.

7.4°. The combination (F, T, F, T) occurs, e.g., for $[\underline{x}, \overline{x}] = [1, 2]$ and $\left[\underline{y}, \overline{y}\right] = [0, 3]$.

7.5°. The combination (F, T, F, F) occurs, e.g., for $[\underline{x}, \overline{x}] = [1, 3]$ and $\left[\underline{y}, \overline{y}\right] = [0, 2]$.

7.6°. The combination (F, F, F, F) occurs, e.g., for $[\underline{x}, \overline{x}] = [2, 3]$ and $\left[\underline{y}, \overline{y}\right] = [0, 1]$.

Proof of Proposition 4.6.

1°. Let us first prove that if the P-relation $\leq_P$ is reasonable, then it is monotonic. Let us assume that $t_{ij} \leq t'_{ij}$ for all i, j, and let us prove that

$$P(t_{--}, t_{-+}, t_{+-}, t_{++}) \leq P(t'_{--}, t'_{-+}, t'_{+-}, t'_{++}).$$

Our proof depends on the truth value of $P(t_{--}, t_{-+}, t_{+-}, t_{++})$.

1.1°. Let us first consider the case when

$$P(t_{--}, t_{-+}, t_{+-}, t_{++}) = F.$$

By definition of the order $\leq$ on the set of truth values, the false value F is smaller than or equal to anything. Thus, in this case, the desired inequality

$$P(t_{--}, t_{-+}, t_{+-}, t_{++}) \leq P(t'_{--}, t'_{-+}, t'_{+-}, t'_{++})$$

is indeed satisfied.

1.2°. Let us now consider the case when

$$P(t_{--}, t_{-+}, t_{+-}, t_{++}) = T.$$

In this case, if $t_{ij} = T$, then, by definition of the order $\le$ on the set of truth values, the inequality $t_{ij} \le t'_{ij}$ implies that $t'_{ij} = T$. Thus, due to the fact that the p-relation is reasonable, we get $P(t'_{--}, t'_{-+}, t'_{+-}, t'_{++}) = T$ and thus,

$$P(t_{--}, t_{-+}, t_{+-}, t_{++}) \le P(t'_{--}, t'_{-+}, t'_{+-}, t'_{++}).$$

2°. Let us now prove that if the p-relation P is monotonic, then it is reasonable. Indeed, let us make the following two assumptions:

- let us assume that P is monotonic, i.e., that $t_{ij} \le t'_{ij}$ implies that

  $$P(t_{--}, t_{-+}, t_{+-}, t_{++}) \le P(t'_{--}, t'_{-+}, t'_{+-}, t'_{++}),$$

  and

- let us also assume that for every i, j, $t_{ij} = T$ implies that $t'_{ij} = T$.

Let us prove that in this case, we have $P(t'_{--}, t'_{-+}, t'_{+-}, t'_{++}) = T$.

To prove this, let us first prove that $t_{ij} \le t'_{ij}$ for all i, j. Indeed, if $t_{ij} = F$, then this inequality is satisfied because the false value F is smaller than or equal to anything. If $t_{ij} = T$, then, by our assumption, we have $t'_{ij} = T$ and thus, $t_{ij} \le t'_{ij}$. Since $t_{ij} \le t'_{ij}$ for all i, j, by monotonicity, we get

$$P(t_{--}, t_{-+}, t_{+-}, t_{++}) \le P(t'_{--}, t'_{-+}, t'_{+-}, t'_{++}).$$

Due to $P(t_{--}, t_{-+}, t_{+-}, t_{++}) = T$, this implies that $P(t'_{--}, t'_{-+}, t'_{+-}, t'_{++}) = T$. The statement is proven, and so is the proposition.

Proof of Proposition 4.7.

1°. To describe a P-relation, we need to describe the propositional formula P, i.e., we need to describe the values of the function P on all six tuples from the set $\mathcal{S}$. We know, from the fact that the P-relation $\leq_P$ extends the original order, that $P(T,T,T,T) = T$ and $P(F,F,F,F) = F$. So, to complete our description, it is sufficient to describe four remaining values: $P(F,T,T,T)$, $P(T,T,F,F)$, $P(F,T,F,T)$, and $P(F,T,F,F)$.

2°. Let us prove, by contradiction, that

$$P(F,T,F,F) = F.$$

Indeed, if we had $P(F,T,F,F) = T$, then we would have $[0,2] \leq [-3,1]$. Indeed, in this case, out of four possible i-relations t_{ij}, only the i-relation t_{-+} $(0 \leq 1)$ is true. Thus, the corresponding tuple is (F,T,F,F), and so,

$$[0,2] \leq [-3,1] \Leftrightarrow P(F,T,F,F) = T.$$

Similarly, we conclude that $[-3,1] \leq [-2,-1]$. So, by transitivity, we would have $[0,2] \leq [-2,-1]$.

However, for the intervals $[0,2]$ and $[-2,-1]$, all four i-relations are false, so we have $P(F,F,F,F) = F$ and

$$[0,2] \leq [-2,-1] \Leftrightarrow P(F,F,F,F) = F,$$

and thus, $[0,2] \not\leq [-2,-1]$. The contradiction shows that our assumption $P(F,T,F,F) = T$ is false, and thus, $P(F,T,F,F) = F$.

3°. Because of Part 2 of this proof, to describe a desired P-relation, it is sufficient to describe three remaining values: $P(T,T,F,T)$, $P(T,T,F,F)$, and $P(F,T,F,T)$.

Let us start with describing the last two values $P(T,T,F,F)$ and $P(F,T,F,T)$. Each of these values can be either true or false, so, in principle, we have four possible combinations of these values: (T,T), (T,F), (F,T), and (F,F). Let us consider these combinations one by one.

3.1°. Let us first consider the case when $P(T,T,F,F) = T$ and $P(F,T,F,T) = T$. We will prove that this case is impossible.

Indeed, in this case, the condition $P(T, T, F, F) = T$ implies that $[0, 3] \leq [1, 1]$, and the condition $P(F, T, F, T) = T$ implies that $[1, 1] \leq [-1, 2]$. Thus, by transitivity, we would conclude that $[0, 3] \leq [-1, 2]$. However, for the intervals $[0, 3]$ and $[-1, 2]$, all four i-relations are false, so due to $P(F, F, F, F) = F$, we should get $[0, 3] \not\leq [-1, 2]$. This contradiction shows that this case is indeed impossible.

3.2°. Let us now consider the case when $P(T, T, F, F) = T$ and $P(F, T, F, T) = F$. In this case, due to monotonicity, we get $P(T, T, F, T) = T$. The corresponding function P is thus fully defined. One can easily see that the corresponding P-relation

$$[\underline{x}, \overline{x}] \leq [\underline{y}, \overline{y}] \Leftrightarrow P\left(\underline{x} \leq \underline{y}, \underline{x} \leq \overline{y}, \overline{x} \leq \underline{y}, \overline{x} \leq \overline{y}\right)$$

corresponds to ordering of lower endpoints.

3.3°. Similarly, when $P(T, T, F, F) = F$ and $P(F, T, F, T) = T$, due to monotonicity, we get $P(T, T, F, T) = T$. The corresponding function P is thus fully defined. One can easily see that the corresponding relation

$$[\underline{x}, \overline{x}] \leq [\underline{y}, \overline{y}] \Leftrightarrow P\left(\underline{x} \leq \underline{y}, \underline{x} \leq \overline{y}, \overline{x} \leq \underline{y}, \overline{x} \leq \overline{y}\right)$$

corresponds to ordering of upper endpoints.

3.4°. The only remaining case is the case when $P(T, T, F, F) = P(F, T, F, T) = F$. In this case, the only value that we still need to define is the value $P(T, T, F, T)$. This value can be either true or false. One can see that:

- when $P(T, T, F, T) = T$, we get the weak order; and
- when $P(T, T, F, T) = F$, we get the strong order.

The proposition is proven.

Proof of Proposition 4.8.

1°. Let us first prove that $\overline{x} \leq \underline{y}$ if and only if $x \leq y$ for all $x \in [\underline{x}, \overline{x}]$ and for all $y \in [\underline{y}, \overline{y}]$.

1.1°. If $\overline{x} \le \underline{y}$, then for every $x \in [\underline{x}, \overline{x}]$ and for every $y \in [\underline{y}, \overline{y}]$, we have $x \le \overline{x} \le \underline{y} \le y$. Thus, by transitivity, we get $x \le y$.

1.2°. Vice versa, if we have $x \le y$ for all $x \in [\underline{x}, \overline{x}]$ and for all $y \in [\underline{y}, \overline{y}]$, then, in particular, this inequality is true for $x = \overline{x} \in [\underline{x}, \overline{x}]$ and $\underline{y} \in [\underline{y}, \overline{y}]$. Thus, we get $\overline{x} \le \underline{y}$.

2°. Let us now prove that $\underline{x} \le \underline{y}$ if and only if there exists an $x \in [\underline{x}, \overline{x}]$ for which $x \le y$ for all $y \in [\underline{y}, \overline{y}]$.

2.1°. If $\underline{x} \le \underline{y}$, then for $x = \underline{x}$ and for all $y \in [\underline{y}, \overline{y}]$, we have $x \le \underline{y} \le y$ and thus, by transitivity, $x \le y$. Thus, there exists an $x \in [\underline{x}, \overline{x}]$ (namely, $x = \underline{x}$) for which $x \le y$ for all $y \in [\underline{y}, \overline{y}]$.

2.2°. Vice versa, let us assume that there exists an $x \in [\underline{x}, \overline{x}]$ for which $x \le y$ for all $y \in [\underline{y}, \overline{y}]$. In particular, this is true for $y = \underline{y} \in [\underline{y}, \overline{y}]$. Thus, we get $x \le \underline{y}$. Since $x \in [\underline{x}, \overline{x}]$, we conclude that $\underline{x} \le x$ and thus, by transitivity, we get $\underline{x} \le \underline{y}$.

3°. Let us prove that $\overline{x} \le \overline{y}$ if and only if there exists an $y \in [\underline{y}, \overline{y}]$ for which $x \le y$ for all $x \in [\underline{x}, \overline{x}]$.

3.1°. If $\overline{x} \le \overline{y}$, then for $y = \overline{y}$ and for all $x \in [\underline{x}, \overline{x}]$, we have $x \le \overline{x} \le y$ and thus, by transitivity, $x \le y$. Thus, there exists a $y \in [\underline{y}, \overline{y}]$ (namely, $y = \overline{x}$) for which $x \le y$ for all $x \in [\underline{x}, \overline{x}]$.

3.2°. Vice versa, let us assume that there exists a $y \in [\underline{y}, \overline{y}]$ for which $x \le y$ for all $x \in [\underline{x}, \overline{x}]$. In particular, this is true for $x = \overline{x} \in [\underline{x}, \overline{x}]$. Thus, we get $\overline{x} \le y$. Since $y \in [\underline{y}, \overline{y}]$, we conclude that $y \le \overline{y}$ and thus, by transitivity, we get $\overline{x} \le \overline{y}$.

4°. Finally, let us prove that $\underline{x} \le \overline{y}$ if and only if there exists an $x \in [\underline{x}, \overline{x}]$ and a $y \in [\underline{y}, \overline{y}]$ for which $x \le y$.

4.1°. If $\underline{x} \le \overline{y}$, then the inequality $x \le y$ holds for $x = \underline{x}$ and for $y = \overline{y}$. Thus, there exist an $x \in [\underline{x}, \overline{x}]$ (namely, $x = \underline{x}$) and a $y \in [\underline{y}, \overline{y}]$ (namely, $y = \overline{y}$) for which $x \le y$.

4.2°. Vice versa, let us assume that there exist $x \in [\underline{x}, \overline{x}]$ and $y \in [\underline{y}, \overline{y}]$ for which $x \le y$. Then, from $\underline{x} \le x$, $x \le y$, and $y \le \overline{y}$, by transitivity, we get $\underline{x} \le \overline{y}$.

Proof of Proposition 4.9. In a (undirected) graph, an edge connecting a vertex v with a vertex v' can be identified with the 2-element set $\{v, v'\}$. As the poset X, let us take the union $X = E \cup (V \times \{-, +\})$ of the set E of all the edges and the set of all the pairs $(v, -)$ and $(v, +)$. On this set, we define the following partial order: $x \le x$ for all $x \in X$ plus the following relations:

- we require that $(v, -) < (v, +)$ for all v;
- for each edge $\{v, v'\} \in E$, we require that

$$(v, -) < (v, v') < (v, +), \quad (v', -) < (v, v') < (v', +),$$

$$(v, -) < (v', +), \text{ and } (v', -) < (v, +).$$

One can check that this relation is transitive and asymmetric, and is, thus, a partial order.

To each element $v \in V$, we put into correspondence an interval $I(v) = [(v, -), (v, +)]$. By definition of our order, the intervals $I(v)$ and $I(v')$, $v \neq v'$, have a non-empty intersection if and only if $\{v, v'\} \in E$, i.e., if and only if the vertices v and v' are connected by an edge in the original graph.

The statement is proven.

Example 4.1 Let us illustrate this construction on the example of a simple fully connected graph with three vertices v_1, v_2, and v_3 described in Figure 4.7.

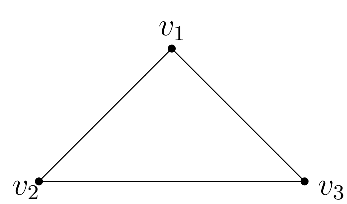

Figure 4.7: An example of a graph G

In this case, the corresponding partially ordered set has the form described in Figure 4.8.

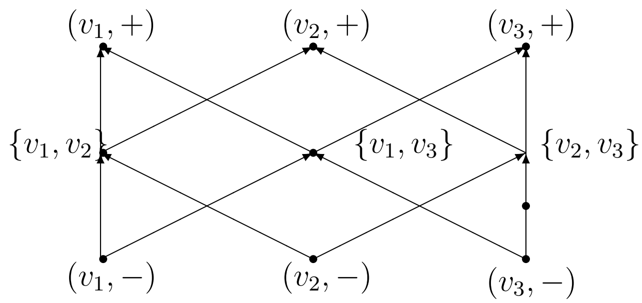

Figure 4.8: A poset for which the interval graph is the original graph G

Chapter 5

Properties of Ordered Spaces: When Is the Resulting Ordered Space a Lattice

Once a new ordered set is defined, we may be interested in properties of these spaces. For example, we may be interested in knowing when such an order is a lattice, i.e., when for every two elements, there is the greatest lower bound and the least upper bound. If this product is not a lattice, we may want to know, e.g., when the order is a *lower semi-lattice*, i.e., when every two elements have the least upper bound, etc. These questions are analyzed in this Chapter.

Specifically, in Section 5.1, for ordered sets coming from considering all possible subsets, we prove the corresponding lattice property. Ordered spaces coming from the special relativity-type definitions are not always lattices; in Section 5.2, we describe necessary and sufficient conditions for such an order to be a lattice. Finally, in Section 5.3, for interval truth values, we provide a natural logical interpretation of the lattice order.

The results of this chapter first appeared in [45, 81, 89].

5.1 Lattice Property Holds for Ordered Sets Coming from Considering All Possible Subsets

Traditional causality relation of special relativity is a lattice in the simplest case of 1-D space (2-D space-time), but it is no longer a lattice in the actual 3-D space (and 4-D space-time). In this section, we show that if we take into account effects of string theory, then we get a lattice-type causality relation even for the 4-D space-time.

Causality in special relativity: a brief reminder. As we have mentioned in Chapter 1, according to Special Relativity Theory, all processes propagate with a speed that does not exceed the speed of light c. Thus, an event $e = (t, x)$ occurring at a spatial location x at moment t can influence an event $e' = (t', x')$ occurring at moment t' at a spatial location x' only if $t \le t'$ and the speed $\dfrac{d(x, x')}{t' - t}$ with which a signal can get from x to x' during the time interval $[t, t']$ does not exceed c:

$$(t \le t')\,\&\,\left(\frac{d(x, x')}{t' - t} \le c\right); \tag{5.1.1}$$

(here, $d(x, x')$ denotes the distance between the two spatial points x and x').

The relation "an event e can influence the event e'" is called a causality relation; we will denote it by $e \preccurlyeq e'$. From the mathematical viewpoint, this is an ordering relation: e.g., if e can influence e' and e' can influence e'', then e can therefore influence e'' – so this relation is transitive.

For each event e, the set $C_e^+ \stackrel{\text{def}}{=} \{e' : e \preccurlyeq e'\}$ of all the events that can be influenced by e is called the *future cone*, and the set $C_e^- \stackrel{\text{def}}{=} \{e' : e' \preccurlyeq e\}$ of all the events that can influence e is called the *past cone*.

In the special relativity theory, causality relation is described by the formula (5.1.1). By multiplying both sides of the equivalent form (5.1.1) by a positive number $t' - t$, we get an equivalent form

$$(t, x) \preccurlyeq (t', x') \Leftrightarrow c \cdot (t' - t) \ge d(x, x'). \tag{5.1.2}$$

Since the distance $d(x, x')$ is always non-negative, this inequality automatically implies that $t' \geq t$.

In the Euclidean space, the distance $d(x, x')$ between the two points $x = (x_1, x_2, x_3)$ and $x' = (x'_1, x'_2, x'_3)$ is determined by the usual formula

$$d(x, x') = \sqrt{(x_1 - x'_1)^2 + (x_2 - x'_2)^2 + (x_3 - x'_3)^2}. \tag{5.1.3}$$

Thus, the formula (5.1.2) takes the form

$$(t, x) \preccurlyeq (t', x') \Leftrightarrow c \cdot (t' - t) \geq \sqrt{(x_1 - x'_1)^2 + (x_2 - x'_2)^2 + (x_3 - x'_3)^2}. \tag{5.1.4}$$

This formula can be simplified if we square both sides of the inequality:

$$(t, x) \preccurlyeq (t', x') \Leftrightarrow ((t' \geq t) \,\&\, (c^2 \cdot (t' - t)^2 - (x_1 - x'_1)^2 - (x_2 - x'_2)^2 - (x_3 - x'_3)^2 \geq 0)). \tag{5.1.5}$$

The set C_e^+ of all the events e' that can be influenced by the event e is thus described by the inequalities

$$(t' \geq t) \,\&\, (c^2 \cdot (t' - t)^2 - (x_1 - x'_1)^2 - (x_2 - x'_2)^2 - (x_3 - x'_3)^2 \geq 0). \tag{5.1.6}$$

From the geometric viewpoint, this set is a *cone*, that is why the set of all the future events is usually called the "future cone"; see Figure 5.1.

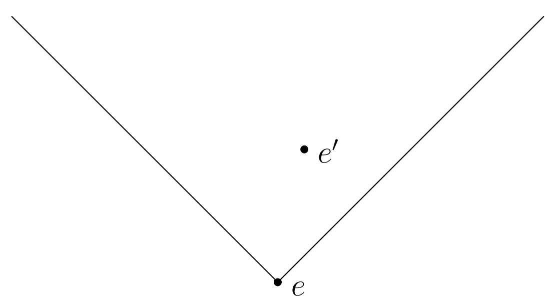

Figure 5.1: Future cone of the event e

Simplest case of the 1-D space (2-D space-time). Let us start the analysis of the causality relation with the simplest case, when we only consider motions in one spatial direction. In other words, we consider a 1-D space in which every spatial location is characterized by a single coordinate x_1. In this description, a point (t, x) in space-time (i.e., an event) is characterized by two parameters t and x_1, so the space-time is 2-dimensional.

In this case of a 1-D space (or, equivalently, of a 2-D space-time), the distance $d(x, x')$ is simply equal to $|x_1 - x'_1|$ and therefore, the causality relation (5.1.2) takes the form

$$c \cdot (t' - t) \geq |x - x'|. \tag{5.1.7}$$

One can easily check that for every real number z, we have $|z| = \max(z, -z)$. Thus, the condition $t' - t \geq |x_1 - x'_1|$ is equivalent to $t' - t \geq \max(x_1 - x'_1, x'_1 - x_1)$. A number is larger than the largest of the two numbers if and only if it is larger than both of them. Thus, the above inequality (5.1.7) is equivalent to two inequalities

$$(t' - t \geq x_1 - x'_1)\,\&\,(t' - t \geq x'_1 - x_1). \tag{5.1.8}$$

In each of these two inequalities, by moving the terms corresponding to $e = (t, x_1)$ to one side and the terms corresponding to $e' = (t', x'_1)$ to the other side, we get the following equivalent form of the causality relation:

$$(t + x_1 \leq t' + x'_1)\,\&\,(t - x_1 \geq t' - x'_1). \tag{5.1.9}$$

Thus, if, instead of the natural coordinates (t, x_1), we use auxiliary coordinates $u = t + x_1$ and $v = t - x_1$ to describe each event, we get a simple formula for the causality relation:

$$(e = (u, v) \preccurlyeq e' = (u', v')) \Leftrightarrow (u \leq u'\,\&\,v \leq v'). \tag{5.1.10}$$

For 1-D space (2-D space-time), the causality relation of special relativity is a lattice. The 2-D space-time has an interesting *lattice property.* To describe this property, let us recall a few definitions.

We say that an element e'' from an ordered set is an *upper bound* of two elements e an e' if $e \preccurlyeq e''$ and $e' \preccurlyeq e''$. For some pairs e and e', there exists the *least upper bound*, i.e., an upper bound e'' that precedes all other upper bounds. An ordered set for which every two elements have the least upper bound is called an *upper semi-lattice*.

Similarly, we say that an element e'' from an ordered set is a *lower bound* of two elements e an e' if $e'' \preccurlyeq e$ and $e'' \preccurlyeq e'$. For some pairs e and e', there exists the *greatest lower bound*, i.e., a lower bound e'' that is preceded by all other lower bounds. An ordered set for which every two elements have the greatest lower bound is called a *lower semi-lattice*.

An ordered set which is both an upper semi-lattice and a lower semi-lattice is called a *lattice*. The formula (5.1.10) shows that the 2-D space-time of special relativity is a lattice. Indeed, let us show that it is an upper semi-lattice. Let $e = (u, v)$ and $e' = (u', v')$ be two arbitrary events. Then, due to (5.1.10), the condition that an event $E = (U, V)$ is an upper bound for e and e' is equivalent to

$$(e \preccurlyeq E \,\&\, e' \preccurlyeq E) \Leftrightarrow (u \le U \,\&\, u' \le U \,\&\, v \le V \,\&\, v' \le V). \tag{5.1.11}$$

The value U is larger than both numbers u and u' if and only if it is larger than the largest of them $u'' \stackrel{\text{def}}{=} \max(u, u')$. Similarly, the value V is larger than both numbers v and v' if and only if it is larger than the largest of them $v'' \stackrel{\text{def}}{=} \max(v, v')$. Thus, the condition (5.1.11) is equivalent to

$$(e \preccurlyeq E \,\&\, e' \preccurlyeq E) \Leftrightarrow (u'' \le U \,\&\, v'' \le V), \tag{5.1.12}$$

i.e., to

$$(e \preccurlyeq E \,\&\, e' \preccurlyeq E) \Leftrightarrow e'' \preccurlyeq E, \tag{5.1.13}$$

where $e'' \stackrel{\text{def}}{=} (u'', v'')$. Thus, the event e'' is the desired least upper bound for the given events e and e' – and thus, the space-time of 2-D special relativity is indeed an upper semi-lattice.

The condition (5.1.13) can be reformulated in terms of future cones. Indeed, the relation $e \preccurlyeq E$ means that E belongs to the future cone C^+_e of the event e, and the relation $e' \preccurlyeq E$ means that E belongs to the future cone $C^+_{e'}$ of the event e'. Thus, the event E satisfies the condition $e \preccurlyeq E \,\&\, e' \preccurlyeq E$ from the left-hand side of this condition (5.1.13) if and only if it belongs to the intersection $C^+_e \cap C^+_{e'}$ of these two future cones. The condition (5.1.13) then states that this intersection coincides with the future cone $C^+_{e''}$ of some event e''.

The condition (5.1.13) can thus be formulated in the following way: for every two events e and e', the intersection $C^+_e \cap C^+_{e'}$ of their future cones is also a future cone, of some event e''; see Figure 5.2.

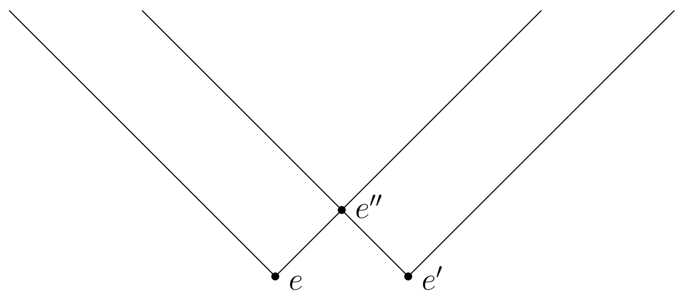

Figure 5.2: The intersection of two future cones is also a future cone: $C^+_e \cap C^+_{e'} = C^+_{e''}$.

Similarly, for every two events $e = (u, v)$ and $e' = (u', v')$, the event $e'' = (\min(u, u'), \min(v, v'))$ is the greatest lower bound. Thus, the space-time of 2-D special relativity is also a lower semi-lattice – and thus, a lattice.

The property of being a lower semi-lattice can also be reformulated in terms of the cones (this time in terms of past cones): for every two events e and e', the intersection $C^-_e \cap C^-_{e''}$ of their past cones is also a past cone, of some event e''; see Figure 5.3.

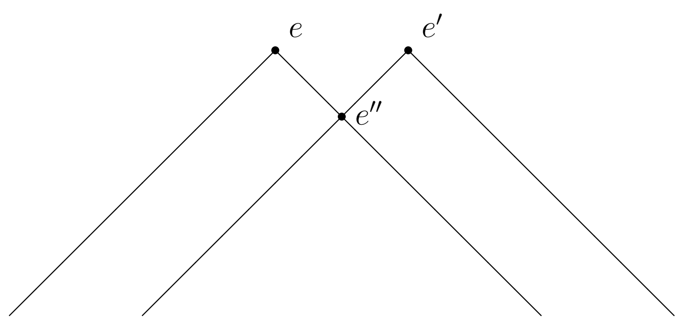

Figure 5.3: The intersection of two past cones is also a past cone: $C^-_e \cap C^-_{e'} = C^-_{e''}$.

In general, the causality relation of special relativity is not a lattice. For 3-D space (= 4-D space-time), the causality relation is no longer a lattice – since for 3-D future (and past) cones, the intersection of two cones is no longer a cone.

In the traditional relativistic physics, elementary particles are points. The *elementary* particle is something that cannot be further divided into parts, something that acts as a whole.

If we have a non-point body, then, due to the fact that any interaction can only spread with a speed of light, affecting one spatial point x in the body does not immediately affect other points $x' \neq x$: it takes time at least $\dfrac{d(x,x')}{c}$ for this disturbance to get to the point x'. Thus, in terms of the reaction to this perturbation, the points x and x' act separately. In line with the above meaning of an elementary particle, this means that the body containing these two points is not an elementary particle. So, an elementary particle should be a single point.

Point-wise particles lead to physically meaningless infinities. It is known that the existence of point particles lead to serious problems of "divergence" (see, e.g., [21]) – i.e., to the occurrence of physically meaningless infinite values.

Let us give one simple example: let us consider an electrically charged elementary particle (e.g., an electron or a proton), and let us estimate the total energy T of its electrostatic field. It is known that the energy density ρ of an electrostatic field E is proportional to E^2: $\rho(x) = c_1 \cdot E^2$ (for some constant c_1). According to Coulomb's law, the field generated by a charged point particle depends on the distance r to this particle as $c_2 \cdot r^{-2}$ (for some constant c_2). Thus, $\rho(x) = c_3 \cdot r^{-4}$, where we denoted $c_3 \stackrel{\text{def}}{=} c_1 \cdot c_2^2$. The total energy of this field can be thus computed by combining the energies of different parts of this field, as

$$\int_{\mathbb{R}^3} c_3 \cdot r^{-4}\, dx = c_3 \cdot \int_{\mathbb{R}^3} r^{-4}\, dx. \tag{5.1.14}$$

Since the field only depends on the distance, it make sense to use polar coordinates. In these coordinates, we have $dx = 4 \cdot \pi \cdot r^2 \cdot dr\, d\Omega$, where $d\Omega$ corresponds to angular coordinates the integral over which is 1, so

$$\int_{\mathbb{R}^3} r^{-4}\,dx = 4\pi \cdot \int_0^\infty r^2 \cdot r^{-4}\,dr = 4\pi \cdot \int_0^\infty r^{-2}\,dr. \tag{5.1.15}$$

This integral can be explicitly computed, as $\int r^{-1}\,dr = -r^{-1} = -\dfrac{1}{r}$, so

$$\int_{\mathbb{R}^3} r^{-4}\,dx = 4\pi \cdot \int_0^\infty r^{-2}\,dr = -4 \cdot \pi \cdot \left.\frac{1}{r}\right|_0^\infty = 4 \cdot \pi \cdot \left(\frac{1}{0} - \frac{1}{\infty}\right) = \infty. \tag{5.1.16}$$

This result is disturbing since, according to General Relativity, energy means mass, so an infinite energy would mean infinite mass – and thus, infinite gravitational attraction.

A similar problem occurs in quantum physics. One may conjecture that the problem appears since we consider classical (non-quantum) particles, thus assuming that we can measure everything – including the location of the elementary particle – with an arbitrary accuracy. In quantum mechanics, due to the uncertainty principle, there are limitations on measurement accuracy, so we may hope that the infinities will disappear. Alas, the same infinities appear in quantum theories as well [21].

Sometimes, there is a solution to this divergence problem. For some physical fields, there is a solution that enables us to avoid meaningless physical infinities. The main idea is that instead of starting with a point-wise particle of charge $q > 0$ and radius $r_0 = 0$, we start with a particle of finite radius r_0 and then tend r_0 to 0.

In electrodynamics, the coefficient c_2 is proportional to the charge q of the particle, so energy density is proportional to q^2. For a particle of radius $r_0 > 0$, the total energy is thus proportional to $\dfrac{q_0^2}{r_0}$. Thus, if we tend r_0 to 0 and then tend q to 0 accordingly, in the limit, we get a finite expression for the total energy. (This is just a raw idea, we also need to change other parameters to make sure that the observed electric charge of the particle is not 0.)

However, this trick (called *renormalization*) only works for some physical fields. For other fields, the equations are more complex, and it is not possible to avoid all the infinities by simply selecting proper values of the parameters.

Main idea of the string theory. String theory (see, e.g. [25, 74]) resolves the divergence problem by assuming that elementary particles are no point-wise, that each elementary particle occupies several different spatial locations.

In the original version of a theory, it was assumed that an elementary particle occupies all the points along a 1-D curve ("string"). Later, it turned out that higher-dimensional locations (called *M-branes*) are also reasonable. So, in general, we can say that an event involving an elementary particle cannot be described by a single spatio-temporal location (t, x), it involves a while *set* of such locations.

How does this assumption affect causality? Since the main reason for considering point particles was to preserve the standard causality, clearly, non-point particles change the causality relation. In this chapter, we analyze how causality is changed.

Causality relation between the particle and an event. Let S be the set of all the points in space-time corresponding to an event in the life of an elementary particle, and let e be a point event. Since the particle is elementary, if we affect any point $s \in S$, we thus influence the particle as a whole. Thus, we can say that e influences S if e can causally influence at least one event from S. So, we arrive at the following definition.

Definition 5.1. *Let $(X, \preccurlyeq)$ be an ordered set. We will call this set X* space-time*, its element* events*, and $\preccurlyeq$* causality relation*. For every event $e \in X$ and for every set $S \subseteq X$, we say that e* can influence *S (and denote it by $e \preccurlyeq S$ if $e \preccurlyeq s$ for some $s \in S$:*

$$e \preccurlyeq S \stackrel{\text{def}}{=} \exists s \in S \, (e \preccurlyeq s). \tag{5.1.17}$$

Our main result is that, with respect to this definition, sets S have the following lattice-like property:

Proposition 5.1. *Let $(X, \preccurlyeq)$ be a space-time. Then, for every two sets S and S', there exists a set S'' for which, for every $e \in X$,*

$$((e \preccurlyeq S) \,\&\, (e \preccurlyeq S')) \Leftrightarrow (e \preccurlyeq S''). \tag{5.1.18}$$

Comment. All the proofs are given in a special proofs section.

Similarly, we can say that S affects e if one of the points $s \in S$ affects e, then we get a similar result:

Definition 5.2. *Let $(X, \preccurlyeq)$ be a space-time. For every event $e \in X$ and for every set $S \subseteq X$, we say that S* can influence *e (and denote it by $S \preccurlyeq e$ if $s \preccurlyeq s$ for some $s \in S$:*

$$S \preccurlyeq e \stackrel{\text{def}}{=} \exists s \in S \, (s \preccurlyeq e). \tag{5.1.19}$$

Proposition 5.2. *Let $(X, \preccurlyeq)$ be a space-time. Then, for every two sets S and S', there exists a set S'' for which, for every $e \in X$,*

$$((S \preccurlyeq e) \,\&\, (S' \preccurlyeq e)) \Leftrightarrow (S'' \preccurlyeq e). \tag{5.1.20}$$

Discussion: this is not exactly the lattice property. In the formulas (5.1.17) and (5.1.18), we only consider relations between events $S \subseteq X$ and point-wise events $e \in X$. In principle, we can also define a similar relation $S \preccurlyeq S'$ between two different events $S, S' \subseteq X$:

$$S \preccurlyeq S' \stackrel{\text{def}}{=} \exists s \in S \, \exists s' \in S' \, (s \preccurlyeq s'). \tag{5.1.21}$$

This relation can be described in terms of the above relations $e \preccurlyeq S$ and $S \preccurlyeq e$:

$$S \preccurlyeq S' \stackrel{\text{def}}{=} \exists s \in S\,(s \preccurlyeq S'); \tag{5.1.22}$$

and

$$S \preccurlyeq S' \stackrel{\text{def}}{=} \exists s' \in S'\,(S \preccurlyeq s'). \tag{5.1.23}$$

However, in general, this new relation $S \preccurlyeq S'$ is no longer transitive (and thus, no longer an order; see, e.g., [78]): for example, even when X is the real line with the usual order, we have:

- $S = \{0, 1\} \preccurlyeq S' = \{-2, 2\}$ since $1 \preccurlyeq 2$ for $1 \in S$ and $2 \in S'$;
- $S' = \{-2, 2\} \preccurlyeq S'' = \{-1\}$ since $-2 \preccurlyeq -1$ for $-2 \in S'$ and $-1 \in S''$;
- however, $S = \{0, 1\} \not\preccurlyeq S'' = \{-1\}$, since neither of the two elements of S (0 and 1) precedes the only element of S'' (-1).

However, from Propositions 5.1 and 5.2, one can easily conclude that similar lattice-like properties hold for the relation $S \preccurlyeq S'$ as well:

Corollary 5.1. *Let $(X, \preccurlyeq)$ be a space-time. Then, for every two sets S and S', there exists a set S'' for which, for every $E \subseteq X$,*

$$((E \preccurlyeq S)\,\&\,(E \preccurlyeq S')) \Leftrightarrow (E \preccurlyeq S''). \tag{5.1.24}$$

Corollary 5.2. *Let $(X, \preccurlyeq)$ be a space-time. Then, for every two sets S and S', there exists a set S'' for which, for every $E \subseteq X$,*

$$((S \preccurlyeq E)\,\&\,(S' \preccurlyeq E)) \Leftrightarrow (S'' \preccurlyeq E). \tag{5.1.25}$$

Specifically, as S'', we can take the exact same sets as in Propositions 5.1 and 5.2.

Possible relation to Berwald-Moor Finsler causality. Several physicists are currently pursuing the idea that the actual causality relation may be different from the causality relation of special relativity, and that this actual relation is a lattice. Specifically, they conjecture that, similarly to the 2-D space-time case, in appropriate coordinates u_1, u_2, u_3, u_4, causality relation between two elements $u = (u_1, u_2, u_3, u_4)$ and $u' = (u'_1, u'_2, u'_3, u'_4)$ takes the form

$$u \preccurlyeq u' \Leftrightarrow ((u_1 \leq u'_1) \,\&\, (u_2 \leq u'_2) \,\&\, (u_3 \leq u'_3) \,\&\, (u_4 \leq u'_4)). \tag{5.1.26}$$

This research is related to a special *Berwald-Moor Finsler* related to this causality relation; see, e.g., [71, 72].

Our result provides a reasonable physical justification for the lattice character of the causality – and thus, for this research direction.

5.2 When a Special-Relativity-Type Ordered Space Is a Lattice

Special relativity: brief reminder. To uniquely describe an event, we need to describe the moment of time t at which it occurs and its spatial location x. In other words, an event can be characterized by a pair (t, x), where $t \in \mathbb{R}$ is a real number and x is an element of the metric space X describing the proper physical space.

Such pairs form a *space-time* $\mathbb{R} \times X$. How can we describe the causality relation $\preccurlyeq$ on this space-time, i.e., the relation $a \preccurlyeq b$ meaning that an event a can causally influence the event b?

According to special relativity, the speed of all processes is limited by the speed of light c. So, an event (t, x) can influenced an event (s, y) if $t \leq s$ and if it is possible for a signal from x to reach y in time $s-t$. During this time, the signal can cover at most the distance $c\cdot(s-t)$, so this condition can be expressed as $(t,x)\preccurlyeq(s,y)\Leftrightarrow d(x,y)\le c\cdot(s-t)$.

This condition can be simplified even further if, instead of using different units for measuring space and time, we use the same units for both, i.e., if we use, as a unit of distance, the distance c that the light covers in one second. In these new units, the numerical value of the speed of light is 1, so the causality relation takes the following simplified form:

$$(t,x)\preccurlyeq(s,y)\Leftrightarrow s-t\ge d(x,y). \tag{5.2.1}$$

Busemann product. In special relativity, the proper space X is a usual Euclidean space. Starting with general relativity, however, physicists realized that the actual space-time is curved. Thus, it is reasonable to consider space-time models $\mathbb{R}\times X$ with non-Euclidean metric spaces X and causality relation (5.2.1). Such models were first considered by H. Busemann [14] and are thus called *Busemann products* of the real line $\mathbb{R}$ and the metric space X (see also [40, 44]).

A natural question: when is the Busemann product $\mathbb{R}\times X$ a lattice? From the viewpoint of ordered spaces, a natural question is: when is the Busemann product a lattice?

In the simplest case of a 1-D Euclidean space (and thus, 2-D space-time) it is a lattice. Indeed, in this case, $d(x,y)=|x-y|$ and since $|z|=\max(z,-z)$, the relation $s-t\ge d(x,y)=|x-y|=\max(x-y,y-x)$ is equivalent to $s-t\ge x-y$ *and* $s-t\ge y-x$. By moving terms t and x related to the event (t,x) to one side of each of these inequalities, and terms s and y related to the event (s,y) to another side, we get an equivalent form: $s+y\ge t+x$ and $s-y\ge t-x$. So, if instead of the original coordinates t and x, we use new ("lightcone") coordinates $u=t+x$ and $v=t-x$, the ordering relation between two events (u,v) and (u',v') takes the form

$$(u,v)\preccurlyeq(u',v')\Leftrightarrow((u\le u')\,\&\,(v\le v')).$$

One can easily check that for this relation, every two elements (u,v) and (u',v') have the greatest lower bound (*meet*) $(u,v)\wedge(u',v')=(\min(u,u'),\min(v,v'))$ and least upper bound (*join*) $(u,v)\vee(u',v')=(\max(u,u'),\max(v,v'))$ – i.e., that it is indeed a lattice.

On the other hand, for the 3-D Euclidean space, the Busemann product – i.e., the causality relation of special relativity – is *not* a lattice. Indeed, for a lattice, the intersection of two *future cones*

$$C_a^+ = \{b : a \preccurlyeq b\} \text{ and } C_{a'}^+ = \{b : a' \preccurlyeq b\}$$

is also a future cone: namely, the future cone of the join $a \vee a'$. For special relativity, the future cone is, from the geometric viewpoint, an actual cone

$$(t, x_1, x_2, x_3) \preccurlyeq (s, y_1, y_2, y_3) \Leftrightarrow (s - t) \geq d(x, y) \Leftrightarrow$$

$$(s \geq t \,\&\, (s-t)^2 \geq d^2(x, y) = (x_1 - y_1)^2 + (x_2 - y_2)^2 + (x_3 - y_3)^2) \Leftrightarrow$$

$$(s \geq t \,\&\, (s-t)^2 - (x_1 - y_1)^2 - (x_2 - y_2)^2 - (x_3 - y_3)^2 \geq 0).$$

It is also easy to see that the intersection of two geometric cones is, in general, *not* a cone – and thus, this ordered space is not a lattice.

It is therefore reasonable to ask: when is a Busemann product a lattice? The definition of a lattice means that for every two elements, we have a meet and a join. If for every two elements, we have a meet, this is called a *lower semi-lattice*; if for every two elements, we have a join, this is called an *upper semi-lattice*. A lattice is thus an ordered space which is at the same time a lower and an upper semi-lattice. We can therefore also ask: when is a Busemann product a lower semi-lattice? an upper semi-lattice? In this chapter, we provide a necessary and sufficient condition for the Busemann product to be a lattice, a lower semi-lattice, and/or an upper semi-lattice.

Main result. The answer comes in terms of *real trees* (*R-trees*), i.e., metric spaces in which every two points x and y are connected by exactly one *arc* – a homeomorphic embedding of an interval into this space, and this arc is *geodesic*, i.e., is formed by points x_α, $\alpha \in [0, d(x, y)]$ for which $d(x_\alpha, x_\beta) = |\alpha - \beta|$; see, e.g., [9].

An example of a real tree is a *hedgehog set* – a collection of several intervals with a common starting point O, in which the distance on each interval is Euclidean, between two points x and y on different intervals is defined as $d(x, y) = d(x, O) + d(O, y)$.

Definition 5.3. *Let X be a metric space with distance d. A set $\mathbb{R} \times X$ with an ordering relation $(t, x) \preccurlyeq (s, y) \Leftrightarrow s - t \geq d(x, y)$ is called a Busemann product.*

Theorem 5.1. *For each metrics space X, the following conditions are equivalent to each other:*

- *the Busemann product $\mathbb{R} \times X$ is a lattice;*
- *the Busemann product $\mathbb{R} \times X$ is a lower semi-lattice;*
- *the Busemann product $\mathbb{R} \times X$ is an upper semi-lattice;*
- *the space X is a real tree.*

First open question: case of quasimetrics. In the main text, we only considered metric spaces X, in which $d(x, y) = d(y, x)$, but a similar construction of a Busemann product order can be described for a *quasimetric*, i.e., a function which is not necessarily symmetric; see, e.g., [44]. It is desirable to extend our results to such quasimetrics.

Second open question: more general Busemann products. The space $\mathbb{R} \times X$ is not just a ordered space: similarly to the case of special relativity, it can be equipped by a function describing proper time [14]:

$$\tau((t, x), (s, y)) = \sqrt[\alpha]{\max((s - t)^\alpha - d^\alpha(x, y), 0)}.$$

This function – called *kinematic metric* – satisfies the following two conditions:

- if $\tau(a, b) > 0$ then $b \preccurlyeq a$, and
- the *anti-triangle inequality:* if $c \preccurlyeq b \preccurlyeq a$, then $\tau(a, c) \geq \tau(a, b) + \tau(b, c)$.

For each ordered space E with a function τ that satisfies these two conditions, and for each metric space X, we can define a Busemann product as the following ordering relation of $E \times X$:

$$(s, y) \preccurlyeq (t, x) \Leftrightarrow \tau(t, s) \geq d(x, y).$$

It is desirable to analyze when this order is a lattice.

5.3 Modal Intervals as a New Logical Interpretation of the Usual Lattice Order Between Interval Truth Values

Before we explain the importance of lattice order in comparing truth values, let us briefly recall the need for truth values in general and for interval truth values in particular.

Traditional numerical truth values in fuzzy logic. In the traditional approach to fuzzy logic (see, e.g., [34, 66]), the degree of confidence ("truth value") of each statement is characterized by a number from the interval $[0, 1]$:

- the value 1 means that the expert is absolutely confident in this statement;
- the value 0 means that the expert is absolutely confident that this statement is false; and
- values between 0 and 1 describe typical situations when the expert has some degree of confidence in the statement, but he or she is not absolutely sure that this statement is true.

These degrees of confidence are easy to compare: if our degree of confidence d in a statement S is larger than our confidence d' in a statement S', this means that we have more confidence in the statement S than in the statement S'.

Need to go beyond numerical truth values. Zadeh's idea of using numerical values to describe degrees of confidence has led to many successful practical applications of fuzzy techniques [34, 66]. Many of these applications start with eliciting the corresponding degrees of confidence form the experts.

There are many different ways to elicit such degrees. For example, we can ask an expert to mark his or her confidence of a scale, e.g., on a scale from 0 to 5, 0 meaning no confidence at all, and 5 meaning absolute confidence. If an expert marks his or her confidence by 3, then we estimate the corresponding degree of confidence as $3/5 = 0.6$.

Another possibility is to poll several experts; if out of 10 experts, 7 believe that the statement is true, we take 7/10 as our degree of confidence in this statement. There are many other ways of eliciting the corresponding degrees.

At first glance, all these techniques provide a number that measures the expert's degree of confidence – the same way as the height in inches or centimeters measures the person's height. However, there is a big difference between these two types of measurements: if we measure the height of a person again and again, by using different rules, we get (more or less) the same value – describing the actual height of this person. In contrast, if we slightly different versions of the same elicitation techniques, we get somewhat different values.

For example, if we ask a person to mark his or her confidence on a scale from 0 to 5, then possible marks are 0, 1, 2, 3, 4, and 5, and the resulting degrees of confidence are $0/5 = 0$, $1/5 = 0.2$, $2/5 = 0.4$, $3/5 = 0.6$, $4/5 = 0.8$, and $5/5 = 1.0$. To get a better estimate, we can use a more detailed scale, e.g., the scale from 0 to 6. However, with the new scale, we get numbers $0/6 = 0$, $1/6$, $2/6 = 1/3$, $3/6 = 0.5$, $4/6 = 2/3$, $5/6$, and $6/6 = 1.0$. With the exception of 0 and 1, none of the previous values can appear in this new scale. So if, e.g., a person selected 3 on a scale from 0 to 5, and we got 0.6 as the degree of confidence, on a new scale, we may get values $3/6 = 0.5$ or $4/6 = 2/3 = 0.66\ldots$, but never the exact same value 0.6.

To avoid this problem, we could ask the expert to make his or her degree of confidence on a scale, for example, from 0 to 100, but this runs into a different problem: that people are rarely able to meaningfully distinguish between, e.g., values of 70 and 71 on this scale.

Similarly, when we poll 10 experts, we can only get values 0, 0.1, 0.2, $\ldots$, 1.0. If we want to get a more accurate estimate, we can ask one more expert, but the resulting values 0, $1/11$, $2/11$, $\ldots$, 1 are all different from the previous values – with the exception, of course, of the values 0 and 1.

In other words, the numerical values depend not only on the actual expert's degree of confidence, they also depend on the technique that was used to elicit these degrees. For example, the same value 0.5 coming from an on-a-scale-from-0-to-something elicitation can mean different things.

- It can mean that we got 1 on a scale from 0 to 2. In this scale, we basically consider three different options: 0 if we are confident that the statement is false, 2 if we are confident that the statement is true, and 1 in cases when we are uncertain. Thus, the fact that the expert selected 1 simply means that the expert is not certain about this statement, and it does not tell us much about the degree of this uncertainty.

- On the other hand, this same value 0.5 could mean that the expert selected 5 on a scale from 0 to 10. This is a completely different story. Here, the expert had 9 values describing uncertainty to choose from: 1, 2, ..., 9, and the fact that the expert selected the midpoint 5 and not any other value means that this expert probably has as many reasons to believe in the original statement as in its negation.

When we make decisions based on the expert's degrees of confidence in different statements, it is definitely desirable to take into account the difference between the above two situations. Since in both situations, we have the exact same numerical value 0.5 of the expert's uncertainty, this means that we need to go beyond the numerical truth values.

Interval truth values. A natural way to go beyond numerical truth values is to use *interval* truth values, when the expert's degree of confidence is described not by a number d from the interval [0,1], but rather by a subinterval $[\underline{d}, \overline{d}]$ of this interval [34, 54, 55, 66].

Indeed, when a person select 3 on a scale from 0 t0 5, this does not necessarily mean that his or her degree of confidence corresponds exactly to the value 3, it simply means that this degree is closer to 3 than to other marks (0, 1, 2, 4, and 5) on scale. Values which are closer to 3 than to all other integers are easy to describe: they form an interval $[2.5, 3.5]$. Based on our scale-from-0-to-5 request, we do not get the actual expert's degree of confidence, we only conclude that this actual (unknown) degree is between $2.5/5 = 0.5$ and $3.5/5 = 0.7$, i.e., that this degree is in the interval $[0.5, 0.7]$. It is therefore reasonable to return this interval as the available information about the expert's degree of confidence in a given statement.

Need to order interval truth values. The ultimate purpose of processing expert knowledge – and, in particular, processing degrees of belief in different statements – is to make decisions. Let us consider a simple example. Suppose that we want to achieve a certain objective. We know of two possible actions each of which can lead to this objective with some confidence, and we need to select the most promising action.

When the degree of confidence is described by a number, this problem is easy to solve: for each of the actions, we estimate the degree of confidence that this particular action will lead to the desired objective, and we select the action for which this degree is the largest possible.

However, when we use intervals to describe degrees of belief, it is not always clear which of the two actions is better. For example, suppose that for one of the actions, we have no information about its possible consequences. In this case, the interval-valued degree of belief is the whole interval $[0, 1]$. Suppose also that for the second action, we have some arguments for and against the success of this action, and we have exactly as many arguments for as we have arguments against. In this case, it is reasonable to take the midpoint 0.5 between 0 ("false") and 1 ("true") as the degree of belief in the second statement. Which one should we prefer?

How to extend functions and operations from numbers to intervals: general idea (a particular case of Zadeh's extension principle). We have an ordering relation between *numbers* a from the interval $[0, 1]$. We need to extend this relation to *subintervals* $[\underline{a}, \overline{a}]$ from this interval.

This is a problem typical in fuzzy techniques: we start with a function $f(x_1, \ldots, x_n)$ defined for real numbers, and we need to extend it to intervals $X_1, \ldots, X_n$ – or, more generally, to fuzzy numbers $X_1, \ldots, X_n$. A natural way towards such extension was developed by Lotfi Zadeh himself and is therefore known as Zadeh's extension principle.

With respect to intervals (and crisp sets in general) Zadeh's extension principle means the following. Suppose that we do not know the exact values x_i of the inputs. For each input i, we only know the set X_i of possible values. Then, a number y is a possible value of the function $f(x_1, \ldots, x_n)$ if and only if there are possible values $x_i \in X_n$ for which $y = f(x_1, \ldots, x_n)$. So, as an answer, we return the set Y of all such numbers y, i.e., the set

$$\{f(x_1, \ldots, x_n) : x_1 \in X_1, \ldots, x_n \in X_n\}.$$

This set is known as a *range* of the functions $f(x_1, \ldots, x_n)$ on intervals $X_1, \ldots, X_n$, and it is usually denoted by $f(X_1, \ldots, X_n)$. The task of computing such a range for different functions and different intervals constitutes so-called *interval computations*; see, e.g., [32, 59].

Let us show how the above idea can help us expand the ordering between numbers to ordering between intervals.

Possible representations of an ordering relation. It turns out that what exactly extension to intervals we get depends on how we represent the order. To show this, let us consider three possible representations.

- The first is the standard representation, in which $\leq$ is a function that transforms two numbers a and b into the truth value of the relation $a \leq b$. In other words, this function returns 1 if $a \leq b$ and it returns 0 if $a \not\leq b$. We call this representation standard, since our ultimate objective is to process all this in computers, and this is how ordering is represented in the computers.

- Instead of the ordering relation itself, we can consider functions $\max(a, b)$ and $\min(a, b)$. Each of these functions is also computer supported. Each of these functions describe the ordering:

  - once we have the function $\max(a, b)$, we can reconstruct the relation $a \leq b$ as $b = \max(a, b)$;

  - similarly, once we have the function $\min(a, b)$, we can reconstruct the relation $a \leq b$ as $a = \min(a, b)$.

Let us show how these representations lead to different interval extensions.

Zadeh's extension principle approach applied to the original ordering relation. The original function $\leq$ starts with two real numbers a and b and produces a (crisp) truth value, i.e., a number from the set $\{0, 1\}$ of crisp truth values. According to the general definition of Zadeh's extension principle, when we start with sets $\mathbf{a} = [\underline{a}, \overline{a}]$ and $\mathbf{b} = [\underline{b}, \overline{b}]$ of possible values of a and b, we thus get a *set* $\leq (\mathbf{a}, \mathbf{b})$ of truth values, i.e., a subset of the set $\{0, 1\}$. Based on the definition, we can distinguish three possible situations:

- if every element $a \in [\underline{a}, \overline{a}]$ is smaller than or equal than every element $b \in [\underline{b}, \overline{b}]$, then the set $\leq (\mathbf{a}, \mathbf{b})$ consists of only one value 1 (corresponding to "true");
- if none of the elements $a \in [\underline{a}, \overline{a}]$ is smaller than or equal than any element $b \in [\underline{b}, \overline{b}]$, then the set $\leq (\mathbf{a}, \mathbf{b})$ consists of only one value 0 (corresponding to "false");
- in all other case, the set $\leq (\mathbf{a}, \mathbf{b})$ contains both values 1 ("true") and 0 ("false'), i.e., we have $\leq (\mathbf{a}, \mathbf{b}) = \{0, 1\}$.

In other words, here, $\mathbf{a} \leq \mathbf{b}$ if and only every element $a \in \mathbf{a}$ is smaller than or equal to every element $b \in \mathbf{b}$:

$$\forall a \in \mathbf{a} \, \forall b \in \mathbf{b} \, (a \leq b).$$

This relation is easy to describe in terms of the endpoints of the intervals $\mathbf{a}$ and $\mathbf{b}$: namely, an element a is smaller than or equal to every element of the interval $[\underline{b}, \overline{b}]$ if and only if it is smaller than or equal to the smallest of these elements, i.e., the element $\underline{b}$.

Thus, the above condition is satisfied if and only if every element a of the interval $\mathbf{a}$ is smaller than or equal to $\underline{a}$.

Similarly, every element a from the interval $[\underline{a}, \overline{a}]$ is smaller than or equal to $\underline{b}$ if and only if the largest of possible values of a, i.e., the element $\overline{a}$, is smaller than or equal to $\underline{b}$. Thus,

$$[\underline{a}, \overline{a}] \preccurlyeq [\underline{b}, \overline{b}] \Leftrightarrow \overline{a} \leq \underline{b}.$$

Zadeh's extension principle applied to the function $\max(a, b)$**.** The function $\max(a, b)$ is non-strictly increasing in a and b, meaning that if $a \leq a'$ and $b \leq b'$, then

$\max(a,b) \le \max(a',b')$. Thus, when a is in the interval $[\underline{a},\overline{a}]$, and b is in the interval $[\underline{b},\overline{b}]$, we can conclude that:

- the smallest possible value of $\max(a,b)$ is attained when both a and b attain their smallest possible values, i.e., when $a=\underline{a}$ and $b=\underline{b}$; the corresponding value of the function $\max(a,b)$ is equal to $\max(\underline{a},\underline{b})$;

- the largest possible value of $\max(a,b)$ is attained when both a and b attain their largest possible values, i.e., when $a=\overline{a}$ and $b=\overline{b}$; the corresponding value of the function $\max(a,b)$ is equal to $\max(\overline{a},\overline{b})$.

Thus, the range $\max([\underline{a},\overline{a}],[\underline{b},\overline{b}])$ of the function $\max(a,b)$ on the intervals $[\underline{a},\overline{a}]$ and $[\underline{b},\overline{b}]$ is equal to

$$\max([\underline{a},\overline{a}],[\underline{b},\overline{b}]) = [\max(\underline{a},\underline{b}),\max(\overline{a},\overline{b})].$$

As we have mentioned, we can now define the relation $\mathbf{a} \le \mathbf{b}$ between intervals as $\mathbf{b} = \max(\mathbf{a},\mathbf{b})$. According to the above formula, this ordering relation has the following form:

$$[\underline{a},\overline{a}] \le [\underline{b},\overline{b}] \Leftrightarrow [\underline{b},\overline{b}] = [\max(\underline{a},\underline{b}),\max(\overline{a},\overline{b})] \Leftrightarrow$$

$$\underline{b} = \max(\underline{a},\underline{b}) \,\&\, \overline{b} = \max(\overline{a},\overline{b}) \Leftrightarrow$$

$$\underline{a} \le \underline{b} \,\&\, \overline{a} \le \overline{b}.$$

This relation – actively used in interval-valued fuzzy logic – is different from what we get by applying Zadeh's extension principle to the original ordering relation.

Zadeh's extension principle applied to the function $\min(a,b)$. The function $\min(a,b)$ is also non-strictly increasing in a and b, meaning that if $a \le a'$ and $b \le b'$, then $\min(a,b) \le \min(a',b')$. Thus, when a is in the interval $[\underline{a},\overline{a}]$, and b is in the interval $[\underline{b},\overline{b}]$, we can conclude that:

- the smallest possible value of $\min(a,b)$ is attained when both a and b attain their smallest possible values, i.e., when $a=\underline{a}$ and $b=\underline{b}$; the corresponding value of the function $\max(a,b)$ is equal to $\min(\underline{a},\underline{b})$;

- the largest possible value of $\min(a, b)$ is attained when both a and b attain their largest possible values, i.e., when $a = \overline{a}$ and $b = \overline{b}$; the corresponding value of the function $\max(a, b)$ is equal to $\min(\overline{a}, \overline{b})$.

Thus, the range $\min([\underline{a}, \overline{a}], [\underline{b}, \overline{b}])$ of the function $\min(a, b)$ on the intervals $[\underline{a}, \overline{a}]$ and $[\underline{b}, \overline{b}]$ is equal to

$$\min([\underline{a}, \overline{a}], [\underline{b}, \overline{b}]) = [\min(\underline{a}, \underline{b}), \min(\overline{a}, \overline{b})].$$

As we have mentioned, we can now define the relation $\mathbf{a} \preccurlyeq \mathbf{b}$ between intervals as $\mathbf{a} = \min(\mathbf{a}, \mathbf{b})$. According to the above formula, this ordering relation has the following form:

$$[\underline{a}, \overline{a}] \preccurlyeq [\underline{b}, \overline{b}] \Leftrightarrow [\underline{a}, \overline{a}] = [\min(\underline{a}, \underline{b}), \min(\overline{a}, \overline{b})] \Leftrightarrow$$

$$\underline{a} = \min(\underline{a}, \underline{b}) \,\&\, \overline{a} = \min(\overline{a}, \overline{b}) \Leftrightarrow \underline{a} \le \underline{b} \,\&\, \overline{a} \le \overline{b}.$$

This relation is exactly the same as we obtained from the function $\max(a, b)$, and it is therefore different from what we get by applying Zadeh's extension principle to the original ordering relation.

Comment. Operations $\max(a, b)$ and $\min(a, b)$ form a lattice, so the corresponding ordering relation can be called a *lattice relation.*

Problem: how to interpret the lattice order in logical terms? Our objective is to develop the corresponding logic. It is therefore desirable to have a logical interpretation of the resulting ordering between intervals. For the first ordering relation – obtained by applying Zadeh's extension principle directly to the order between real numbers – we have a straightforward logical interpretation. However, for the lattice order, we do not have such a direct logical interpretation.

What we do in this section. In this section, we show that *modal intervals* (see, e.g., [24, 42]) – a practice-motivated generalization of intervals – provide the desired logical explanation for the lattice order. To provide such an explanation, we first need to recall what are modal intervals.

Traditional interval computations: reminder. Let us assume that a quantity z depends on quantities $x = (x_1, \ldots, x_n)$, and that we know the exact form of this dependence, i.e., we know a continuous function $z = f(x) = f(x_1, \ldots, x_n)$. In practice, we often do not know the exact values of the quantities x_i, we only know the intervals $X_i = [\underline{x}_i, \overline{x}_i]$ that contain these values.

These intervals may come from *measurements:* when the measurement result is $\widetilde{x}_i$ and we know the upper bound Δ_i on (absolute value of) the measurement error $\Delta x_i \stackrel{\text{def}}{=} \widetilde{x}_i - x_i$, this means that the actual (unknown) value x_i can take any value from the interval $[\widetilde{x}_i - \Delta_i, \widetilde{x}_i + \Delta_i]$. These intervals can also come from *manufacturing tolerances*, when we recommend the value $\widetilde{x}_i$ of the corresponding quantity but allow deviations $\pm\Delta_i$ from this recommended value. In this case also, the resulting the resulting quantity x_i can take any value from the interval $[\widetilde{x}_i - \Delta_i, \widetilde{x}_i + \Delta_i]$.

In both cases, the only information that we have about z is that z belongs to the interval

$$Z = \{f(x_1, \ldots, x_n) : x_1 \in X_1, \ldots, x_n \in X_n\} = \left[\min_{x \in X} f(x), \max_{x \in X} f(x)\right],$$

where we denoted $X \stackrel{\text{def}}{=} X_1 \times \ldots X_n$. This interval Z is called the *result of applying the function* f to the intervals $X_1, \ldots, X_n$ and denoted by $f(X_1, \ldots, X_n)$.

In many practical situations, it is desirable to make the interval Z as narrow as possible. For example, z may be the direction of the airplane flight, and we want to maintain this direction as accurately as possible. In the above setting, if we want to decrease the width Z, we have to decrease the width of the original intervals – e.g., measure the values x_i more accurately, or impose stricter tolerances on the manufacturing process.

Logical reformulation of the traditional interval computation. First, we need to make sure that for all possible combinations of $x_i \in X_i$, the value $z = f(x_1, \ldots, x_n)$ is contained in the interval Z. In other words, we want to make sure that

$$\forall x_1 \in X_1 \ldots \forall x_n \in X_n \, \exists z \in Z \, (z = f(x_1, \ldots, x_n)).$$

Second, we need to make sure that Z is the narrowest interval with this property. These two requirements guarantee that Z is equal to the above range: $Z = f(X)$.

Beyond the main problem of (traditional) interval computations – possibility of controlled variables: formulation of the problem. In the traditional approach, we have no control over the values of the input variables x_i, we only know that these values belong to the corresponding intervals X_i. In practice, often, the desired value $z = f(x, u)$ depends not only the variables $x = (x_1, \ldots, x_n)$ over which we have no control, it also depends on the additional variables $u = (u_1, \ldots, u_m)$ that we *can* control. Specifically, for each of these additional variables u_j, there is a range U_j, and we can set up any value within this range. We can use these additional variables to narrow down the range $Z = [\underline{z}, \overline{z}]$ of the values z that can be achieved.

In precise terms, we want to select an interval $Z = [\underline{z}, \overline{z}]$ for which, for each combination $x \in X$, there exists a control u that would lead to the value $f(x, u) \in Z$. Among all such intervals Z, we want to select the one that is the narrowest. In other words, we want to make sure that

$$\forall x \in X\, \exists u \in U\, (f(x, u) \in Z),$$

i.e., that

$$\forall x \in X\, \exists u \in U\, \exists z \in Z\, (z = f(x, u)),$$

and that Z is the narrowest interval with this property.

How can we find such an interval Z?

Possibility of controlled variables: towards a solution to the problem. For each $x \in X$, the set of all possible values $f(x, u)$ forms an interval

$$F(x) \stackrel{\text{def}}{=} \left[\min_{u \in U} f(x, u), \max_{u \in U} f(x, u)\right].$$

The existence of a control u for which one of these values is from the interval Z is equivalent to requiring that that the intervals $F(x)$ and Z have a common point. One can easily check that the two intervals $[\underline{a}, \overline{a}]$ and $[\underline{b}, \overline{b}]$ have a common point if and only if $\underline{a} \le \overline{b}$ and $\underline{b} \le \overline{a}$. For intervals $F(x)$ and Z, this means that we must have

$$\min_{u \in U} f(x, u) \le \overline{z} \text{ and } \underline{z} \le \max_{u \in U} f(x, u).$$

These two inequalities much hold for every $x \in X$. For $\overline{z}$, this means that the value $\overline{z}$ must be larger than or equal to $\min\limits_{u \in U} f(x, u)$ for all $x \in X$. This is equivalent to requiring that $\overline{z}$ is larger than or equal to the largest of these values, i.e., that

$$\overline{z} \geq \max_{x \in X} \min_{u \in U} f(x, u).$$

Similarly, the requirement that $\underline{z}$ must be smaller than or equal to $\max\limits_{u \in U} f(x, u)$ for all $x \in X$ is equivalent to requiring that $\underline{z}$ is smaller than or equal to the smallest of these values, i.e., that

$$\underline{z} \leq \min_{x \in X} \max_{u \in U} f(x, u).$$

Among all the intervals that satisfy these two inequalities, we need to find the narrowest. It turns out that the selection of the narrowest interval depends on the relation between the two bounds. If

$$\min_{x \in X} \max_{u \in U} f(x, u) \leq \max_{x \in X} \min_{u \in U} f(x, u),$$

then the narrowest interval is when $\overline{z}$ is equal to its lower bound and $\underline{z}$ is equal to its upper bound, i.e., when

$$Z = [\underline{z}, \overline{z}] = \left[\min_{x \in X} \max_{u \in U} f(x, u), \max_{x \in X} \min_{u \in U} f(x, u)\right].$$

On the other hand, if the opposite inequality is satisfied, i.e., if

$$\min_{x \in X} \max_{u \in U} f(x, u) > \max_{x \in X} \min_{u \in U} f(x, u),$$

then we can have intervals Z with the desired property which have width 0: namely, for any value z between these two bounds, i.e., for any value z from the interval

$$Z = \left[\max_{x \in X} \min_{u \in U} f(x, u), \min_{x \in X} \max_{u \in U} f(x, u)\right],$$

the one-point interval $Z' = [z, z]$ satisfies the desired property.

Thus, we arrive at the following solution.

Case of controlled variables: solution. Once we have a function $f(x, u)$ and the ranges X and U, we compute the two values

$$z^- = \min_{x \in X} \max_{u \in U} f(x, u) \text{ and } z^+ = \max_{x \in X} \min_{u \in U} f(x, u).$$

If $z^- \le z^+$, then the interval $Z = [z^-, z^+]$ is the narrowest interval for which

$$\forall x \in X \, \exists z \in Z \, \exists u \in U \, (z = f(x, u)).$$

If $z^- > z^+$, then we have many such narrowest intervals – namely, every interval $[z, z]$ for $z \in [z^+, z^-]$ is a one. This can be described as follows:

$$\forall x \in X \, \forall z \in Z \, \exists u \in U \, (z = f(x, u)).$$

Comment. The above solution is presented in [24], where the pair consisting of the values z^- and z^+ is called an f^*-extension of the original function $f(x, u)$.

Reformulation in terms of modal intervals. In [24], logical terms are used to distinguish between intervals X_i over which we have no control and intervals U_j in which we can select whichever value $u_i \in U_i$ we choose. To guarantee that the value z of the desired quantity is within the given range, we need to make sure that this property holds *for all* possible values $x_i \in X_i$, while for the controlled intervals, it is sufficient to require that *there exist* values $u_j \in U_j$ that make this property true. To emphasize this distinction, the authors of [24] treat each interval as a pair of the interval itself and of the corresponding quantifier:

- a traditional interval X_i is considered as a pair $\langle X_i, \forall \rangle$, while
- a controlled interval is considered as a pair $\langle U_j, \exists \rangle$.

Such pairs are called *modal intervals.*

In these terms, the condition

$$\forall x_1 \in X_1 \, \ldots \, \forall x_n \in X_n$$

$$\exists u_1 \in U_1 \, \ldots \, \exists u_m \in U_m \; \exists z \in Z \, (z = f(x, u))$$

can be reformulated as

$$Q_1 x_1 \in X_1 \, \ldots \, Q_n x_n \in X_n$$

$$Q'_1 u_1 \in U_1 \, \ldots \, Q'_m u_m \in U_m \; \exists z \in Z \, (z = f(x, u)),$$

where Q_i and Q'_j are the quantifiers attached to the corresponding intervals. For the case when all the intervals are traditional (non-controlled), we get the usual expression for the range. Because of this example, we can treat the resulting interval Z as the range defined over modal intervals:

$$Z = f(\langle X_1, \forall \rangle, \ldots, \langle X_n, \forall \rangle, \langle U_1, \exists \rangle, \ldots, \langle U_m, \exists \rangle).$$

The difference between the cases $z^- \le z^+$ and $z^- > z^+$ translates, as we have seen, into the difference between $\exists z \in Z$ and $\forall z \in Z$ in the corresponding formulas. So, the authors of [24] say that when $z^- \le z^+$, the range is the usual interval $\langle Z, \forall \rangle$, while for $z^- > z^+$, the range is the interval $\langle Z, \exists \rangle$.

Relation to Kaucher intervals. The above example shows that the difference between the two types of intervals can also be represented as the difference between the usual intervals, for which $z^- \le z^+$, and the "new" intervals for which $z^- > z^+$. It is therefore reasonable to represent these "new intervals" as $[z^-, z^+]$.

For example, the interval $Z = \langle [2, 4], \forall \rangle$ is represented as a usual interval $[2, 4]$, while an interval $\langle [2, 4], \exists \rangle$ is represented as $[4, 2]$. Such intervals have been previously introduced by Kaucher.

This connection with Kaucher intervals is not accidental: indeed, for arithmetic operations $f(x, u)$, the f^*-extensions coincide with the operations of Kaucher arithmetic.

Modal intervals explain lattice order: main idea. As we have mentioned, when we apply Zadeh's extension principle – i.e., the usual range estimation formula – to the function $\le (a, b)$, we get the relation $\overline{a} \le \underline{b}$ that corresponds to the logical formula

$$\forall a \in \mathbf{a} \, \forall b \in \mathbf{b} \, (a \le b).$$

Our main idea is to consider situations when, instead of one the original intervals $[\underline{a}, \overline{a}]$ and $[\underline{b}, \overline{b}]$, we consider the "dual" intervals $[\overline{a}, \underline{a}]$ and $[\overline{b}, \underline{b}]$.

As we have mentioned earlier, replacing an interval by a dual one means that we replace the corresponding universal quantifier with an existential one. Thus, we get the following two formulas:

$$\forall a \in \mathbf{a}\, \exists b \in \mathbf{b}\, (a \leq b)$$

and

$$\forall b \in \mathbf{b}\, \exists a \in \mathbf{a}\, (a \leq b).$$

Let us consider these formulas one by one.

First formula. For each a, the existence of $b \in [\underline{b}, \overline{b}]$ for which a is smaller than or equal to b is equivalent to a being smaller than or equal to the largest possible element $\overline{b}$ of the b-interval. Indeed:

- if $a \leq b$ for some b for which $\underline{b} \leq b \leq \overline{b}$ then, by transitivity, we get $a \leq \overline{b}$;
- vice versa, if $a \leq \overline{b}$, then $a \leq b$ for some $b \in [\underline{b}, \overline{b}]$: namely, for $b = \overline{b}$.

Now, the first formula can be equivalently formulated as follows: every value a from the interval $[\underline{a}, \overline{a}]$ is smaller than or equal to $\overline{b}$. Similarly to the previous paragraph, it is sufficient to check this property for the largest possible value $\overline{a}$ of the quantity a. Indeed:

- if $\overline{a} \leq \overline{b}$, this implies that for every value $a \leq \overline{a}$, we have $a \leq \overline{b}$;
- vice versa, if every number a from the interval $[\underline{a}, \overline{a}]$ satisfies the inequality $a \leq \overline{b}$, then, in particular, this inequality holds for the value $\overline{a} \in [\underline{a}, \overline{a}]$, i.e., we have

$$\overline{a} \leq \overline{b}.$$

Thus, the first formula is equivalent to $\overline{a} \leq \overline{b}$.

Second formula. For each b, the existence of $a \in [\underline{a}, \overline{a}]$ for which a is smaller than or equal to b is equivalent to b being larger than or equal to the smallest possible element $\overline{a}$ of the a-interval. Indeed:

- if $a \le b$ for some a for which $\underline{a} \le a \le \overline{a}$ then, by transitivity, we get $\underline{a} \le b$;
- vice versa, if $\underline{a} \le b$, then $a \le b$ for some $a \in [\underline{a}, \overline{a}]$: namely, for $a = \overline{a}$.

Now, the first formula can be equivalently formulated as follows: every value b from the interval $[\underline{b}, \overline{b}]$ is larger than or equal to $\underline{a}$. Similarly to the previous paragraph, it is sufficient to check this property for the smallest possible value $\underline{b}$ of the quantity b. Indeed:

- if $\underline{a} \le \underline{b}$, this implies that for every value $b \ge \underline{b}$, we have $\underline{a} \le b$;
- vice versa, if every number b from the interval $[\underline{b}, \overline{b}]$ satisfies the inequality $\underline{a} \le b$, then, in particular, this inequality holds for the value $\underline{b} \in [\underline{b}, \overline{b}]$, i.e., we have
$$\underline{a} \le \underline{b}.$$

Thus, the second formula is equivalent to $\underline{a} \le \underline{b}$.

Combining the two formulas: the resulting logical interpretation. The first formula is equivalent to $\overline{a} \le \overline{b}$, the second formula is equivalent to $\underline{a} \le \underline{b}$. Thus, the two formulas together are equivalent to lattice order. So, we get the desired logical interpretation of the lattice order.

This interpretation can be described – as with modal logic – in control-type terms. Namely, the order $\underline{a} \le \overline{b}$ means that every element $a \in \mathbf{a}$ is smaller than or equal to every element $b \in \mathbf{b}$. In contrast, the lattice order is equivalent to the following two statements:

- no matter what the actual value $a \in \mathbf{a}$ is, once we know this value, we can always select $b \in \mathbf{b}$ for which $a \le b$;
- vice versa, no matter what the actual value $b \in \mathbf{b}$ is, once we know this value, we can always select $a \in \mathbf{a}$ for which $a \le b$.

Comment: possible generalizations of this interpretation. In the above text, we considered intervals from the real line. In this case, the relation

$$[\underline{a}, \overline{a}] \preccurlyeq [\underline{b}, \overline{b}] \Leftrightarrow (\underline{a} \le \underline{b} \,\&\, \overline{a} \le \overline{b})$$

forms a *lattice* – in the sense that for every two intervals, there is the least upper bound and the greatest lower bound. A similar definition can be formulated for a more general case, when we consider intervals

$$[a,b] \stackrel{\text{def}}{=} \{x : a \preccurlyeq x \preccurlyeq b\}$$

over an arbitrary partially ordered set. In this case, the above relation is not longer a lattice, but we can still prove that it is equivalent to

$$\forall a \in \mathbf{a}\, \exists b \in \mathbf{b}\, (a \preccurlyeq b) \text{ and } \forall b \in \mathbf{b}\, \exists a \in \mathbf{a}\, (a \preccurlyeq b).$$

5.4 Proofs

Proofs of Propositions 5.1 and 5.2. Without losing generality, it is sufficient to prove Proposition 5.1. For every set S, let us define its *past cone* as the set of all the events that precede S: $C^-_S \stackrel{\text{def}}{=} \{e : e \preccurlyeq S\}$. Let us prove that the desired formula (5.1.18) holds for $S'' = C^-_S \cap C^-_{S'}$.

If $e \preccurlyeq S$ and $e \preccurlyeq S'$, then, by definition of the past cone, we conclude that $e \in C^-_S$ and $e \in C^-_{S'}$. Thus, the event e belongs to the intersection $S'' = C^-_S \cap C^-_{S'}$ of these past cones: $e \in S''$. Since $\preccurlyeq$ is an order relation, we have $e \preccurlyeq e$, so $e \preccurlyeq s''$ for some $s'' \in S''$ – namely, for $s'' = e$. Thus, $e \preccurlyeq S''$.

Vice versa, let us assume that $e \preccurlyeq S''$. By definition, this means that $e \preccurlyeq s''$ for some element $s'' \in S'' = C^-_S \cap C^-_{S'}$. Since the element s'' belongs to the intersection of the two past cones, it thus belongs to both of them: $s'' \in C^-_S$ and $s'' \in C^-_{S'}$. By definition of the past cone, the condition $s'' \in C^-_S$ means that $s'' \preccurlyeq S$, i.e., that $s'' \preccurlyeq s$ for some $s \in S$. Now, from $e \preccurlyeq s''$ and $s'' \preccurlyeq s$, we conclude that $e \preccurlyeq s$ for $s \in S$. This means that $e \preccurlyeq S$. Similarly, from $e \preccurlyeq s''$ and $s'' \in C^-_{S'}$, we conclude that $e \preccurlyeq S'$.

The proposition is proven.

Proof of Theorem 5.1.

1°. In this proof, we will use the following equivalent characterization of real trees: a metric space X is a real tree if and only if the following two conditions are satisfied:

- every two points x and y can be connected by a geodesic arc, and
- for every point x_α on the geodesic arc connecting x and y, and for every other point z, either x_α lies on a geodesic arc connecting x and z, or x_α lies on a geodesic arc connecting y and z.

These conditions are intuitively clear: when we go from x to z in a tree, we may follow the geodesic arc from x to y for a while, but there is a branching point at which the geodesic arcs deviate.

- If this branching point is after x_α, then x_α is on the geodesic arc from x to y; see Figure 5.4.

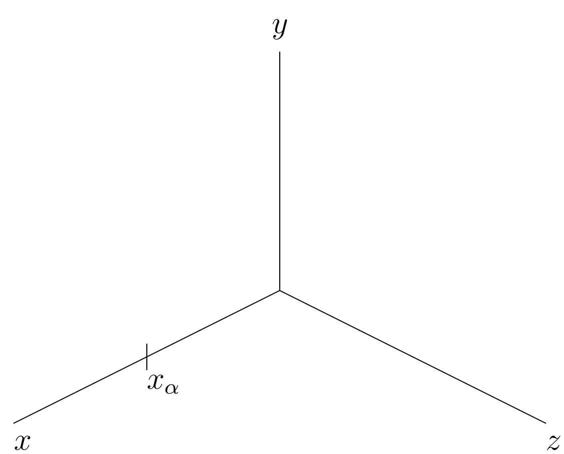

Figure 5.4: x_α is on the geodesic arc from x to y

- If the branching happens before x_α, then the geodesic arc from z to y should go pass x_α – otherwise, the geodesic arcs from x to y, from y to z, and from x to z would form a loop, which cannot happen in a tree; see Figure 5.5.

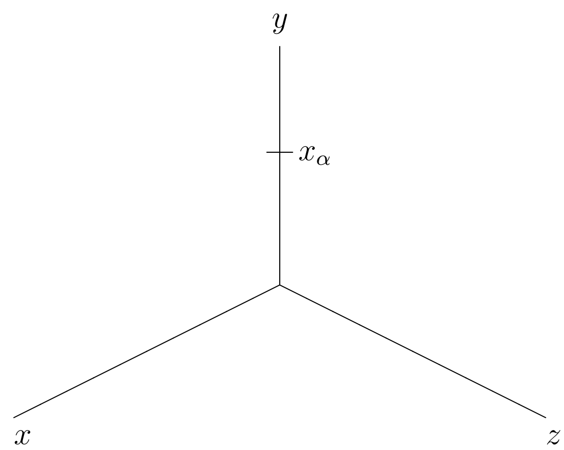

Figure 5.5: x_α is on the geodesic arc from z to y

2°. First, let us prove that if $\mathbb{R} \times X$ is a lower semi-lattice, then X is a real tree. For the upper semi-lattice, the proof is similar.

3°. Let us first prove that for every two points $x, y \in X$, and for every $\alpha \in (0, d(x,y))$, there exists a point x_α for which $d(x, x_\alpha) = \alpha$ and $d(x_\alpha, y) = d(x,y) - \alpha$.

Indeed, let us consider the following four points: (α, x), $(d(x,y) - \alpha, y)$, $(-\alpha, x)$, and $(\alpha - d(x,y), y)$. By using the definition of the Busemann product order, we can easily check that each of the first two points follows each of the second two points:

$$(-\alpha, x) \preccurlyeq (\alpha, x), \quad (\alpha - d(x,y), y) \preccurlyeq (\alpha, x),$$

$$(-\alpha, x) \preccurlyeq (d(x,y) - \alpha, y), \quad (\alpha - d(x,y), y) \preccurlyeq (d(x,y) - \alpha, y).$$

Since the Busemann product $\mathbb{R} \times X$ is a lower semi-lattice, the first two points (α, x) and $(d(x,y) - \alpha, y)$ have a meet, i.e., a point $(s, z) \stackrel{\text{def}}{=} (\alpha, x) \wedge (d(x,y) - \alpha, y)$ which precedes both of them and which follows both of the points $(-\alpha, x)$ and $(\alpha - d(x,y), y)$:

$$(s, z) \preccurlyeq (\alpha, x), \quad (s, z) \preccurlyeq (d(x,y) - \alpha, y),$$

$$(-\alpha, x) \preccurlyeq (s, z), \quad (\alpha - d(x, y), y) \preccurlyeq (s, z).$$

These ordering relations mean that the following four inequalities are satisfied:

$$\alpha - s \geq d(x, z); \quad d(x, y) - \alpha - s \geq d(y, z);$$

$$s + \alpha \geq d(x, z); \quad s + d(x, y) - \alpha \geq d(y, z).$$

Adding the first and the third inequalities and dividing the sum by two, we conclude that $\alpha \geq d(x, z)$. Similarly, by adding the second and the fourth inequalities and dividing the sum by two, we conclude that $d(x, y) - \alpha \geq d(y, z)$.

We cannot have strict inequality in any of these two inequalities, because if, e.g., $\alpha > d(x, z)$, then by adding it to $d(x, y) - \alpha \geq d(y, z)$, we could conclude that $d(x, y) > d(x, z) + d(y, z)$ – which contradicts to the triangle inequality. Thus, we must have equality, i.e., we must have $d(x, z) = \alpha$ and $d(y, z) = d(x, y) - \alpha$. The statement is proven: the point z is our desired point x_α.

In this case, from $d(x, z) = \alpha$ and $\alpha - s \geq d(x, z)$, we conclude that $s \leq 0$. Similarly, from $d(x, z) = \alpha$ and $\alpha + s \geq d(x, z)$, we conclude that $s \geq 0$. Since $s \leq 0$ and $s \geq 0$, we have $s = 0$. Thus, $(\alpha, x) \wedge (d(x, y) - \alpha, y) = (0, z)$.

4°. Let us now prove that for every two points $x, y \in X$, and for every $\alpha \in (0, d(x, y))$, there exists only one point x_α for which $d(x, x_\alpha) = \alpha$ and $d(x_\alpha, y) = d(x, y) - \alpha$.

Indeed, we have already shown that one such point exists – the point x_α for which $(\alpha, x) \wedge (d(x, y) - \alpha, y) = (0, x_\alpha)$. Let us assume that for some point $x'_\alpha \neq x_\alpha$, we have $d(x, x'_\alpha) = \alpha$ and $d(x'_\alpha, y) = d(x, y) - \alpha$. Then, by definition of the Busemann product order, we have $(0, x'_\alpha) \preccurlyeq (\alpha, x)$ and $(0, x'_\alpha) \preccurlyeq (d(x, y) - \alpha, y)$. By the definition of a meet, we then conclude that $(0, x'_\alpha) \preccurlyeq (0, x_\alpha)$. By definition of the Busemann product order, this means that $0 \geq d(x_\alpha, x'_\alpha)$, i.e., that $d(x_\alpha, x'_\alpha) = 0$ and $x_\alpha = x'_\alpha$. Uniqueness is proven.

5°. Now, we can conclude that every two points $x, y \in X$ are connected by a geodesic arc.

We have already shown that for every α, there exists a unique point x_α for which $d(x, x_\alpha) = \alpha$ and $d(x_\alpha, y) = d(x, y) - \alpha$. We want to prove that these points x_α form a geodesic arc, i.e., that for every $\alpha < \beta$, we have $d(x_\alpha, x_\beta) = \beta - \alpha$. Indeed, due to Part 1 of this proof, if we take the points x_α and y with $d(x_\alpha, y) = d(x, y) - \alpha$, then there exists a point x'_β for which $d(x_\alpha, x'_\beta) = \beta - \alpha$ and

$$d(x'_\beta, y) = d(x_\alpha, y) - (\beta - \alpha) = (d(x, y) - \alpha) - (\beta - \alpha) = d(x, y) - \beta.$$

Due to the triangle inequality,

$$d(x, x'_\beta) \leq d(x, x_\alpha) + d(x_\alpha, x'_\beta) \leq \alpha + (\beta - \alpha) = \beta,$$

so $d(x, x'_\beta) \leq \beta$. We cannot have $d(x, x'_\beta) < \beta$, since then we would have

$$d(x, y) \leq d(x, x'_\beta) + d(x'_\beta, y) < \beta + (d(x, y) - \beta) < d(x, y),$$

i.e., $d(x, y) < d(x, y)$, a contradiction. Thus, we have $d(x, x'_\beta) = \beta$ and $d(x'_\beta, y) = d(x, y) - \beta$. Due to Part 2 of our proof, this means that $x'_\beta = x_\beta$. Thus, $d(x_\alpha, x'_\beta) = \beta - \alpha$ implies that $d(x_\alpha, x_\beta) = \beta - \alpha$. The statement is proven.

6°. Let us now prove that for every $x, y \in X$, for every $\alpha \in (0, d(x, y))$, and for every point $z \in X$, we have either $d(x, z) = d(x, x_\alpha) + d(x_\alpha, z)$ or $d(y, z) = d(y, x_\alpha) + d(x_\alpha, z)$. In other words, x_α either lies on a geodesic arc connecting x and z or on a geodesic arc connecting y and z. This would mean that X is a real tree.

Indeed, we know that $(\alpha, x) \wedge (d(x, y) - \alpha, y) = (0, x_\alpha)$. Let us find s for which $(-s, z) \preccurlyeq (\alpha, x)$ and $(-s, z) \preccurlyeq (d(x, y) - \alpha, y)$. The first desired relation means that $\alpha + s \geq d(x, z)$, i.e., that

$$s \geq d(x, z) - \alpha = d(x, z) - d(x, x_\alpha).$$

The second relation means that $d(x, y) - \alpha + s \geq d(y, z)$, i.e., that

$$s \geq d(y, z) - (d(x, y) - \alpha) = d(y, z) - d(y, x_\alpha).$$

So, if we take

$$s = \max(d(x, z) - d(x, x_\alpha), d(y, z) - d(y, x_\alpha)),$$

both inequalities will be satisfied and thus, we will have $(-s,z)\preccurlyeq(\alpha,x)$ and

$$(-s,z)\preccurlyeq(d(x,y)-\alpha,y).$$

By definition of the meet, this means that $(-s,z)\preccurlyeq(0.x_\alpha)$, i.e., that $s\ge d(x_\alpha,z)$. The value s is defined as the largest of the two expressions, so it is equal to one of them.

If s is equal to the first expression $s=d(x,z)-d(x,x_\alpha)$, then the above inequality $s\ge d(x_\alpha,z)$ takes the form $d(x,z)-d(x,x_\alpha)\ge d(x_\alpha,z)$, i.e., equivalently, $d(x,z)\ge d(x,x_\alpha)+d(x_\alpha,z)$. Since by the triangle inequality, we have $d(x,z)\le d(x,x_\alpha)+d(x_\alpha,z)$, we thus conclude that $d(x,z)=d(x,x_\alpha)+d(x_\alpha,z)$.

If s is equal to the second expression $s=d(y,z)-d(y,x_\alpha)$, then the above inequality $s\ge d(x_\alpha,z)$ takes the form $d(y,z)-d(y,x_\alpha)\ge d(x_\alpha,z)$, i.e., equivalently, $d(y,z)\ge d(y,x_\alpha)+d(x_\alpha,z)$. Since by the triangle inequality, we have $d(y,z)\le d(y,x_\alpha)+d(x_\alpha,z)$, we thus conclude that $d(y,z)=d(y,x_\alpha)+d(x_\alpha,z)$.

The statement is proven.

7°. To complete our proof, we need to show that if X is a real tree, then the Busemann product is a lattice.

Let us assume that the metric space X is a real tree, and let us consider two points (t,x) and (s,y) in the Busemann product $\mathbb{R}\times X$. Let us show that the meet of these points exists (for the join, the proof is similar).

7.1°. If $t-s\ge d(x,y)$, then $(s,y)\preccurlyeq(t,x)$, so the smaller point (s,y) is the desired meet.

7.2°. If $s-t\ge d(x,y)$, then $(t,x)\preccurlyeq(s,y)$, so the smaller point (t,x) is the desired meet.

7.3°. Let us now consider the remaining case when $d(x,y)>|t-s|$. In this case, $-d(x,y)\le t-s\le d(x,y)$ hence $0\le t-s+d(x,y)\le 2d(x,y)$ and thus, $0\le\alpha\le d(x,y)$, where we denoted $\alpha\stackrel{\text{def}}{=}\dfrac{t-s+d(x,y)}{2}$. We will prove that in this case, the desired meet is the element (t_0,x_α), where $t_0\stackrel{\text{def}}{=}\dfrac{t+s-d(x,y)}{2}$ and x_α is a point on the geodesic arc connecting x and y for which $d(x,x_\alpha)=\alpha$.

Note that indeed $(t_0,x_\alpha)\preccurlyeq(t,x)$ and $(t_0,x_\alpha)\preccurlyeq(s,y)$.

We need to prove that for every q and z, if $(q, z) \preccurlyeq (t, x)$ and $(q, z) \preccurlyeq (s, y)$ then $(q, z) \preccurlyeq (t_0, x_\alpha)$. By the property of a real tree,

- either x_α lies on a geodesic arc connecting x and z,
- or x_α lies on a geodesic arc connecting y and z.

Without losing generality, let us consider the first case, in which $d(x, z) = d(x, x_\alpha) + d(x_\alpha, z)$. We know that $(q, z) \preccurlyeq (t, x)$ and $(q, z) \preccurlyeq (s, y)$, i.e., that $t - q \geq d(x, z)$ and $s - q \geq d(y, z)$. We need to prove that $(q, z) \preccurlyeq (t_0, x_\alpha)$, i.e., that $t_0 - q \geq d(x_\alpha, z)$. Since we are in the first case, we have $d(x_\alpha, z) = d(x, z) - d(x, x_\alpha) = d(x, z) - \alpha$. By definition of α, this means that

$$d(x_\alpha, z) = d(x, z) - \frac{t - s + d(x, y)}{2}.$$

Substituting this expression for $d(x_\alpha, z)$ and the definition of t_0 into the desired inequality $t_0 - q \geq d(x_\alpha, z)$, we get an equivalent inequality

$$\frac{t + s - d(x, y)}{2} - q \geq d(x, z) - \frac{t - s + d(x, y)}{2} = d(x, z) + \frac{s - t - d(x, y)}{2}.$$

By canceling identical terms $s/2$ and $-d(x, y)/2$ on both sides, and by moving $t/2$ into the left-hand side of this inequality, we get an equivalent inequality $t - q \geq d(x, z)$ which we assumed to be true. The statement is proven, and so is the theorem.

Chapter 6

How to Combine Ordered Sets

In physics, a system often consists of several subsystems; we need to combine the information related to each subsystem into a single description. In mathematical terms, we need to combine the corresponding ordered sets.

Similarly, in describing uncertainty, we may have different experts who provide different descriptions of what is more certain and what is less certain. It is desirable to combine the descriptions of several experts into a single description.

In all these cases, we need to combine several partial orders on different sets X_1 and X_2 into a single partial order on the set $X_1 \times X_2$ of all the pairs (x_1, x_2), where $x_i \in X_i$. This set of pairs is called a *product* of the sets X_i; in these terms, our question becomes a question of describing possible orders on the product of two ordered sets. Such operations are described in Chapter 6.

The results of this chapter first appeared in [83], [84], [85], and [86].

6.1 Formulation of the Problem

Partially ordered sets (posets) in space-time geometry. As we have mentioned in Chapter 1, partially ordered sets (posets) are very important in space-time geometry, where the causality is the corresponding partial order. These posets are especially important when we analyze quantum effects.

Comment. In this chapter (and in the next chapter), $a \prec b$ will denote $a \preccurlyeq b$ and $a \neq b$, i.e., equivalently, that $a \preccurlyeq b$ and $b \not\preccurlyeq a$.

Products of space-time posets. Sometimes, we need to consider *pairs* of events – e.g., in situations like quantum entanglement, situations of importance to quantum computing [68]. How to extend partial orders on posets A_1 and A_2 to a partial order on the set $A_1 \times A_2$ of all such pairs?

Posets in uncertainty logic: need for products. A similar partial order $\preccurlyeq$ is useful in describing degrees of expert's certainty, where $a \preccurlyeq a'$ means that a corresponds to less certainty than a'; see, e.g., [34, 66].

Sometimes, two (or more) experts evaluate a statement S. Then, our certainty in S is described by a *pair* (a_1, a_2), where $a_i \in A_i$ is the i-th expert's degree of certainty. When our certainty in S is described by a *pair* $(a_1, a_2) \in A_1 \times A_2$, we must define a *partial order* on the set $A_1 \times A_2$ of all pairs.

What we do in this chapter. In this chapter, we consider the general problem of how to combine two ordered spaces A_1 and A_2 into one.

6.2 Products of Partially Ordered Sets: What Is Known

Known examples of product operations. At present, two product operations are known [14, 73]:

- *Cartesian product*: $(a_1, a_2) \preccurlyeq (a'_1, a'_2) \Leftrightarrow (a_1 \preccurlyeq_1 a'_1 \,\&\, a_2 \prec_2 a'_2)$, and
- *lexicographic product*

$$(a_1, a_2) \preccurlyeq (a'_1, a'_2) \Leftrightarrow ((a_1 \preccurlyeq a'_1 \,\&\, a_1 \neq a'_1) \vee (a_1 = a'_1 \,\&\, a_2 \preccurlyeq a'_2)).$$

Physical meaning of lexicographic order. For space-time models, a possible meaning of a lexicographic product $A_1 \times A_2$ is that A_1 is *macroscopic* space-time, and A_2 is *microscopic* space-time. When a_1' macroscopically precedes a_1, i.e., when $a_1' \prec_1 a_1$, then, of course, the microscopic events should not matter – and we should have $(a_1', a_2') \preccurlyeq (a_1, a_2)$.

On the other hand, when $a_1' = a_1$, i.e., when, from the macroscopic viewpoint, the two events a_1' and a_1 are indistinguishable, we need to go to the microscopic level to see which of these two events causally influences another one, i.e., $(a_1, a_2') \preccurlyeq (a_1, a_2) \Leftrightarrow a_2' \preccurlyeq_2 a_2$; see Figure 6.1

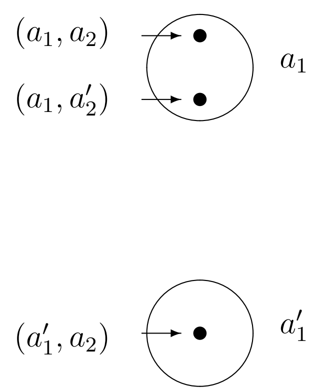

Figure 6.1: Physical meaning of lexicographic order

Logical meaning of Cartesian product. The Cartesian product means that our confidence in S is higher than in S' if and only if it is higher for both experts. In other words, the Cartesian product corresponds to a *maximally cautious* approach.

Logical meaning of lexicographic product. In contrast, a lexicographic product means that we have *absolute confidence* in the first expert, and we only use the opinion of the 2nd expert when, to the 1st expert, the degrees of certainty are equivalent.

6.3 Result: a General Description of Product Operations on Partially Ordered Sets

A natural question. A natural question is: what other operations are possible?

What we prove in this section. In this chapter, we prove that every non-degenerate product operation satisfying the above properties coincides with one of these two products.

Reasonable assumptions on the product. It is reasonable to assume that the validity of the relation $(a_1, a_2) \preccurlyeq (a'_1, a'_2)$ depends only on whether $a_1 \preccurlyeq_1 a'_1$, $a'_1 \preccurlyeq_1 a_1$, $a_2 \preccurlyeq_2 a'_2$, and/or $a'_2 \preccurlyeq_2 a_2$.

It is also reasonable to assume that if $a_1 \preccurlyeq_1 a'_1$ and $a_2 \preccurlyeq_2 a'_2$ then

$$(a_1, a_2) \preccurlyeq (a'_1, a'_2).$$

Definition 6.1.

- *By a* product operation*, we mean a Boolean function* $P : \{T, F\}^4 \to \{T, F\}$.
- *For every two partially ordered sets* A_1 *and* A_2*, we define the following relation on* $A_1 \times A_2$*:*

$$(a_1, a_2) \preccurlyeq (a'_1, a'_2) \stackrel{\text{def}}{=} P(a_1 \preccurlyeq_1 a'_1, a'_1 \preccurlyeq_1 a_1, a_2 \preccurlyeq_2 a'_2, a'_2 \preccurlyeq_2 a_2).$$

- *We say that a product operation is* consistent *if* $\preccurlyeq$ *is always a partial order, and*

$$(a_1 \preccurlyeq_1 a'_1 \,\&\, a_2 \preccurlyeq_2 a'_2) \Rightarrow (a_1, a_2) \preccurlyeq (a'_1, a'_2).$$

Theorem 6.1. *Every consistent product operation is the Cartesian or the lexicographic product.*

6.4 Proofs

Proof of Theorem 6.1.

1°. According to the definition, whether $(a_1, a_2) \preccurlyeq (a_1', a_2)$ depends on the two relations: the relation between a_1 and a_1' and on the relation between a_2 and a_2'. For each pair a_i and a_i', we have four possible relations:

- the relation $a_i \prec_i a_i'$; we will denote this case by $+$;
- the relation $a_i' \prec_i a_i$; we will denote this case by $-$;
- the relation $a_i = a_i'$; we will denote this relation by $=$; and
- the relation $a_i \not\preccurlyeq_i a_i'$ and $a_i' \not\preccurlyeq_i a_i$; we will denote this relation by $\|$.

The case when we have relation R_1 for a_1 and a_1' and relation R_2 for a_2 and a_2' will be denoted by R_1R_2. So, we have 16 possible pairs of relations: $++$, $+-$, $+=$, $+\|$, $-+$, $--$, etc. To describe the product, it is sufficient to describe which of these 16 pairs correspond to $(a_1, a_2) \preccurlyeq (a_1', a_2)$.

Due to the consistency requirement, pairs $++$, $+=$, $=+$, and $==$ always result in $\preccurlyeq$, so it is sufficient to classify the remaining 12 pairs. If only these four pairs result in $\preccurlyeq$, then we have the Cartesian product. So, to prove our theorem, it is sufficient to prove that if at least one other pair leads to $\preccurlyeq$, then we get a lexicographic product. To prove this, let us consider the remaining 12 pairs one by one.

2°. Let us first consider pairs that contain $-$.

2.1°. Let us prove that the pair $--$ cannot lead to $\preccurlyeq$. Indeed, when both A_1 and A_2 are real lines $\mathbb{R}$ with the usual order, due to the fact that $++$ leads to $\preccurlyeq$, we get $(0,0) \preccurlyeq (1,1)$, while due to the fact that $--$ leads to $\preccurlyeq$, we get $(1,1) \preccurlyeq (0,0)$. Hence, we have $(0,0) \preccurlyeq (1,1)$ and $(1,1) \preccurlyeq (0,0)$ but $(0,0) \neq (1,1)$ – a contradiction to antisymmetry.

2.2°. Similarly, the pair $- =$ cannot lead to $\preccurlyeq$ because otherwise, for the same example $A_1 = A_2 = \mathbb{R}$, we would get $(0,0) \preccurlyeq (1,0)$ and $(1,0) \preccurlyeq (0,0)$ but $(0,0) \neq (1,0)$ – also a contradiction to antisymmetry.

2.3°. Let us now consider the pair $- \parallel$.

To prove that it cannot lead to $\preccurlyeq$, we consider $A_1 = \mathbb{R}$ and $A_2 = \mathbb{R} \times \mathbb{R}$ with Cartesian order. In this case,

$$(0,0) \parallel_2 (1,-2)$$

and $(1,-2) \parallel_2 (-1,-1)$. Thus, if $- \parallel$ leads to $\preccurlyeq$, we have $(0,(0,0)) \preccurlyeq (-1,(1,-2))$ and $(-1,(1,-2)) \preccurlyeq (-2,(-1,-1))$. Thus, due to transitivity of $\preccurlyeq$, we get $(0,(0,0)) \preccurlyeq (-2,(-1,-1))$. On the other hand, due to consistency, from $-2 \preccurlyeq_1 0$ and $(-1,-1) \preccurlyeq_2 (0,0)$, we conclude that $(-2,(-1,-1)) \preccurlyeq (0,(0,0))$ – a contradiction with antisymmetry.

2.4°. Similarly, pairs $= -$ and $\parallel -$ cannot lead to $\preccurlyeq$. Thus, the only pairs containing $-$ that can potentially lead to $\preccurlyeq$ are pairs containing a $+$.

3°. Let us prove a similar property for pairs containing $\parallel$. We already know that pairs $\parallel -$ and $- \parallel$ cannot lead to $\preccurlyeq$, so it is sufficient to consider pairs $\parallel=$, $=\parallel$, and $\parallel\parallel$.

3.1°. To prove that the pair $=\parallel$ cannot lead to $\preccurlyeq$, let us consider the same case $A_1 = \mathbb{R}$ and $A_2 = \mathbb{R} \times \mathbb{R}$. In this case, due to

$$(0,0) \parallel_2 (1,-2)$$

and $(1,-2) \parallel_2 (-1,-1)$, if $=\parallel$ leads to $\preccurlyeq$, we have $(0,(0,0)) \preccurlyeq (0,(1,-2))$ and $(0,(1,-2)) \preccurlyeq (0,(-1,-1))$. Thus, due to transitivity of $\preccurlyeq$, we get

$$(0,(0,0)) \preccurlyeq (0,(-1,-1)).$$

On the other hand, due to consistency, from $0 \preccurlyeq_1 0$ and $(-1,-1) \preccurlyeq_2 (0,0)$, we conclude that $(0,(-1,-1)) \preccurlyeq (0,(0,0))$ – a contradiction with antisymmetry.

3.2°. Similarly, it is possible to prove that the pair $\parallel=$ cannot lead to $\preccurlyeq$.

3.3°. To prove that the pair $\|\|$ cannot lead to $\preccurlyeq$, let us consider the case when $A_1 = A_2 = \mathbb{R} \times \mathbb{R}$. In this case, due to

$$(0,0) \parallel_i (1,-2)$$

and $(1,-2) \parallel_i (-1,-1)$, if $\|\|$ leads to $\preccurlyeq$, we have $((0,0),(0,0)) \preccurlyeq ((1,-2),(1,-2))$ and $((1,-2),(1,-2)) \preccurlyeq ((-1,-1),(-1,-1))$. Thus, due to transitivity of $\preccurlyeq$, we get $((0,0),(0,0)) \preccurlyeq ((-1,-1),(-1,-1))$. On the other hand, due to consistency, from $(-1,-1) \preccurlyeq_i (0,0)$, we conclude that $((-1,-1),(-1,-1)) \preccurlyeq ((0,0),(0,0))$ – a contradiction with antisymmetry.

4°. Thus, due to Part 2 and 3 of this proof, the only additional pairs that can, in principle, lead to $\preccurlyeq$ are pairs containing $+$, i.e., pairs $+-$, $+\parallel$, $-+$, and $-\parallel$.

5°. Let us prove that the pair $+-$ leads to $\preccurlyeq$ if and only if the pair $+\parallel$ leads to $\preccurlyeq$.

5.1°. Let us first prove that if the pair $+-$ leads to $\preccurlyeq$, then the pair $+\parallel$ also leads to $\preccurlyeq$.

Indeed, let us consider the case when $A_1 = \mathbb{R}$ and $A_2 = \mathbb{R} \times \mathbb{R}$. If $+-$ leads to $\preccurlyeq$, then $0 \prec_1 1$ and $(-1,-1) \prec_2 (0,0)$ imply $(0,(0,0)) \preccurlyeq (1,(-1,-1))$. Due to consistency, $1 \preccurlyeq_1 1$ and $(-1,-1) \preccurlyeq_2 (-1,1)$ lead to $(1,(-1,-1)) \preccurlyeq (1,(-1,1))$. Due to transitivity of $\preccurlyeq$, we get $(0,(0,0)) \preccurlyeq (1,(-1,1))$. In this case, $\preccurlyeq$ holds for a pair for which $0 \prec_1 1$ and $(0,0) \parallel_2 (-1,1)$, i.e., for a pair of type $+\parallel$. By our definition of an order on the product, this means that $\preccurlyeq$ must hold for all pairs of this type, i.e., that the pair $+\parallel$ indeed leads to $\preccurlyeq$.

5.2°. Let us now prove that if the pair $+\parallel$ leads to $\preccurlyeq$, then the pair $+-$ also leads to $\preccurlyeq$.

Let us consider the same case $A_1 = \mathbb{R}$ and $A_2 = \mathbb{R} \times \mathbb{R}$. If $+\parallel$ leads to $\preccurlyeq$, then $0 \prec_1 1$ and $(1,-2) \parallel_2 (-1,-1)$ imply $(0,(0,0)) \preccurlyeq (1,(1,-2))$, and $1 \prec_1 2$ and $(0,0) \parallel_2 (1,-2)$ imply and $(1,(1,-2)) \preccurlyeq (2,(-1,-1))$. Due to transitivity of $\preccurlyeq$, we get $(0,(0,0)) \preccurlyeq (2,(-1,-1))$. In this case, $\preccurlyeq$ holds for a pair for which $0 \prec_1 2$ and $(-1,-1) \prec_2 (0,0)$, i.e., for a pair of type $+-$. By our definition of an order on the product, this means that $\preccurlyeq$ must hold for all pairs of this type, i.e., that the pair $+-$ indeed leads to $\preccurlyeq$.

6°. Similarly, we can prove that the pair $-+$ leads to $\preccurlyeq$ if and only if the pair $\| +$ leads to $\preccurlyeq$. Thus, adding $+-$ is equivalent to adding $+ \|$, and adding $-+$ is equivalent to adding $\| +$.

If we add $+-$ (and hence $+ \|$), we get the lexicographic product $A_1 \times A_2$. If we add $-+$ (and hence $\| +$), we get the lexicographic product $A_2 \times A_1$. Thus, to complete the proof, it is sufficient to show that we cannot simultaneously add $+-$ and $-+$.

7°. Let us prove that $+-$ and $-+$ cannot simultaneously lead to $\preccurlyeq$.

We will prove this by contradiction. Let us assume that adding both $+-$ and $-+$ always leads to a consistent partial order. In this case, let us take $A_1 = A_2 = \mathbb{R}$. Since $+-$ leads to $\preccurlyeq$, the conditions $0 \prec_1 1$ and $-2 \prec_2 0$ lead to $(0,0) \preccurlyeq (1,-2)$. Similarly, since $-+$ leads to $\preccurlyeq$, from $-1 \prec_1 1$ and $-2 \prec_2 -1$, we conclude that $(1,-2) \preccurlyeq (-1,-1)$. By transitivity of $\preccurlyeq$, we can now conclude that $(0,0) \preccurlyeq (-1,-1)$. However, due to consistency, $(-1,-1) \preccurlyeq (0,0)$ – a contradiction to anti-symmetry.

The statement is proven, and so is the theorem.

Chapter 7

How to Tell When a Product of Two Partially Ordered Spaces Has a Certain Property

In this chapter, we describe how checking whether a given property F is true for a product $A_1 \times A_2$ of partially ordered spaces can be reduced to checking several related properties of the original spaces A_i.

This result can be useful in the analysis of properties of intervals $[a, b] \stackrel{\text{def}}{=} \{x : a \preccurlyeq x \preccurlyeq b\}$ over general partially ordered spaces – such as the space of all vectors with component-wise order or the set of all functions with component-wise ordering $f \leq g \Leftrightarrow \forall x\,(f(x) \leq g(x))$. When we consider sets of pairs of such objects $A_1 \times A_2$, it is natural to define the order on this set in terms of orders in A_1 and A_2 – this is, e.g., how ordering and intervals are defined on the set $\mathbb{R}^2$ of all 2-D vectors.

This result can also be useful in the analysis of ordered spaces describing different degrees of certainty in expert knowledge.

This result first appeared in [84, 85].

7.1 Formulation of the Main Problem

Interval uncertainty for numbers, vectors, functions, etc. In many practical situations, we do not know the exact value x of a physical quantity, we only know the lower bound $\underline{x}$ and the upper bound $\overline{x}$. In this case, the only information that we have about the unknown value x is that x belongs to the interval $\{x : \underline{x} \le x \le \overline{x}\}$.

For example, if we have a measurement result $\widetilde{x}$ and an upper bound Δ on the measurement error $\Delta x \stackrel{\text{def}}{=} \widetilde{x} - x$, then we can conclude that the actual (unknown) value x belongs to the interval $[\underline{x}, \overline{x}] = [\widetilde{x} - \Delta, \widetilde{x} + \Delta]$; see, e.g., [32, 59, 75].

If we are interested in the values of two different quantities x_1 and x_2, then, to describe the actual values of these two quantities, we need a tuple $x = (x_1, x_2)$. In practice, we usually do not know the exact value of x. Instead, we have a tuple $\underline{x} = (\underline{x}_1, \underline{x}_2)$ which is a "lower bound" for the actual tuple x and a tuple $\overline{x} = (\overline{x}_1, \overline{x}_2)$ which is an "upper bound" for the actual tuple x. This informal description of bounds can be formalized if we introduce a natural component-wise ordering relation between tuples:

$$(x_1, x_2) \preccurlyeq (x_1', x_2') \Leftrightarrow ((x_1 \le x_1') \,\&\, (x_2 \le x_2')).$$

In terms of this ordering, the set of all possible tuples x can also be described as an interval $\{x : \underline{x} \preccurlyeq x \preccurlyeq \overline{x}\}$.

In this case, we started with the ordering relations of two different sets – the set X_1 of possible values of x_1 and the set X_2 of possible values of x_2 – and we defined the corresponding ordering relation on the set $X_1 \times X_2$ of all possible pairs (x_1, x_2). In the above case, X_1 and X_2 were sets of real numbers with usual linear order, but a same construction can be useful in more complex cases as well.

For example, when both x_1 and x_2 are vectors, the ordering relation on each set X_i is a partial order, so we need to analyze the product of partial orders.

When we are interested in the function $f(x)$ – e.g., the function that describes the dependence of one physical quantity on another one – we rarely know the exact function, we usually know some lower and upper bounds $\underline{f}(x)$ and $\overline{f}(x)$. If we consider a pair of functions, or a pair consisting of a function and a number, then we need to define an appropriate ordering relation on the set of all possible pairs.

In the above examples, we had a component-wise order, but in principle, we could have a more complex ordering relation on the product set $X_1 \times X_2$.

Need to analyze properties of products of partially ordered spaces. The above examples show that we need to consider ordering relations on the product $X_1 \times X_2$ of two partially ordered sets. It is therefore desirable to analyze when this new ordering relation satisfies certain property: e.g., when it is linearly ordered, when it is a lattice, etc.

What we do in this chapter. In this chapter, we provide a general algorithm that reduces the question whether a certain property is satisfied for a product to several properties of component spaces.

Fuzzy modeling and fuzzy techniques: an area of AI where interval methods help. Interval techniques originated from situations in which the interval uncertainty comes from the bounds on measurement errors. However, the same techniques are useful in more general situations. For example, it is known that interval techniques can be very helpful in analyzing expert information, in so-called *fuzzy logic* (see, e.g., [34, 66]). In this technique, to describe informal expert statements like "x is small", we assign, to each value x, a number $\mu(x)$ from the interval $[0, 1]$ that describes the expert's confidence that this particular value x is small.

From the computational viewpoint, it is often convenient to describe the corresponding function $\mu(x)$ (called *membership function*) by the sets

$$\mathbf{x}(\alpha) = \{x : \mu(x) \geq \alpha\}$$

called *α-cuts*. When we increase α, the α-cut decreases: if $\alpha < \alpha'$, then $\mathbf{x}(\alpha) \supseteq \mathbf{x}(\alpha')$. Thus, in this representation, expert knowledge is described by "nested" sets $\mathbf{x}(\alpha)$ each of which is the set of all the values which are, according to the expert, possible with the degree $\geq \alpha$. In many cases, e.g., for terms like "medium", "approximately 0.3", the expert's degree of confidence first grows with x then decreases; in such situations, each α-cut is an interval.

Operations with expert knowledge can be naturally reformulated in terms of the α-cuts. In particular, we have a problem similar to interval computations: we have expert knowledge about the quantities $x_1, \ldots, x_n$, we know the relation $y = f(x_1, \ldots, x_n)$ between x_i and y, and we want to make conclusions about y. In this case, a usual fuzzy way of finding the membership function for y can be equivalently described in terms of α-cuts as

$$\mathbf{y}(\alpha) = f(\mathbf{x}_1(\alpha), \ldots, \mathbf{x}_n(\alpha)) \stackrel{\text{def}}{=} \{f(x_1, \ldots, x_n) : x_1 \in \mathbf{x}_1(\alpha), \ldots, x_n \in \mathbf{x}_n(\alpha)\}.$$

Thus, processing fuzzy data can be reduced (and is often reduced) to processing interval data – i.e., to solving interval computation problems corresponding to several possible values of α; see, e.g., [18, 34, 58, 64, 66].

Since the value α describes the expert's degree of confidence, it is not known with any high accuracy: e.g., hardly anyone can say that his or her degree of confidence is some statement is 0.71 but not 0.72. So, it is sufficient to take only values $\alpha = 0$, 0.1, ..., 0.9, 1.0.

Degrees of certainty: from $[0,1]$ to general partially ordered sets. Traditionally, fuzzy logic uses values from the interval $[0,1]$ to describe uncertainty. In this interval, the order is total (linear) in the sense that for every two elements $a, a' \in [0,1]$, either $a \le a'$ or $a' \le a$. As we have mentioned in Chapter 1, often, partial orders provide a more adequate description of the expert's degree of confidence. For example, since an expert cannot describe her degree of certainty by an exact number, it makes sense to describe this degree by an *interval* $[\underline{d}, \overline{d}]$ of possible numbers (see, e.g., [54, 65]) – and intervals are only partially ordered; e.g., the intervals $[0.5, 0.5]$ and $[0,1]$ are not easy to compare.

More complex sets of possible degrees are also sometimes useful. Not to miss any new options, in this chapter, we consider general partially ordered spaces.

Need for product operations. Often, two (or more) experts evaluate a statement S. Then, our certainty in S is described by a pair (a_1, a_2), where $a_i \in A_i$ is the i-th expert's degree of certainty. To compare such pairs, we must therefore define a partial order on the set $A_1 \times A_2$ of all such pairs.

First example of a product operation: Cartesian product. One example of a partial order on $A_1 \times A_2$ is a *Cartesian* product:

$$(a_1, a_2) \preccurlyeq (a'_1, a'_2) \Leftrightarrow ((a_1 \preccurlyeq a'_1) \,\&\, (a_2 \preccurlyeq a'_2)).$$

Logical meaning of Cartesian product. This product corresponds to a *cautious* approach, when our confidence in S' is higher than in S if and only if it is higher for both experts.

Second example of a product operation: lexicographic product. Another example is a *lexicographic* product:

$$(a_1, a_2) \preccurlyeq (a'_1, a'_2) \Leftrightarrow ((a_1 \preccurlyeq a'_1) \,\&\, a_1 \neq a'_1) \vee ((a_1 = a'_1) \,\&\, (a_2 \preccurlyeq a'_2))).$$

Logical meaning of lexicographic product. This product corresponds to the case when we have the absolute confidence in the first expert; then, we only use the opinion of the second expert when, to the first expert, the degrees of certainty are indistinguishable.

We can have other product operations in which the relation between the pairs (a_1, a_2) and (a'_1, a'_2) is defined in terms of the relations between the elements $a_1, a'_1 \in A_1$ and between the elements $a_2, a'_2 \in A_2$.

A natural question. Once a product is defined, it is reasonable to ask when the resulting partially ordered set $A_1 \times A_2$ it satisfies a certain property: is it a total order? is it a lattice order? etc. It is desirable to have some criteria that would transform the question about the product space into questions about related properties of component spaces.

Some such criteria are known (see, e.g., [83, 84] and references therein). For example:

- A Cartesian product is a total order if and only if one of the components is a total order, and the other consists of a single element.
- A lexicographic product is a total order if and only if both components are totally ordered.

Applications beyond logic. Similar questions arise in other applications of ordered sets, e.g., in space-time geometry where the causality ordering relation $a \preccurlyeq b$ means that an event a can influence the event b.

Applications beyond orders. Our algorithm does not use the fact that the original relations are orders (i.e., transitive antisymmetric relations). Thus, our algorithm is applicable to a general case when we have an arbitrary binary relation – equivalence, similarity, etc. Moreover, this algorithm can be applied to the case when we have a space with *several* binary relations – e.g., an order relation and a similarity relation.

7.2 Definitions and the Main Algorithmic Result

In the following text, we fix a positive integer m; this integer will be called a *number of binary relations.* Our main case is $m = 1$, when we consider a single binary relation, and this binary relation is an order. However, our result is applicable to an arbitrary finite set of binary relations.

Definition 7.1. *By a* space*, we mean a set A with m binary relations $P_1(a, a')$, ..., $P_m(a, a')$.*

Clarification. In this definition and in the following definitions, we only consider *crisp* relations – such as an order between the traditional fuzzy degrees of belief, i.e., between the numbers from the interval $[0, 1]$.

Terminological comment. Strictly speaking, a space is thus defined as a tuple $(A, P_1, \ldots, P_m)$. Following the usual mathematical practice, we will, however, usually simplify our notations and simply talk about a space A – implicitly meaning the relations as well.

Definition 7.2. *By a* first order property *(or simply* property*, for short), we mean a (closed) formula F that is obtained from formulas $P_i(x, x')$ by using logical connectives $\vee$, &, $\neg$, and $\to$, and quantifiers $\exists x$ and $\forall x$.*

Comment. Most properties in which we may be interested are first order properties. For example, the property to be a total order has the form

$$\forall a \forall a' \,((a \preccurlyeq a') \vee (a' \preccurlyeq a)).$$

The property to be a lattice L means that for every two elements a and a' there is a least upper bound and a greatest lower bound: $L \Leftrightarrow \overline{L} \,\&\, \underline{L}$, where

$$\overline{L} \Leftrightarrow \forall a \forall a' \exists a^+ \left((a \preccurlyeq a^+) \,\&\, (a' \preccurlyeq a^+) \,\&\, \forall a''(((a \preccurlyeq a'') \,\&\, (a' \preccurlyeq a'')) \rightarrow a^+ \preccurlyeq a'')\right),$$

and

$$\underline{L} \Leftrightarrow \forall a \forall a' \exists a^- \left((a^- \preccurlyeq a) \,\&\, (a^- \preccurlyeq a') \,\&\, \forall a''(((a'' \preccurlyeq a) \,\&\, (a'' \preccurlyeq a')) \rightarrow a'' \preccurlyeq a^-)\right).$$

Notations. When a property F is true for a space X, we will denote it by $F(X)$.

Definition 7.3. *By a* product operation*, we mean a collection of m propositional formulas that describe the relation $P_i((a_1, a_2), (a'_1, a'_2))$ between the elements $(a_1, a_2), (a'_1, a'_2) \in A_1 \times A_2$ in terms of the relations between the components $a_1, a'_1 \in A_1$ and $a_2, a'_2 \in A_2$ of these elements, i.e., in terms of the relations $P_1(a_1, a'_1)$, …, $P_m(a_1, a'_1)$, $P_1(a'_1, a_1)$, …, $P_m(a'_1, a_1)$, $P_1(a_2, a'_2)$, …, $P_m(a_2, a'_2)$, $P_1(a'_2, a_2)$, …, $P_m(a'_2, a_2)$.*

Comment. The above formulas that define Cartesian and lexicographic products of partially ordered sets show that these two product operations are examples of product operations in the sense of Definition 7.3.

Notational comment. For each operation, the space of all the elements is the set of all pairs $A_1 \times A_2$; so, in line with the above terminological comment, we will simply talk about the space $A_1 \times A_2$.

Proposition 7.1. *There exists an algorithm that, given a product operation and a property F, generates a finite list of properties F_{11}, F_{12}, F_{21}, F_{22}, …, F_{p1}, F_{p2}, such that*

$$F(A_1 \times A_2) \Leftrightarrow ((F_{11}(A_1) \,\&\, F_{12}(A_2)) \vee \ldots \vee (F_{p1}(A_1) \,\&\, F_{p2}(A_2))).$$

Comment. The above examples of checking when a Cartesian product or a lexicographic product are total orders, are examples of such equivalences. For example, for the Cartesian product, we have $p = 2$,

- $F_{11}(A_1)$ meaning that A_1 is a total order,

- $F_{12}(A_2)$ meaning that A_2 is a one-element set,
- $F_{21}(A_1)$ meaning that A_1 is a one-element set, and
- $F_{22}(A_2)$ meaning that A_2 is a total order.

Generalizations. As we will see from the proof, a similar algorithm can be formulated for a product of three or more spaces, and for the case when we allow ternary and higher order operations in the definition of a space.

7.3 Description of the Algorithm and Proof of Its Correctness

1°. Let us start with the desired property F. This property uses basic relations $P_i(a, a')$ between elements $a, a' \in A_1 \times A_2$ and quantifiers $\forall a$ and $\exists a$ over elements $a \in A_1 \times A_2$.

2°. Every element $a \in A_1 \times A_2$ is, by definition, a pair (a_1, a_2) in which a_1 is an element of the set A_1 and a_2 is an element of the set A_2.

Let us explicitly replace each variable with such a pair.

3°. By definition of a product operation, each relation $P_i(a, a')$ – i.e., each relation $P_i((a_1, a_2), (a'_1, a'_2))$ – can be replaced by a propositional combination of relations between elements $a_1, a'_1 \in A_1$ and between elements $a_2, a'_2 \in A_2$.

Let us perform this replacement.

4°. Each quantifier can also be replaced by two quantifiers corresponding to components:

- $\forall (a_1, a_2)$ is equivalent to $\forall a_1 \forall a_2$, and
- $\exists (a_1, a_2)$ is equivalent to $\exists a_1 \exists a_2$.

Let us perform this replacement as well.

5°. As a result, we get an equivalent reformulation of the original formula F in which elementary formulas are relations between elements of A_1 or between A_2 and quantifiers are over A_1 or over A_2.

We want to reduce this formula to the desired form

$$((F_{11}(A_1) \,\&\, F_{12}(A_2)) \vee \ldots \vee (F_{p1}(A_1) \,\&\, F_{p2}(A_2))). \tag{7.3.1}$$

We will reduce this by induction. Elementary formulas are already of the desired form – provided, of course, that we allow free variables.

We will show that if we apply a propositional connective or a quantifier to a formula of this type, then we can reduce the result again to the formula of this type.

6°. When we apply propositional connectives to formulas of type (7.3.1), we thus get a propositional combination of the formulas of the type $F_{ij}(A_j)$. It is known that an arbitrary propositional combination can be described in a Disjunctive Normal Form (DNF), i.e., as a disjunction of conjunctions. Each conjunction combines properties related to A_1 and properties related to A_2, i.e., has the form

$$G_1(A_1) \,\&\, \ldots \,\&\, G_p(A_1) \,\&\, G_{p+1}(A_2) \,\&\, \ldots \,\&\, G_q(A_2).$$

Thus, each conjunction has the from $G(A_1) \,\&\, G'(A_2)$, where

$$G(A_1) \Leftrightarrow (G_1(A_1) \,\&\, \ldots \,\&\, G_p(A_1))$$

and

$$G'(A_2) \Leftrightarrow (G_{p+1}(A_2) \,\&\, \ldots \,\&\, G_q(A_2)).$$

Thus, the disjunction of such properties has the desired form (7.3.1).

7°. When we apply an existential quantifier, e.g., $\exists a_1$, then we get a formula

$$\exists a_1\, ((F_{11}(A_1) \,\&\, F_{12}(A_2)) \vee \ldots \vee (F_{p1}(A_1) \,\&\, F_{p2}(A_2))).$$

It is known that $\exists a\,(A \vee B)$ is equivalent to $\exists a\, A \vee \exists a\, B$. Thus, the above formula is equivalent to a disjunction

$$\exists a_1\, (F_{11}(A_1) \,\&\, F_{12}(A_2)) \vee \ldots \vee \exists a_1\, (F_{p1}(A_1) \,\&\, F_{p2}(A_2)).$$

If we prove that each term in this disjunction can be transformed into the desired form (7.3.1), then, by using the Part 6 of this proof, we will be able to conclude that the entire disjunction has the desired form. Thus, it is sufficient to prove that each formula

$$\exists a_1\,(F_{i1}(A_1)\,\&\,F_{i2}(A_2)) \tag{7.3.2}$$

has the desired form. The term $F_{i2}(A_2)$ does not depend on a_1 at all, it is all about elements of A_2. Thus, the formula (7.3.2) is equivalent to

$$(\exists a_1\,F_{i1}(A_1))\,\&\,F_{i2}(A_2),$$

i.e., to the formula

$$F'_{i1}(A_1)\,\&\,F_{i2}(A_2),$$

where

$$F'_{i1} \Leftrightarrow \exists a_1\,F_{i1}(A_1)$$

is a formula depending only on the space A_1.

The reduction is proven.

8°. When we apply a universal quantifier, e.g., $\forall a_1$, then we can use the fact that $\forall a_1\,F$ is equivalent to $\neg\exists a_1\,\neg F$. We have assumed that the formula F is of the desired type (7.3.1). Thus,

- by using Part 6 of this proof, we can conclude that the formula $\neg F$ can be reduced to the desired type;
- now, by applying Part 7 of this proof, we can conclude that the formula $\exists a_1\,(\neg F)$ can also be reduced to the desired type;
- finally, by using Part 6 again, we conclude that the formula $\neg(\exists a_1\,\neg F)$ can be reduced to the desired type.

9°. By induction, we can now conclude that the original formula can be reduced to the desired type. The main result is proven.

7.4 Example of Applying the Algorithm

To clarify our algorithm, let us apply it to the above simple case of checking whether a Cartesian product is totally ordered. In this case, the formula F that we want to check has the form

$$\forall a \forall a' \, ((a \preccurlyeq a') \vee (a' \preccurlyeq a)).$$

According to our algorithm, we first explicitly replace each variable $a, a' \in A_1 \times A_2$ with the corresponding pair. As a result, we get the following formula:

$$\forall (a_1, a_2) \forall (a'_1, a'_2) \, (((a_1, a_2) \preccurlyeq (a'_1, a'_2)) \vee ((a'_1, a'_2) \preccurlyeq (a_1, a_2))).$$

Replacing the ordering relation on the Cartesian product with its definition, we get

$$\forall (a_1, a_2) \forall (a'_1, a'_2) \, ((a_1 \preccurlyeq a'_1 \,\&\, a_2 \preccurlyeq a'_2)) \vee ((a'_1 \preccurlyeq a_1 \,\&\, a'_2 \preccurlyeq a_2))).$$

Replacing quantifiers over pairs with individual quantifiers, we get

$$\forall a_1 \forall a_2 \forall a'_1 \forall a'_2 \, ((a_1 \preccurlyeq a'_1 \,\&\, a_2 \preccurlyeq a'_2)) \vee ((a'_1 \preccurlyeq a_1 \,\&\, a'_2 \preccurlyeq a_2))).$$

By using the relation $\forall \Leftrightarrow \neg \exists \neg$, we get an equivalent form

$$\neg \exists a_1 \exists a_2 \exists a'_1 \exists a'_2 \, \neg ((a_1 \preccurlyeq a'_1 \,\&\, a_2 \preccurlyeq a'_2) \vee (a'_1 \preccurlyeq a_1 \,\&\, a'_2 \preccurlyeq a_2))).$$

Moving negation inside the propositional formula, we get

$$\neg \exists a_1 \exists a_2 \exists a'_1 \exists a'_2 \, ((a_1 \not\preccurlyeq a'_1 \vee a_2 \not\preccurlyeq a'_2) \,\&\, (a'_1 \not\preccurlyeq a_1 \vee a'_2 \preccurlyeq a_2))).$$

The propositional formula

$$(a_1 \not\preccurlyeq a'_1 \vee a_2 \not\preccurlyeq a'_2)) \,\&\, (a'_1 \not\preccurlyeq a_1 \vee a'_2 \not\preccurlyeq a_2)$$

must now be transformed into a DNF form. The result is

$$(a_1 \not\preccurlyeq a'_1 \,\&\, a'_1 \not\preccurlyeq a_1) \vee (a_1 \not\preccurlyeq a'_1 \,\&\, a'_2 \not\preccurlyeq a_2) \vee (a_2 \not\preccurlyeq a'_2 \,\&\, a'_1 \not\preccurlyeq a_1) \vee (a_2 \not\preccurlyeq a'_2 \,\&\, a'_2 \not\preccurlyeq a_2).$$

Thus, the formula

$$\exists a_1 \exists a_2 \exists a_1' \exists a_2' \, \neg((a_1 \preccurlyeq a_1' \,\&\, a_2 \preccurlyeq a_2') \vee (a_1' \preccurlyeq a_1 \,\&\, a_2' \preccurlyeq a_2)))$$

is equivalent to

$$F_1 \vee F_2 \vee F_3 \vee F_4,$$

where

$$F_1 \Leftrightarrow \exists a_1 \exists a_2 \exists a_1' \exists a_2' \, (a_1 \not\preccurlyeq a_1' \,\&\, a_1' \not\preccurlyeq a_1), F_2 \Leftrightarrow \exists a_1 \exists a_2 \exists a_1' \exists a_2' \, (a_1 \not\preccurlyeq a_1' \,\&\, a_2' \not\preccurlyeq a_2),$$

$$F_3 \Leftrightarrow \exists a_1 \exists a_2 \exists a_1' \exists a_2' \, (a_2 \not\preccurlyeq a_2' \,\&\, a_1' \not\preccurlyeq a_1), F_4 \Leftrightarrow \exists a_1 \exists a_2 \exists a_1' \exists a_2' \, (a_2 \not\preccurlyeq a_2' \,\&\, a_2' \not\preccurlyeq a_2).$$

By applying the quantifiers to the corresponding parts of the formulas, we get

$$F_1 \Leftrightarrow \exists a_1 \exists a_1' \, (a_1 \not\preccurlyeq a_1' \,\&\, a_1' \not\preccurlyeq a_1), F_2 \Leftrightarrow (\exists a_1 \exists a_1' \, (a_1 \not\preccurlyeq a_1')) \,\&\, (\exists a_2 \exists a_2' \, (a_2' \not\preccurlyeq a_2)),$$

$$F_3 \Leftrightarrow (\exists a_1 \exists a_1' \, (a_1' \not\preccurlyeq a_1)) \,\&\, (\exists a_2 \exists a_2' \, (a_2 \not\preccurlyeq a_2')), F_4 \Leftrightarrow \exists a_2 \exists a_2' \, (a_2 \not\preccurlyeq a_2' \,\&\, a_2' \not\preccurlyeq a_2).$$

Then, we again reduce

$$\neg(F_1 \vee F_2 \vee F_3 \vee F_4)$$

to DNF.

The result is more complex than the above criterion – because our algorithm does not use the fact that $\preccurlyeq$ is an order relation.

7.5 Auxiliary Results

General idea. For each property of an ordered set A, we analyze which properties need to be satisfied for A_1 and A_2 so that the corresponding property is satisfies in $A_1 \times A_2$. The following results are examples of the reduction that we have been aiming at in the main result of this chapter.

In contrast to all the other results from this thesis, which are all new, the "reduction" results from this section seem to be known to specialists; however, since we were unable to find good references where the proofs of all these results can be found, we decided to provide explicit proofs for these known results.

First example: connectedness property. As a first example, let us consider the following

- *Connectedness property (CP):* for every two points $a, a' \in A$, there exists an interval that contains a and a': $\forall a\, \forall a'\, \exists a^-\, \exists a^+\, (a^- \preccurlyeq a, a' \preccurlyeq a^+)$; see Figure 7.1.

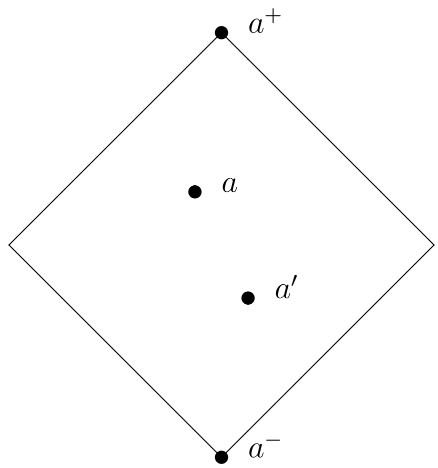

Figure 7.1: Connectedness property

Connectedness property: analysis. One can easily see that a partially ordered set A satisfies the connectedness property if and only if it satisfies the following two properties:

- A is *upward-directed*: $\forall a\, \forall a'\, \exists a^+\, (a, a' \preccurlyeq a^+)$; see Figure 7.2.

- A is *downward-directed*: $\forall a\, \forall a'\, \exists a^-\, (a^- \preccurlyeq a, a')$; see Figure 7.3.

So, to check when the product satisfies the connectivity property, it is sufficient to check when the product is upward- and downward-directed.

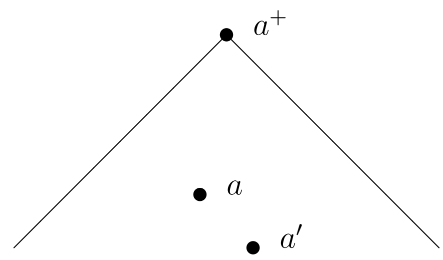

Figure 7.2: Upward-directed partially ordered set

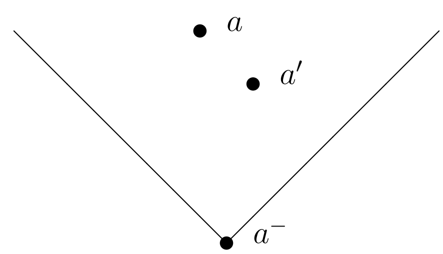

Figure 7.3: Downward-directed partially ordered set

Results for Cartesian product. For both $\preccurlyeq$- and $\prec$-Cartesian products, we get the following results.

Proposition 7.2. *A Cartesian product $A_1 \times A_2$ is upward-directed if and only if both A_1 and A_2 are upward-directed.*

Proposition 7.3. *A Cartesian product $A_1 \times A_2$ is downward-directed if and only if both A_1 and A_2 are downward-directed.*

Results for lexicographic product. For the lexicographic product, we get the following results:

Definition 7.4.

- *An element* $\overline{a} \in A$ *is called* maximal *if there are no elements* a *with* $\overline{a} \prec a$.

- *An element* $\underline{a} \in A$ *is called* minimal *if there are no elements* a *with* $a \prec \underline{a}$.

Proposition 7.4. *A lexicographic product* $A_1 \times A_2$ *is upward-directed* $\Leftrightarrow$ *the following two conditions hold:*

- *the set* A_1 *is upward-directed, and*

- *if* A_1 *has a maximal element* $\overline{a}_1$*, then* A_2 *is upward-directed.*

Proposition 7.5. *A lexicographic product* $A_1 \times A_2$ *is downward-directed* $\Leftrightarrow$ *the following two conditions hold:*

- *the set* A_1 *is downward-directed, and*

- *if* A_1 *has a minimal element* $\underline{a}_1$*, then* A_2 *is downward-directed.*

Second example: intersection property. As a second example, let us consider the following

- *Intersection property:* the intersection of every two intervals is an interval; see Figure 7.4.

Comment. This property is satisfied for intervals on the real line.

Intersection property: analysis. Similarly to the connectivity property, the intersection property can also be reduced to two properties:

- the intersection of every two future cones $C_a^+ \stackrel{\text{def}}{=} \{b : a \preccurlyeq b\}$ is a future cone; see Figure 7.5;

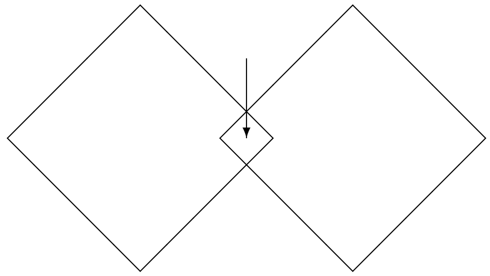

Figure 7.4: Intersection property

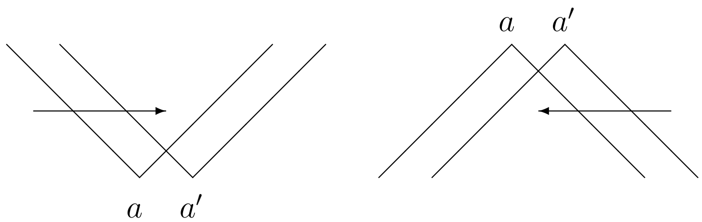

Figure 7.5: Intersection property: analysis

- the intersection of every two past cones $C_a^- \stackrel{\text{def}}{=} \{b : b \preccurlyeq a\}$ is a past cone; see Figure 7.5.

If both properties hold, then a non-empty intersection of every two intervals $[a, b] = C_a^+ \cap C_b^-$ is an interval.

Definition 7.5.

- *An ordered set for which the intersection $C_a^+ \cap C_{a'}^+$ of every two future cones C_a^+ and $C_{a'}^+$ is a future cone is called an* upper semi-lattice.

- *For every two elements a, a', the element a'' for which $C_a^+ \cap C_{a'}^+ = C_{a''}^+$ is called a* join *of a and a' and is denoted by $a \vee a'$.*

- *An ordered set for which the intersection $C_a^- \cap C_{a'}^-$ of every two past cones C_a^- and $C_{a'}^-$ is a past cone is called a* lower semi-lattice.

- *For every two elements a, a', the element a'' for which $C^-_a \cap C^-_{a'} = C^-_{a''}$ is called a* meet *of a and a' and is denoted by $a \wedge a'$.*

What we do in this section. In this section, we analyze when the Cartesian and lexicographic products are upper and lower semi-lattices.

Proposition 7.6. *A Cartesian product $A_1 \times A_2$ is an upper semi-lattice if and only if both A_1 and A_2 are upper semi-lattices.*

Proposition 7.7. *A Cartesian product $A_1 \times A_2$ is a lower semi-lattice if and only if both A_1 and A_2 are upper semi-lattices.*

To describe when a lexicographic product is an upper semi-lattice, we need two introduce the following auxiliary notions:

Definition 7.6.

- *We say that an ordered set A is* linearly (totally) ordered *if for every two elements $a, a' \in A$, we have either $a \preccurlyeq a'$ or $a' \preccurlyeq a$.*
- *We say that an ordered set is a* conditional upper semi-lattice *if for all a and a' for which the future cones C^+_a and $C^+_{a'}$ intersect, this intersection is also a future cone.*
- *We say that an ordered set is a* conditional lower semi-lattice *if for all a and a' for which the past cones C^-_a and $C^-_{a'}$ intersect, this intersection is also a past cone.*
- *We say that an element a^+ is the* next *element to a if for every a', the condition $a \prec a'$ is equivalent to $a^+ \preccurlyeq a'$.*
- *We say that an ordered set is* sequential up *if every element has a next one.*
- *We say that an element a^- is the* previous *element to a if for every a', the condition $a' \prec a$ is equivalent to $a' \preccurlyeq a^-$.*

- *We say that an ordered set is* sequential down *if every element has a previous one.*

Proposition 7.8. *The lexicographic product $A_1 \times A_2$ is an upper semi-lattice if and only A_1 is an upper semi-lattice and one of the following conditions holds:*

- *A_1 is linearly ordered and A_2 is an upper semi-lattice;*
- *A_2 is an upper semi-lattice that has the smallest element;*
- *A_1 is sequential up, A_2 is a conditional upper semi-lattice, and A_2 has the smallest element.*

Proposition 7.9. *The lexicographic product $A_1 \times A_2$ is a lower semi-lattice if and only A_1 is a lower semi-lattice and one of the following conditions holds:*

- *A_1 is linearly ordered and A_2 is a lower semi-lattice;*
- *A_2 is a lower semi-lattice that has the largest element;*
- *A_1 is sequential down, A_2 is a conditional lower semi-lattice, and A_2 has the largest element.*

7.6 Proofs

Proof of Proposition 7.2.

1°. Let us assume that the $\preccurlyeq$-Cartesian product $A_1 \times A_2$ is upward-directed. We want to prove that A_1 is upward-directed. (For A_2, the proof of similar.)

In other words, we want to prove that for every $a_1, a'_1 \in A_1$, there exists an element $a_1^+ \in A_1$ for which $a_1, a'_1 \preccurlyeq_1 a_1^+$. Let us take any $a_1, a'_1 \in A_1$, and any $a_2 \in A_2$. Then, since the product $A_1 \times A_2$ is upward-directed, there exists an element $a^+ = (a_1^+, a_2^+) \in A_1 \times A_2$ for which

$$(a_1, a_2) \preccurlyeq a^+ = (a_1^+, a_2^+) \text{ and } (a'_1, a_2) \preccurlyeq a^+ = (a_1^+, a_2^+).$$

By definition of an order on $A_1 \times A_2$, we thus conclude that $a_1 \preccurlyeq_1 a_1^+$ and $a_1' \preccurlyeq_1 a_1^+$. Thus, A_1 is indeed upward-directed.

2°. Let us now assume that both A_i are upward-directed. We want to prove that $A_1 \times A_2$ is upward-directed, i.e., that for any two elements $a = (a_1, a_2) \in A_1 \times A_2$ and $a' = (a_1', a_2') \in A_1 \times A_2$, there exists an element a^+ for which $a \preccurlyeq a^+$ and $a' \preccurlyeq a^+$.

Indeed, since the set A_1 is upward-directed, there exists an element a_1^+ for which $a_1 \preccurlyeq_1 a_1^+$ and $a_1' \preccurlyeq_1 a_1^+$. Similarly, since the set A_2 is upward-directed, there exists an element a_2^+ for which $a_2 \preccurlyeq_2 a_2^+$ and $a_2' \preccurlyeq_2 a_2^+$. By definition the order on the Cartesian product, we can now conclude that $(a_1, a_2) \preccurlyeq (a_1^+, a_2^+)$ and $(a_1', a_2') \preccurlyeq (a_1^+, a_2^+)$. Thus, the set $A_1 \times A_2$ is upward-directed.

Proof of Proposition 7.3 is similar to the proof of Proposition 7.2.

Proof of Proposition 7.4.

1°. Let us assume that $A_1 \times A_2$ is upward-directed.

1.1°. Let us prove that A_1 is upward-directed, i.e., that for every $a_1 \in A_1$ and $a_1' \in A_1$, there exists an element a_1^+ for which $a_1 \preccurlyeq_1 a_1^+$ and $a_1' \preccurlyeq_1 a_1^+$.

Indeed, since the product $A_1 \times A_2$ is upward-directed, for any $a_2 \in A_2$, there exists an element $a^+ = (a_1^+, a_2^+)$ for which $(a_1, a_2) \preccurlyeq a^+$ and $(a_1', a_2) \preccurlyeq a^+$. By definition of the lexicographic product, this implies that $a_1 \preccurlyeq_1 a_1^+$ and $a_1' \preccurlyeq_1 a_1^+$. Thus, the set A_1 is upward-directed.

1.2°. Let us now assume that, in addition to $A_1 \times A_2$ being upward-directed, the set A_1 has a maximal element $\overline{a}_1$. Let us prove that under these assumptions, the set A_2 is also upward-directed, i.e., for every $a_2, a_2' \in A_2$, there exists an element a_2^+ for which $a_2 \preccurlyeq_2 a_2^+$ and $a_2' \preccurlyeq_2 a_2^+$.

Indeed, since the product $A_1 \times A_2$ is upward-directed, there exists an element $a^+ = (a_1^+, a_2^+) \in A_1 \times A_2$ for which $(\overline{a}_1, a_2) \preccurlyeq a^+$ and $(\overline{a}_1, a_2') \preccurlyeq a^+$. By definition of the lexicographic product, this implies that $\overline{a}_1 \preccurlyeq_1 a_1^+$, i.e., that either $\overline{a}_1 \prec_1 a_1^+$ or $\overline{a}_1 = a_1^+$. Since the element $\overline{a}_1$ is maximal, we cannot have $\overline{a}_1 \prec_1 a_1^+$, so we have $a_1^+ = \overline{a}_1$. In this case, $(\overline{a}_1, a_2) \preccurlyeq a^+ = (\overline{a}_1, a_2^+)$, so by definition of the lexicographic order, we get $a_2 \preccurlyeq_2 a_2^+$. Similarly, we get $a'_2 \preccurlyeq_2 a_2^+$, so the set A_2 is indeed upward-directed.

2°. Let us now assume that A_1 is upward-directed, and that if A_1 has a maximal element, then A_2 is upward-directed. Let us prove that $A_1 \times A_2$ is upward-directed.

Indeed, let us take any two elements $a = (a_1, a_2)$ and $a' = (a'_1, a'_2)$ from $A_1 \times A_2$, and let us show that there exists an element a^+ for which $a \preccurlyeq a^+$ and $a' \preccurlyeq a^+$.

Since A_1 is upward-directed, there exists an element $a_1^+ \in A_1$ for which $a_1 \preccurlyeq_1 a_1^+$ and $a'_1 \preccurlyeq_1 a_1^+$.

Since $a_1 \preccurlyeq_1 a_1^+$, we have either $a_1 \prec_1 a_1^+$ or $a_1 = a_1^+$. Similarly, since $a'_1 \preccurlyeq_1 a_1^+$, we have either $a'_1 \prec_1 a_1^+$ or $a'_1 = a_1^+$. Thus, by considering both a_1 and a'_1, we have four possible situations:

- situation when $a_1 \prec_1 a_1^+$ and $a'_1 \prec_1 a_1^+$;
- situation when $a_1 \prec_1 a_1^+$ and $a'_1 = a_1^+$;
- situation when $a_1 = a_1^+$ and $a'_1 \prec_1 a_1^+$; and
- situation when $a_1 = a_1^+$ and $a'_1 = a_1^+$,

Let us consider these four situations one by one.

When $a_1 \prec_1 a_1^+$ and $a'_1 \prec_1 a_1^+$, then, by the definition of lexicographic order, we have $(a_1, a_2) \preccurlyeq (a_1^+, a'_2)$ and $(a'_1, a_2) \preccurlyeq (a_1^+, a'_2)$. So, in this case, we can take (a_1^+, a'_2) as the desired element a^+.

When $a_1 \prec_1 a_1^+$ and $a'_1 = a_1^+$, then, by the definition of lexicographic order, we have $(a_1, a_2) \preccurlyeq (a_1^+, a'_2)$. We also have $(a'_1, a_2) = (a_1^+, a'_2)$ hence $(a'_1, a_2) \preccurlyeq (a_1^+, a'_2)$. So, in this case, we can also take (a_1^+, a'_2) as the desired element a^+.

When $a_1 = a_1^+$ and $a'_1 \prec_1 a_1^+$, then, by the definition of lexicographic order, we have $(a'_1, a'_2) \preccurlyeq (a_1^+, a_2)$. We also have $(a_1, a_2) = (a_1^+, a_2)$ hence $(a_1, a_2) \preccurlyeq (a_1^+, a_2)$. So, in this case, we can also take (a_1^+, a_2) as the desired element a^+.

Finally, let us consider the situation when $a_1 = a_1^+ = a_1'$. In this situation, there are two possible sub-situations: when this element a_1 is maximal and when it is not maximal.

If the element a_1 is not maximal, then there exists a value a_1^+ for which $a_1 \prec_1 a_1^+$. In this case, by definition of the lexicographic product, we have $(a_1, a_2) \preccurlyeq (a_1^+, a_2)$ and $(a_1', a_2) \preccurlyeq (a_1^+, a_2)$. So, we can take (a_1^+, a_2) as the desired element a^+.

If the element a_1 is maximal, then A_2 is upward-directed. Thus, there exists an element $a_2^+ \in A_2$ for which $a_2 \preccurlyeq_2 a_2^+$ and $a_2' \preccurlyeq_2 a_2^+$. For this element, we have $(a_1, a_2) \preccurlyeq (a_1, a_2^+)$ and $(a_1, a_2') \preccurlyeq (a_1, a_2^+)$. So, we can take (a_1, a_2^+) as the desired element a^+.

In all four situations, there exists an element a^+ for which $a \preccurlyeq a^+$ and $a' \preccurlyeq a^+$. The statement is proven, and so is the proposition.

Proof of Proposition 7.5 is similar to the proof of Proposition 7.4.

Proof of Proposition 7.6.

1°. Let us assume that the Cartesian product $A_1 \times A_2$ is an upper semi-lattice. Let us prove that in this case, A_1 is also an upper semi-lattice (for A_2, the proof is the same).

We need to prove that for every two elements $a_1, a_1' \in A_1$, there exists an element a_1^+ for which $C^+_{a_1} \cap C^+_{a_1'} = C^+_{a_1^+}$, i.e., for which, for every element b_1, we have

$$(a_1 \preccurlyeq_1 b_1 \,\&\, a_1' \preccurlyeq_1 b_1) \Leftrightarrow a_1^+ \preccurlyeq_1 b_1. \tag{7.6.1}$$

To prove it, let us take an arbitrary element $a_2 \in A_2$ and consider two elements $a = (a_1, a_2)$ and $a' = (a_1', a_2)$. Since the set $A_1 \times A_2$ is an upper semi-lattice, there exists an element $a^+ = (a_1^+, a_2^+)$ which is a join of the elements a and a', i.e., for which, for every $b_1 \in A_1$ and $b_2 \in A_2$, we have

$$((a_1, a_2) \preccurlyeq (b_1, b_2) \,\&\, (a_1', a_2) \preccurlyeq (b_1, b_2)) \Leftrightarrow (a_1^+, a_2^+) \preccurlyeq (b_1, b_2). \tag{7.6.2}$$

Let us prove that the first component a_1^+ of this join a^+ is the desired join of a_1 and a_1', i.e., that for this first component, the condition (7.6.1) holds.

$\Leftarrow$ If $a_1^+ \preccurlyeq_1 b_1$, then by the definition of the Cartesian product, we have $(a_1^+, a_2^+) \le (b_1, a_2^+)$. Applying the $\Leftarrow$ part of the condition (7.6.2) with $b_2 = a_2^+$, we conclude that $(a_1, a_2) \preccurlyeq (b_1, a_2^+)$ and $(a_1', a_2) \preccurlyeq (b_1, a_2^+)$. By the definition of the Cartesian product, the first condition implies that $a_1 \preccurlyeq_1 b_1$, and the second condition implies that $a_1' \preccurlyeq_1 b_1$.

$\Rightarrow$ Vice versa, let us assume that $a_1 \preccurlyeq_1 b_1$ and $a_1' \preccurlyeq_1 b_1$. We need to prove that $a_1^+ \preccurlyeq_1 b_1$. In this case, by the definition of the Cartesian product, we have $(a_1, a_2) \le (b_1, a_2)$ and $(a_1', a_2) \le (b_1, a_2)$. Applying the $\Rightarrow$ part of the condition (7.6.2) with $b_2 = a_2$, we conclude that $(a_1^+, a_2^+) \preccurlyeq (b_1, a_2)$. By the definition of the Cartesian product, this implies that $a_1^+ \preccurlyeq_1 b_1$.

2°. It is easy to prove that if A_1 and A_2 are upper semi-lattices, then their Cartesian product $A_1 \times A_2$ is also an upper semi-lattice, with $(a_1, a_2) \vee (a_1', a_2') = (a_1 \vee a_1', a_2 \vee a_2')$.

The proposition is proven.

Proof of Proposition 7.7 is similar to the proof of Proposition 7.6.

Proof of Proposition 7.8.

1°. Let us assume that $A_1 \times A_2$ is an upper semi-lattice. Let us prove that in this case, A_1 is an upper semi-lattice, and one of the three properties described in the formulation of Proposition 7 holds.

1.1°. Let us prove that A_1 is an upper semi-lattice, i.e., that for every two elements $a_1, a_1' \in A_1$, there exists an element a_1^+ for which $C_{a_1}^+ \cap C_{a_1'}^+ = C_{a_1^+}^+$, i.e., for which, for every element b_1, we have

$$(a_1 \preccurlyeq_1 b_1 \,\&\, a_1' \preccurlyeq_1 b_1) \Leftrightarrow a_1^+ \preccurlyeq_1 b_1. \tag{7.6.3}$$

Let us take an arbitrary element $a_2 \in A_2$ and consider two elements $a = (a_1, a_2)$ and $a' = (a_1', a_2)$. Since the set $A_1 \times A_2$ is an upper semi-lattice, there exists an element

$a^+ = (a_1^+, a_2^+)$ which is a join of the elements a and a', i.e., for which, for every $b_1 \in A_1$ and $b_2 \in A_2$, we have

$$((a_1, a_2) \preccurlyeq (b_1, b_2)\,\&\,(a'_1, a_2) \preccurlyeq (b_1, b_2)) \Leftrightarrow (a_1^+, a_2^+) \preccurlyeq (b_1, b_2). \tag{7.6.4}$$

Let us prove that the first component a_1^+ of this join a^+ is the desired join of a_1 and a'_1, i.e., that for this first component, the condition (7.6.3) holds.

$\Leftarrow$ If $a_1^+ \preccurlyeq_1 b_1$, then, by the definition of the lexicographic product, we have $(a_1^+, a_2^+) \le (b_1, a_2^+)$. Applying the $\Leftarrow$ part of the condition (7.6.4) with $b_2 = a_2^+$, we conclude that $(a_1, a_2) \preccurlyeq (b_1, a_2^+)$ and $(a'_1, a_2) \preccurlyeq (b_1, a_2^+)$. By the definition of the lexicographic product, the first condition implies that $a_1 \preccurlyeq_1 b_1$, and the second condition implies that $a'_1 \preccurlyeq_1 b_1$.

$\Rightarrow$ Vice versa, let us assume that $a_1 \preccurlyeq_1 b_1$ and $a'_1 \preccurlyeq_1 b_1$. We need to prove that $a_1^+ \preccurlyeq_1 b_1$. To prove this statement, we will consider four possible cases:

- case when $a_1 \prec_1 b_1$ and $a'_1 \prec_1 b_1$;
- case when $a_1 \prec_1 b_1$ and $a'_1 = b_1$;
- case when $a_1 = b_1$ and $a'_1 \prec_1 b_1$; and
- case when $a_1 = b_1$ and $a'_1 = b_1$.

1) When $a_1 \prec_1 b_1$ and $a'_1 \prec_1 b_1$, then, by the definition of the lexicographic order, we have $(a_1, a_2^+) \preccurlyeq (b_1, a_2^+)$ and $(a'_1, a_2^+) \preccurlyeq (b_1, a_2^+)$. Thus, due to the $\Rightarrow$ part of (7.6.4), with $b_2 = a_2^+$, we get $(a_1^+, a_2^+) \preccurlyeq (b_1, a_2^+)$. So, by the definition of the lexicographic product, we have $a_1^+ \preccurlyeq_1 b_1$.

2) When $a_1 \prec_1 b_1$ and $a'_1 = b_1$, we have $a_1 \prec_1 a'_1$. Here, $(a'_1, a_2) = (a'_1, a_2)$ hence $(a'_1, a_2) \preccurlyeq (a'_1, a_2)$, and $a_1 \prec_1 a'_1$ implies that $(a_1, a_2) \prec (a'_1, a_2)$. Thus, due to the $\Rightarrow$ part of (7.6.4), with $b_1 = a'_1$ and $b_2 = a_2$, we get $(a_1^+, a_2^+) \preccurlyeq (a'_1, a_2)$. So, by the definition of the lexicographic product, we have $a_1^+ \preccurlyeq_1 a'_1 = b_1$.

3) When $a_1 = b_1$ and $a'_1 \prec_1 b_1$, we have $a'_1 \prec_1 a_1$. Here, $(a_1, a_2) = (a_1, a_2)$ hence $(a_1, a_2) \preccurlyeq (a_1, a_2)$, and $a'_1 \prec_1 a_1$ implies that $(a'_1, a_2) \prec (a_1, a_2)$. Thus, due to the $\Rightarrow$ part of (7.6.4), with $b_1 = a_1$ and $b_2 = a_2$, we get $(a_1^+, a_2^+) \preccurlyeq (a_1, a_2)$. So, by the definition of the lexicographic product, we have $a_1^+ \preccurlyeq_1 a_1 = b_1$.

4) Finally, when $a_1 = b_1 = a'_1$, we have $a_1 = a'_1$. In this case, $(a_1, a_2) = (a'_1, a_2)$, so the condition $(a_1, a_2) \preccurlyeq (b_1, b_2)$ is simply equivalent to $(a'_1, a_2) \preccurlyeq (b_1, b_2)$. In this case, the condition (7.6.4) simply means that

$$(a_1, a_2) \preccurlyeq (b_1, b_2) \Leftrightarrow (a_1^+, a_2^+) \preccurlyeq (b_1, b_2). \tag{7.6.5}$$

For $(b_1, b_2) = (a_1, a_2)$, the $\Rightarrow$ part of the formula (7.6.5) implies that $(a_1^+, a_2^+) \preccurlyeq (a_1, a_2)$. For $(b_1, b_2) = (a_1^+, a_2^+)$, the $\Leftarrow$ part the formula (7.6.5) implies that $(a_1, a_2) \preccurlyeq (a_1^+, a_2^+)$. Thus, $(a_1^+, a_2^+) = (a_1, a_2)$, and hence, $a_1^+ = a_1$ $(= a'_1)$. Since $a_1^+ = a_1 = a'_1$, the condition (7.6.3) is clearly satisfied.

In all four cases, we have the desired proof, so the statement is proven. So, A_1 is indeed an upper semi-lattice.

1.2°. Let us prove that if the set A_1 is not linearly ordered, then the set A_2 has the smallest element.

Indeed, the fact that the set A_1 is not linearly ordered means that there exist values a_1 and a'_1 for which $a_1 \not\preccurlyeq_1 a'_1$ and $a'_1 \not\preccurlyeq_1 a_1$.

Let us take an arbitrary $a_2 \in A_2$. Since $A_1 \times A_2$ is an upper semi-lattice, for $a = (a_1, a_2)$ and $a' = (a'_1, a_2)$, there exists an element $a^+ = (a_1^+, a_2^+)$ for which $C_a^+ \cap C_{a'}^+ = C_{a^+}^+$, i.e., for which, for every $b_1 \in A_1$ and $b_2 \in A_2$, we have

$$((a_1, a_2) \preccurlyeq (b_1, b_2) \,\&\, (a'_1, a_2) \preccurlyeq (b_1, b_2)) \Leftrightarrow (a_1^+, a_2^+) \preccurlyeq (b_1, b_2). \tag{7.6.6}$$

Let us prove that a_2^+ is the smallest element of the set A_2.

Indeed, the formula (7.6.6) implies, for $(b_1, b_2) = (a_1^+, a_2^+)$, we have $(a_1^+, a_2^+) = (b_1, b_2)$ and hence, $(a_1^+, a_2^+) \preccurlyeq (b_1, b_2)$. Thus, due to the $\Leftarrow$ part of the formula (7.6.6), we conclude that $(a_1, a_2) \preccurlyeq (a_1^+, a_2^+)$ and $(a_1', a_2) \preccurlyeq (a_1^+, a_2^+)$. By definition of the lexicographic order, we conclude that $a_1 \preccurlyeq_1 a_1^+$ and $a_1' \preccurlyeq_1 a_1^+$.

We cannot have $a_1 = a_1^+$, since then we would have $a_1' \preccurlyeq_1 a_1$ – which contradicts to our choice of a_1 and a_1'. Thus, $a_1 \preccurlyeq_1 a_1^+$ implies that $a_1 \prec_1 a_1^+$.

Similarly, we cannot have $a_1' = a_1^+$, since then we would have $a_1 \preccurlyeq_1 a_1'$ – which also contradicts to our choice of a_1 and a_1'. Thus, $a_1' \preccurlyeq_1 a_1^+$ implies that $a_1' \prec_1 a_1^+$.

Let a_2' be an arbitrary element of the set A_2. Since $a_1 \prec_1 a_1^+$ and $a_1' \prec_1 a_1^+$, we have $(a_1, a_2) \prec (a_1^+, a_2')$ and $(a_1', a_2) \prec (a_1^+, a_2')$. Thus, by applying the $\Rightarrow$ part of the formula (7.6.6) to $b_1 = a_1^+$ and $b_2 = a_2'$, we conclude that $(a_1^+, a_2^+) \preccurlyeq (a_1^+, a_2')$. By the definition of the lexicographic order, this means that $a_2^+ \preccurlyeq_2 a_2'$. Thus, a_2^+ precedes all the elements of A_2, so it is indeed the smallest element of A_2.

1.3°. Let us now prove that A_2 is a conditional semi-lattice, i.e., that for every $a_2 \in A_2$ and $a_2' \in A_2$ for which $a_2 \preccurlyeq_2 a_2''$ and $a_2' \preccurlyeq_2 a_2''$ for some a_2'', there exists an element a^+2 for which $C^+_{a_2} \cap C^+_{a_2'} = C^+_{a_2^+}$, i.e., for which, for every $b_1 \in A_2$, we have

$$(a_2 \preccurlyeq_2 b_2 \,\&\, a_2' \preccurlyeq_2 b_2) \Leftrightarrow a_2^+ \preccurlyeq_2 b_2. \tag{7.6.7}$$

Indeed, let us take an arbitrary element $a_1 \in A_1$. Since the set $A_1 \times A_2$ is an upper semi-lattice, for elements (a_1, a_2) and (a_1', a_2), there exists an element $a^+ = (a_1^+, a_2^+)$ for which

$$C^+_{(a_1,a_2)} \cap C^+_{(a_1',a_2)} = C^+_{(a_1^+,a_2^+)}.$$

In other words, for every $b_1 \in A_1$ and $b_2 \in A_2$, we have

$$((a_1, a_2) \preccurlyeq (b_1, b_2) \,\&\, (a_1, a_2') \preccurlyeq (b_1, b_2) \Leftrightarrow (a_1^+, a_2^+) \preccurlyeq (b_1, b_2). \tag{7.6.8}$$

Let us show that the second component a_2^+ of this element is the desired join of a_2 and a_2'.

For that, let us first prove that $a_1^+ = a_1$. Indeed, from $(a_1^+, a_2^+) = (a_1^+, a_2^+)$, we conclude that $(a_1^+, a_2^+) \preccurlyeq (a_1^+, a_2^+)$. So, from the $\Leftarrow$ part of the formula (7.6.8) for $(b_1, b_2) = (a_1^+, a_2^+)$, we conclude that $(a_1, a_2) \preccurlyeq (b_1, b_2)$, i.e., that $(a_1, a_2) \preccurlyeq (a_1^+, a_2^+)$. By the definition of the lexicographic order, this implies that $a_1 \preccurlyeq_1 a_1^+$.

Now, $a_2 \preccurlyeq_2 a_2''$ and $a_2' \preccurlyeq_2 a_2''$. So, by definition of the lexicographic order, we have $(a_1, a_2) \preccurlyeq (a_1, a_2'')$ and $(a_1, a_2') \preccurlyeq (a_1, a_2'')$. Thus, due to the $\Rightarrow$ part of the formula (7.6.8), with $(b_1, b_2) = (a_1, a_2'')$, we conclude that $(a_1^+, a_2^+) \preccurlyeq (a_1, a_2'')$. Thus, by definition of the lexicographic order, we conclude that $a_1^+ \preccurlyeq_1 a_1$. From $a_1 \preccurlyeq_1 a_1^+$ and $a_1^+ \preccurlyeq_1 a_1$, we conclude that $a^+1 = a_1$. Thus, the formula (7.6.8) takes the form

$$((a_1, a_2) \preccurlyeq (b_1, b_2) \,\&\, (a_1, a_2') \preccurlyeq (b_1, b_2)) \Leftrightarrow (a_1, a_2^+) \preccurlyeq (b_1, b_2). \tag{7.6.9}$$

Now, we are ready to prove that $a_2^+ = a_2 \vee a_2'$, i.e., that the formula (7.6.7) holds.

$\Leftarrow$ Let $a_2^+ \preccurlyeq_2 b_2$. Then, by definition of the lexicographic order, we have $(a_1, a^+2) \preccurlyeq (a_1, b_2)$. Due to the $\Leftarrow$ part of the formula (7.6.9), with $b_1 = a_1$, we get $(a_1, a_2) \preccurlyeq (b_1, b_2) = (a_1, b_2)$. By definition of the lexicographic order, this implies that $a_2 \preccurlyeq_2 b_2$. Similarly, we conclude that $a_2' \preccurlyeq_2 b_2$.

$\Rightarrow$ Let $a_2 \preccurlyeq_2 b_2$ and $a_2' \preccurlyeq_2 b_2$. Then, by definition of the lexicographic order, we have $(a_1, a_2) \preccurlyeq (a_1, b_2)$ and $(a_1, a_2') \preccurlyeq (a_1, b_2)$. Due to the $\Rightarrow$ part of the formula (7.6.9), with $b_1 = a_1$, we get $(a_1, a_2^+) \preccurlyeq (b_1, b_2) = (a_1, b_2)$. By definition of the lexicographic order, this implies that $a_2^+ \preccurlyeq_2 b_2$.

The formula (7.6.7) is proven.

1.4°. Let us now prove that is A_2 is a not an upper semi-lattice, then A_1 is sequential up and A_2 has the smallest element.

We have already proven that A_2 is a conditional upper semi-lattice, i.e., that every two elements a_2 and a_2' for which there exists an element a_2'' with $a_2 \preccurlyeq_2 a_2''$ and $a_2' \preccurlyeq_2 a_2''$ have a joint $a_1 \vee a_2'$. Since A_2 is not an upper semi-lattice, this means that there exist elements a_2 and a_2' for which no element from A_2 follows both these elements.

Let us prove that A_2 has the smallest element and that A_1 is sequential up. For that, let us select any element $a_1 \in A_1$. Since $A_1 \times A_2$ is an upper semi-lattice, for the elements

$a = (a_1, a_2)$ and $a' = (a_1, a_2')$, there exists a join $a^+ = (a_1^+, a_2^+)$, i.e., an element for which, for every $b_1 \in A_1$ and $b_2 \in A_2$, we have

$$((a_1, a_2) \preccurlyeq (b_1, b_2) \,\&\, (a_1, a_2') \preccurlyeq (b_1, b_2) \Leftrightarrow (a_1^+, a_2^+) \preccurlyeq (b_1, b_2). \qquad (7.6.10)$$

Let us show a_1^+ is the next element to a_1 and that a^+2 is the smallest element of the set A_2. This will prove that A_1 is sequential up and that the set A_2 has the smallest element.

1.4.1°. To prove that a_1^+ is the next element to a_1, we need to prove that for every $a_1' \in A_1$, we have $a_1 \prec_1 a_1' \Leftrightarrow a_1^+ \preccurlyeq_1 a_1'$.

$\Rightarrow$ Let $a_1 \prec_1 a_1'$. Then, for an arbitrary $b_2 \in A_2$, by definition of the lexicographic order, we have $(a_1, a_2) \preccurlyeq (a_1', b_2)$ and $(a_1, a_2') \preccurlyeq (a_1', b_2)$. Thus, due to the $\Rightarrow$ part of the formula (7.6.10), with $b_1 = a_1'$, we conclude that $(a_1^+, a_2^+) \preccurlyeq (a_1', b_2)$. Thus, by definition of the lexicographic order, we have $a_1^+ \preccurlyeq_1 a_1'$.

$\Leftarrow$ Let $a_1^+ \preccurlyeq_1 a_1'$. Then, for $b_2 = a_2^+$, by definition of the lexicographic order, we have $(a_1^+, a_2^+) \preccurlyeq (a_1', a_2^+)$. Thus, due to the $\Leftarrow$ part of the formula (7.6.10), with $b_1 = a_1'$ and $b_2 = a_2^+$, we conclude that $(a_1, a_2) \preccurlyeq (a_1', a_2^+)$ and $(a_1, a_2) \preccurlyeq (a_1', a_2^+)$. By definition of the lexicographic order, we thus have $a_1 \preccurlyeq_1 a_1'$, i.e., either $a_1 \prec_1 a_1'$ or $a_1 = a_1'$.

We cannot have $a_1' = a_1$ because then, by definition of the lexicographic order, we would have $a_2 \prec_2 a_2^+$ and $a_2' \preccurlyeq_2 a_2^+$, which contradicts to our choice of a_2 and a_2' as the elements for which there is no elements following both. Thus, we have $a_1 \prec_1 a_1'$.

1.4.2°. To prove that a_2^+ is the smallest element of the set A_2, we must prove that for every $b_2 \in A_2$, we have $a_2^+ \preccurlyeq_2 b_2$.

Indeed, since $a_1 \prec_1 a_1^+$, for every $b_2 \in A_2$, we have $(a_1, a_2) \preccurlyeq (a_1^+, b_2)$ and $(a_1, a_2') \preccurlyeq (a_1^+, b_2)$. Thus, due to the $\Rightarrow$ part of the formula (7.6.10), with $b_1 = a_1^+$, we conclude that $(a_1^+, a_2^+) \preccurlyeq (a_1^+, b_2)$. By definition of the lexicographic order, this implies that $a_2^+ \leq_2 b_2$. The statement is proven.

2°. Let us now prove that in all three cases described in the formulation of Proposition 7, the lexicographic product $A_1 \times A_2$ is an upper semi-lattice, i.e., for every two elements

$a = (a_1, a_2)$ and $a' = (a'_1, a'_2)$, there exists a join $(a_1, a_2) \vee (a'_1, a'_2)$.

In this proof, we simply describe the corresponding joins. Proving that the described elements are indeed joins is (reasonably) easy.

2.1°. Let us first consider the case when A_1 is linearly ordered and A_2 is an upper semi-lattice.

In this case, since the set A_1 is linearly ordered, for every two elements $a = (a_1, a_2)$ and $a' = (a'_1, a'_2)$, we have either $a_1 \prec_1 a'_1$, or $a'_1 \prec_1 a_1$, or $a_1 = a'_1$. Let us consider these three cases one by one.

- When $a_1 \prec_1 a'_1$, by definition of the lexicographic product, we have $(a_1, a_2) \prec (a'_1, a'_2)$, hence $(a_1, a_2) \vee (a'_1, a'_2) = (a'_1, a'_2)$.
- When $a'_1 \prec_1 a_1$, by definition of the lexicographic product, we have $(a'_1, a'_2) \prec (a_1, a_2)$, hence $(a_1, a_2) \vee (a'_1, a'_2) = (a_1, a_2)$.
- Finally, when $a_1 = a'_1$, we have $(a_1, a_2) \vee (a_1, a'_2) = (a_1, a_2 \vee a'_2)$.

2.2°. Let us now consider the case when A_1 is an upper semi-lattice, and A_2 is an upper semi-lattice that has the smallest element $\underline{a}_2$.

In this case, similarly to the previous case,

- When $a_1 \prec_1 a'_1$, we have $(a_1, a_2) \prec (a'_1, a'_2)$, hence
$$(a_1, a_2) \vee (a'_1, a'_2) = (a'_1, a'_2).$$
- When $a_1 \prec_1 a'_1$, we have $(a_1, a_2) \prec (a'_1, a'_2)$, hence
$$(a_1, a_2) \vee (a'_1, a'_2) = (a'_1, a'_2).$$
- When $a_1 = a'_1$, we have
$$(a_1, a_2) \vee (a_1, a'_2) = (a_1, a_2 \vee a'_2).$$

When $a_1 \not\preccurlyeq_1 a'_1$ and $a'_1 \not\preccurlyeq_1 a_1$, then, since A_1 is an upper semi-lattice, there exists an element $a_1 \vee a'_1$. It is easy to prove that this element is different from a_1 and a'_1. We can then take $(a_1, a_2) \vee (a_1, a'_2) = (a_1 \vee a'_1, \underline{a}_2)$.

2.3°. Finally, let us consider the case when A_1 is an upper semi-lattice which is sequential up, A_2 is a conditional upper semi-lattice, and A_2 has the smallest element $\underline{a}_2$.

To describe the join operation, let us consider all possible relations between a_1 and a'_1.

- When $a_1 \prec_1 a'_1$, we have $(a_1, a_2) \prec (a'_1, a'_2)$, hence

$$(a_1, a_2) \vee (a'_1, a'_2) = (a'_1, a'_2).$$

- When $a_1 \prec_1 a'_1$, we have $(a_1, a_2) \prec (a'_1, a'_2)$, hence

$$(a_1, a_2) \vee (a'_1, a'_2) = (a'_1, a'_2).$$

- When $a_1 \not\preccurlyeq_1 a'_1$ and $a'_1 \not\preccurlyeq_1 a_1$, we have

$$(a_1, a_2) \vee (a_1, a'_2) = (a_1 \vee a'_1, \underline{a}_2).$$

- When $a_1 = a'_1$, and for elements $a_2, a'_2 \in A_2$, these exists an element a''_2 that follows both a_2 and a'_2, then (since A_2 is a conditional upper semi-lattice), there exists a join $a_2 \vee a'_2$ of these two elements. So, we can take

$$(a_1, a_2) \vee (a_1, a'_2) = (a_1, a_2 \vee a'_2).$$

- Finally, if $a_1 = a'_1$, and there are no elements a''_2 which follow both a_2 and a'_2, then we take

$$(a_1, a_2) \vee (a_1, a'_2) = (a_1^+, \underline{a}_2),$$

  where a_1^+ is the next element to a_1.

Proof of Proposition 7.9 is similar to the proof of Proposition 7.8.

Chapter 8

Conclusions and Future Work

Motivations. One of the main objectives of science and engineering is to help people select the most beneficial decisions. To make these decisions, we must know people's preferences, we must have the information about different possible consequences of different decisions. Since information is never absolutely accurate and precise, we must also have information about the degree of certainty of different parts on information. All these types of information naturally lead to partial orders:

- For preferences, $a \preccurlyeq b$ means that b is preferable to a. This relation is used in decision theory.
- For events, $a \preccurlyeq b$ means that a can influence b. This causality relation is one of the fundamental notions of physics, especially of physics of space-time.
- For uncertain statements, $a \preccurlyeq b$ means that a is less certain than b. This relation is used in logics describing uncertainty, such as fuzzy logic.

In each of these areas, there is abundant research about studying the corresponding partial orders. This research has shown that *some ideas are common in all three applications of partial orders*. In this dissertation, we analyzed general properties, operations, and algorithms related to partial orders for representing uncertainty, causality, and decision making, with a special emphasis on uncertainty.

Results. Under uncertainty, instead of a *single* partial order, we have a *class* C of possible partial orders. In such situations, it makes sense to ask when it is *possible* that a precedes b (i.e., when a precedes b according to *some* of these orders), and when it is *necessary* that

a precedes b (i.e., when a precedes b according to *all* these orders). In Chapter 2, we gave a general characterization of such "possible order" and "necessary order" relations.

In Chapter 3, we considered a special case of such a situation, when different partial orders result from measurements with different accuracy. In this case, we can distinguish between the original ("closed") partial order $\preccurlyeq$ and the "open" partial order relation $a \prec b$ meaning that $a \preccurlyeq b$ and we can verify this based on measurements (i.e., $a \preccurlyeq \widetilde{b}$ for all $\widetilde{b}$ from some neighborhood of b). It has been proven that once we know the open order, then we can uniquely reconstruct the closed order. Whether it is possible, vice versa, to reconstruct the open order from the closed one was an open problem. In Chapter 3, we proved that, under reasonable conditions, such a reconstruction is indeed possible.

In Chapter 4, we moved from *potentially* detectable (measurable) orders to orders which can be detected for a given accuracy. A typical example is when we only know the lower bound $\underline{a}$ and the upper bound $\overline{a}$ for an object a; in this case, we only know that a belongs to the *interval* $[\underline{a}, \overline{a}]$. In Chapter 4, we describe all possible relations between such intervals.

Once an order is defined, we are interested in its *properties*, e.g., whether the order is a *lattice*. For special-relativity-type partial orders, a new necessary and sufficient criterion for being a lattice was described in Chapter 5.

In many practical applications, we need to *combine* different partial orders. In Chapter 6, we described all possible *combination operations*, and in Chapter 7, we provided a *general algorithm* that reduces the analysis of properties of such combined spaces to properties of individual partially ordered spaces.

Future work: practical applications of results related to partial orders. While most of the results from this dissertation were motivated by applications – to physics, to fuzzy logic, etc. – at this stage, most of these results are *mostly theoretical*. It is therefore important to develop our ideas and results further, so that they can be applied to solving *practical problems*.

For example, in considering combination operations relevant for expert systems and fuzzy logic, we mostly analyzed the situations when we combine *two* spaces – which corresponds to combining opinions of *two* experts. We considered two cases:

- the case when we are equally confident in both experts, and
- the case when we have more confidence in one of these experts.

In practice, we often have *more than two* experts. In such situations, the relation "more confident" – describing when we have more confidence in one expert in comparison to another one – is, in general, a *partial order*. It is desirable to generalize our combination operations to such a general case, when:

- we combine *several* partially ordered sets, and
- there is a partial order *between* the labels describing these sets.

Another example when such a multiple combination is desirable is *group decision making*, when:

- we start with individual preferences of different members of the group, and
- we need to combine these individual preferences into a coherent group decision.

Future work: possible applications beyond partial orders. In practice, as we have mentioned in this dissertation, not all important binary relations are partial orders. This was the main reason why some of our results have been formulated and proven not only for partial orders, but also for *relations* that are *more general* than partial orders. For example, the general algorithm for analyzing properties of product spaces (described in Chapter 7) is applicable to general binary (and even ternary) relations. It is therefore desirable to develop applications for such more general results.

Possible applications include *software testing*, where:

- we start with test suites describing tested values of each parameter, and

- we need to combine them to come up with a set of pairs (triples, etc.), which would allow us to test all possible values of each parameter, and/or all possible pairs of values of two parameters, etc.

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Curriculum Vitae

Francisco Zapata was born in Chihuahua, México. The first son of Elisa González Morgado and Francisco Adolfo Zapata Herrera, he earned his high school diploma from Colegio de Bachilleres #3 in Chihuahua, México. He obtained Bachelor's and Master's of Engineering in Computer Systems in 2001 and 2007 from the Universidad Autónoma de Chihuahua in México. While pursuing his Masters, he worked as a Technical Support Engineer in the Network Operations Center, as a Computer Center Coordinator, and also taught several courses in Java Programming, Networking, and Computer Maintenance in the School of Engineering of the Universidad Autónoma de Chihuahua. In the industry, he has worked as Information Technology Consultant for Camionera de Chihuahua, S.A. de C.V. Mercedes Benz, and Freightliner Comercial Vehicles México helping them deploy and operate their information systems across the Chihuahua state.

In the Fall of 2008, he entered the Graduate School at the University of Texas at El Paso and started the Ph.D. program in Computer Science. While pursuing his Ph.D., he worked as a Teaching Assistant for Elementary Algorithms and Data Structures, Computer Architecture, and Operating Systems courses, and as a Research Assistant in the Research Institute for Manufacturing and Engineering Systems.

He is a member of the El Paso Student Chapter of the Association for Computing Machinery (ACM) and an Associate member of the Scientific Research Society Sigma Xi.

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