Consistency Techniques over Real Numbers

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Reduction and Propagation

- Compute the solution set of \( \{ y = x^2, x \geq y + 1 \} \) in the box \([-10, 10]^2\)

- Propagation = sequence of reductions

\[
\begin{align*}
y &= x^2 & \implies & y \in [0, 10] \\
x &\geq y + 1 & \implies & x \in [1, 10] \\
y &\leq x - 1 & \implies & y \in [0, 9] \\
y &= x^2 & \implies & y \in [1, 9] \\
y &= x^2 & \implies & x \in [1, 3] \\
\ldots & \ldots & \ldots & \ldots
\end{align*}
\]
Projections of a Constraint

- Constraint $c(x_1, \ldots, x_n) \subseteq \mathbb{R}^n$
- Box $d = d_1 \times \cdots \times d_n$
- Projection of $c$ over $x_i$

\[
\{ a_i \in d_i \mid \exists a_1 \in d_1, \ldots, \\
\exists a_{i-1} \in d_{i-1}, \\
\exists a_{i+1} \in d_{i+1}, \ldots, \\
\exists a_n \in d_n : \\
c(a_1, \ldots, a_n) \}
\]
Approximation of Projections

- Projections cannot be represented by floating-point intervals
- Approximation

- Reduction is computation of some yellow box
Reduction: Inversion

- Cleary 87
- Reduce the domains of $x$ and $y$ s.t. $y = \exp(x)$

\[
\begin{align*}
  y &= \exp(x) \\
  x &\in \mathbf{x} = [u, v] \\
  y &\in \mathbf{y} = [a, b] \\
  \mathbf{y} &:= \mathbf{y} \cap \exp([u, v]) \\
  \mathbf{x} &:= \mathbf{x} \cap \exp^{-1}([a, b])
\end{align*}
\]
Inversion for Complex Constraints

- Cleary 87
- Decomposition into primitives

\[ 2x = z - y^2 \iff \begin{cases} 
\alpha &= 2 \times x \\
\beta &= y^2 \\
\alpha &= z - \beta 
\end{cases} \]

for some \( \alpha, \beta \in \mathbb{R} \)

- Slow compilation and propagation processes
Inversion for Complex Constraints

- Benhamou et al. 99

\[ [0, 16] \times [0, 40] \]

\[ 2 \times [2, 2] [0, 8] x [0, 20] \]

\[ \alpha = 2 \times x \]

\[ x \in [0, 20], \alpha \in [0, 16] \]

\[ \Rightarrow x \in [0, 8] \]
The inversion technique can be weak

\[ x + x = 0, \; x \in d = [-1, 1] \]

\[ d \; := \; d \cap (-d) \]
\[ := \; [-1, 1] \cap [-1, 1] \]
\[ := \; [-1, 1] \]

Solutions
- symbolic transformation
- bisection-evaluation process
Reduction: Box-consistency

- Benhamou & McAllester & Van Hentenryck 94
- Reduce the domain of $x$ st. $f(x) = 0$
Box-co. for Multivariate Constraints

- Consider a constraint \( f(x_1, \ldots, x_n) = 0 \) and an interval form \( f \) of \( f \)
- Given a box \( d \) box-consistency is computed for the set of interval functions

\[
\begin{align*}
  f_1(x_1, d_2, d_3, \ldots, d_n) \\
  f_2(d_1, x_2, d_3, \ldots, d_n) \\
  \quad \vdots \\
  f_n(d_1, d_2, \ldots, d_{n-1}, x_n)
\end{align*}
\]
Reduction Model

- Let $\mathbb{I}$ be the set of closed intervals
- A reduction operator is a function $\mathbb{I}^n \rightarrow \mathbb{I}^n$ which is for all $x, y \in \mathbb{I}$
  - monotonic: $x \subseteq y \implies \theta(x) \subseteq \theta(y)$
  - contracting: $\theta(x) \subseteq x$
- Consistency techniques are monotonic and contracting
- These techniques are also verified!
Propagation Model


- Lemma: given a finite set of reduction operators and a domain \( d \) the greatest common fixed-point of the operators included in \( d \) exists

- Propagation = application of operators
  - fair strategy \( \iff \) convergence
  - finiteness of domain \( \iff \) termination
  - independence wrt. the strategy
Fixed-point Propagation Algorithm

\[ S := \{O_1, \ldots, O_n\} \]

Box \( d \)

repeat

\[ \text{choose } O \text{ in } S \]

\[ \text{save := } d \]

\[ d := O(d) \]

\[ \text{if } d = \text{save then} \]

\[ S := S - \{O\} \]

else

\[ S := \{O_j | O_j \text{ depends on a modified domain}\} \]

until \( S \) is empty
Solver Cooperation

1. Generation of reduction operators
   - Inversion
   - Box-consistency
   - Interval methods
   - Simplex

2. Propagation over the set of operators
   - good strategy (choose) $\implies$ efficiency
A Strategy

\[ S := \text{empty} \]

\text{for each constraint } C \text{ do}

\text{if there exists } X \text{ in } C \text{ occurring once then } S := S \cup \{ \text{inversion}(c) \} \]

\text{for each variable } X \text{ in } C \text{ do}

\text{if } X \text{ occurs more than once in } C \text{ then } S := S \cup \{ \text{box}(C,X) \} \]

\text{od}

\text{if a square system of equations } E \text{ can be computed then } S := S \cup \{ \text{IntervalNewton}(E) \} \]
Locality Problem

- Consider a difficult problem for inversion or box-consistency

- The intersection of projections is weaker than the projection of the intersection
Reduction: 3B-consistency

- Lhomme 1993
- Prove the inconsistency of a sub-domain at one bound of a variable domain
Bisection Algorithm

- Classical algorithm
  - precision of boxes: $10^{-8}$
  - choice of variable: round-robin
  - bisection in 3 parts

- Reductions = cooperation of
  - I: inversion
  - B: box-consistency
  - N: interval Newton
## Results

<table>
<thead>
<tr>
<th>Problem</th>
<th>( v )</th>
<th>( n )</th>
<th>I-B-N</th>
<th>I-B</th>
<th>B-N</th>
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</table>
Conclusion

- Consistency techniques are used in projects
  - automatic control (CNRS group on set-based methods)
  - computer-aided design (EPFL, RNTL CO2)
  - global optimization (COCONUT project)
  - image synthesis (PRIAMM project)

- Research directions: new types of constraints, specific propagation algorithms, inner approximations, quantified constraints, mixed-problems...