Interval Computations in Maple

using intpakX v1.0

Schloss Dagstuhl, 01/2003
1. intpakX - History

2. intpakX - Functional Range and Realization
   - Real Interval Arithmetic
     - Types and operations
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   - Complex Disc Arithmetic
     - Types and operations
     - An Application

3. intpakX - Examples
   - Interval Newton Method
   - Range Enclosure (2D and 3D)
   - Complex Disc Arithmetic
• 1993: \textit{intpak} by R. Corless and A. Connell
  – Interval Type
  – Basic Arithmetic operations
  – Standard Functions
• 1999: \textit{intpakX} by W. Krämer and I. Geulig (Add-on)
  – Interval Newton Method
  – Range Enclosure
  – Complex Disc Arithmetic
  – Graphical output
intpakX V1.0

- Update for Maple from Version 6
- Integrates intpak and intpakX (formerly separate packages)
- Redesign as Maple-Module
- Language integration
- Published as a Maple PowerTool Interval Arithmetic in 12/2002
Real Intervals

Interval type:

- Maple List with additional properties
  (i.e. intervals \([x_1, x_2]\),
  \(x_1, x_2\) float or \(\pm\) infinity, \(x_1 \leq x_2\))

- `construct`, `convert` and `inapply` (`"interval unapply"`)
Operators and standard functions

- Operators \&+, \&−, \&*, \&/ for intervals
- Additional function for extended division
- Interval intersection/union
- Power, square/root, logarithm and exponential functions
- Trigonometric functions
- Rounding functions included

→ Verified computations with guaranteed interval bounds possible (provided appropriate use of the Maple variable \texttt{Digits})
Applicability

- suited for numerical computations as the operators described above or as the applications presented later

Effects to be paid attention to:

- Computations that need procedures other than the ones mentioned or that need to identify special operators; i.e. the degree function does not work with interval polynomials like
  \[ [1.0,2.0] \times x^2 + [-1.5,-1.0] \]

- No automatical simplification of functions:
  \[ A + 2 \times A \] is not simplified into \[ 3 \times A \].
  (can affect the success of symbolical computations).
Iteration:

\[ [x]^0 := [x], \quad [x]^{k+1} := \left( m([x]^k) - \frac{f(m([x]^k)))}{f'(m([x]^k))} \right) \cap [x]^k \]

- Computes enclosing intervals for all zeros of a cont. diff. function \( f \) with guaranteed bounds
- arbitrary start interval
- automatical use of interval operators/functions
- adjustable precision (digits, diameter of solution interval) and number of iterations
- graphical output of iteration steps (optional)
Applications(II)  Range Enclosure

- for real-valued functions of one or two real variables
- in 2D, interval evaluation and mean value form are combined
- in 3D, only interval evaluation is done
- iterative interval subdivision
- automatical use of interval operators/functions
- adjustable number of iterations
Complex Intervals

- Disc intervals: **List** with three numerical values, interpreted as midpoint and radius (non-negative) of the disc
- Arithmetic operations: \&cadd, \&csup, \&cmult, \&cdiv \ (centered)
- Additional: Area-optimal multiplication and division
- Exponential function for complex discs
An Application:
Range Enclosure for Complex Polynomials

- Range Enclosure based on a Horner-Scheme using centered multiplication
  \&horner\_eval\_cent

- Range Enclosure based on a Horner-Scheme using area-optimal multiplication
  \&horner\_eval\_opt

- Range Enclosure using centered forms (similar to mean value form for real numbers)
  \&centred\_form\_eval
Enclosure of the zeros of

\[ f := x \rightarrow \sin(\exp(\sqrt{x - 2})) \; \]

on the interval \([8.,10.]\):

Load \texttt{intpakX}:

\begin{verbatim}
> restart;
> libname:="/opt/lib/mymaplelib", libname;
> with(intpakX);
\end{verbatim}

Enclose zeros:

\begin{verbatim}
> f:=x->sin(exp(sqrt(x-2)));
> X:=[8.,10.];
> compute_all_zeros_with_plot(f,X,0.001);
\end{verbatim}
\[ f(x) = \sin(e^{\sqrt{x-2}}) \]

\[
\text{compute_all_zeros_with_plot}(f, X, 0.001, 10, 10); \\
\text{Digits} = 10 \\
\text{Iteration step 1} \\
x_{old} = [8., 10.] \\
x_{new1} = [9.289288473, 10.] \\
x_{new2} = [8., 8.71071527] \\
\]

\[
\text{Iteration step 2} \\
x_{old} = [9.289288473, 10.] \\
x_{new1} = [9.462634907, 9.590102305] \\
\]

\[
\text{Iteration step 3} \\
x_{old} = [9.462634907, 9.590834649] \\
x_{new1} = [9.584440425, 9.590102305] \\
\]
Iteration step 4
\[ x_{\text{old}} = [8., \ 8.710711527] \]
\[ x_{\text{new1}} = [8.401353571, \ 8.456507702] \]

Iteration step 5
\[ x_{\text{old}} = [8.401353571, \ 8.456507702] \]
\[ x_{\text{new1}} = [8.405771237, \ 8.406299401] \]
Example 2: Range Enclosure (2D)

Range Enclosure of

\[ f := x \rightarrow \exp(-x^2) \times \sin(\pi x^3); \]

\[
\begin{align*}
& f := x \rightarrow \exp(-x^2) \times \sin(\pi x^3); \\
& X := [0., 2.]; \\
& \text{compute\_range}(f, X, 3);
\end{align*}
\]

Start range enclosure = \([-1.000000002, 1.000000002]\]
Range enclosure after step 3 = \([-0.3678794418, 0.7554611004]\)

Iterative range enclosure of \( f \)
> compute_range(f,X,5);

Start range enclosure = [-1.000000002, 1.000000002]
Range enclosure after step 5 = [-0.2820629525, 0.5815483768]
Example 2: Range Enclosure (2D)

- Reasonable results displayed after few iteration steps
- Numerical values available in variables
  - `list_of_intervals` and `list_of_ranges`

```
> list_of_ranges;
[[.2980341562, .3644580713], [.3644580699, .4296256944], [.4296256931, .4877217319],
 [.4877217307, .5314384278], [.5314384254, .5523340343], [.5348860212, .5680755514],
 [.4922683804, .5417224422], [.4003403046, .4922683817], [.268894620, .4003403067],
 [.1101516556, .268894645], [-.5312103680e-1, .1101516586], [-.1896402434, -.5312103191e-1],
 [-.2663280673, -.1896402391], [-.2920842800, -.2460035260], [-.2599399733, -.1706423650],
 [-.1706423681, -.3075637699e-1], [-.3075637973e-1, .1002647343], [.1002647315, .1599864556],
 [.1197822643, .1643652721], [.1173607362e-1, .1197822664], [-.8297860010e-1, .1173607573e-1],
 [-.1104304755, -.7204027092e-1], [-.9203610994e-1, -.1800481925e-1],
 [-.1800482181e-1, .5649041403e-1], [.4840145700e-1, .7131668286e-1],
 [-.1254422882e-1, .5228321860e-1], [-.5212547117e-1, -.1065394717e-1],
 [-.4426684404e-1, -.5350768968e-2], [-.5350770488e-2, .3072094080e-1],
 [.6012399421e-2, .3151551268e-1], [-.2177923802e-1, .6012400536e-2],
 [-.2204438709e-1, .4408877421e-9]]
```
Example 3: Range Enclosure (3D)

\[ g := (x, y) \to \exp(-x*y) \cdot \sin(Pi \cdot x^2 \cdot y^2); \]

Evaluation on the interval \( \left[ \frac{\pi}{8}, \frac{\pi}{2} \right] \times \left[ \frac{\pi}{8}, \frac{\pi}{2} \right]; \)

\[ > \text{compute\_range3d}(g, T, S, 2); \]

Start range enclosure = \([-0.8570898115, 0.8570898115]\]
Range enclosure after step 1 = \([-0.8570898115, 0.8570898115]\]
Range enclosure after step 2 = \([-0.6800891261, 0.8570898115]\]
\[ g := (x, y) \rightarrow \exp(-x \times y) \times \sin(\pi \times x^2 \times y^2); \]

> compute_range3d(g,T,S,4);

Start range enclosure = [ -0.8570898115, 0.8570898115 ]
Range enclosure after step 1 = [ -0.8570898115, 0.8570898115 ]
Range enclosure after step 2 = [ -0.6800891261, 0.8570898115 ]
Range enclosure after step 3 = [ -0.6800891261, 0.8486122905 ]
Range enclosure after step 4 = [ -0.5093193828, 0.7559256232 ]
Example 4: Complex Arithmetic

Multiplication of $<1,1>$ and $<-1+i,1>$

- with centered multiplication
- with area-optimal multiplication
Range Enclosure for Complex Polynomials

\[ p := (0.1 + 0.1i)z^5 + 0.2iz^4 - 0.1iz^3 + (-0.2 - 0.1i)z + 2.0 + 1.0i; \]

Results using Horner-Sheme with centered and area-optimal multiplication and centered form.
Tasks and Questions

- Adaptation to new concept of type definitions in Maple 8
- Output capabilities?
- Operator overloading?
- Symbolical computation capabilities?
Available from Waterloo Maple™ as

**Maple PowerTool* Interval Arithmetic**

http://www.mapleapps.com/powertools/interval/Interval.shtml

or directly from our Research Group’s website at

http://www.math.uni-wuppertal.de/wrswt/software/inpakX/

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