Why Monotonicity in Interval Computations?
A Remark

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Abstract. Monotonicity of functions has been successfully used in many problems of interval computations. However, in the context of interval computations, monotonicity seems somewhat ad hoc. In this paper, we show that monotonicity can be reformulated in interval terms and is, therefore, a natural condition for interval mathematics.

1 Formulation of the Problem

One of the main problems of interval computations it to compute the range

$$y = f^n(x_1, ..., x_n) = \{f(x_1, ..., x_n) \mid x_1 \in x_1, ..., x_n \in x_n\}$$

of a given continuous function $f(x_1, ..., x_n)$ on given intervals $x_1, ..., x_n$. The solution of this problem often requires long computations. It is known that these computations can be made faster if the function $f(x_1, ..., x_n)$ is strictly monotonic (strictly increasing or strictly decreasing) in some of the variables.

From a practical viewpoint, this is a great idea that has been successfully applied to many real-life problems (see, e.g., [1, 2, 5, 11, 4, 8, 10, 9, 6, 3, 7]).

But from the theoretical viewpoint, the idea of monotonicity sounds too ad hoc, unrelated to intervals.

In this short paper, we make monotonicity more theoretically acceptable by showing that monotonicity can be easily reformulated in interval terms.

2 Results

Proposition 1. For an arbitrary continuous function $f : R \to R$, the following two conditions are equivalent to each other:

1) $f$ is either strictly increasing or strictly decreasing;

2) for every two intervals $a$ and $b$, if $a \subset b$ then $f^n(a) \subset f^n(b)$. 
Comments.

- In this formulation, $A \subset B$ means that $A$ is a proper subset of $B$, i.e., that $A \subset B$ and $A \neq B$.

- This result is intuitively evident and very simple to prove. However, to the best of our knowledge, it is new; since we believe that our result provides an answer to an important foundational question, we decided to present it as a short paper.

- A remark to avoid possible confusion: This result is of pure foundational interest; it is not a practical test for monotonicity, because it is impossible to check $f^\circ(a) \subset f^\circ(b)$ for all intervals $a$ and $b$.

- This result can be easily generalized to a multi-dimensional case, with:
  
  - multi-dimensional intervals ($= \text{"boxes"} = \text{Cartesian products of intervals}$) $A = a_1 \times \ldots \times a_n$ instead of standard intervals; and
  
  - functions that are either strictly increasing or strictly decreasing in each variable; a function is strictly increasing in $x_i$ if for every $x_1, \ldots, x_{i-1}, x_i', x_{i+1}, \ldots, x_n$, $x_i < x_i'$ implies $f(x_1, \ldots, x_{i-1}, x_i', x_{i+1}, \ldots, x_n) < f(x_1, \ldots, x_{i-1}, x_i, x_{i+1}, \ldots, x_n)$; a strictly decreasing function is defined similarly.

Proposition 2. Let $n \geq 1$. For an arbitrary continuous function $f : R^n \to R$, the following two conditions are equivalent to each other:

1) $f$ is either strictly increasing or strictly decreasing in each of its variables;

2) for every two boxes ($n$-dimensional intervals) $A$ and $B$, if $A \subset B$ then $f^\circ(A) \subset f^\circ(B)$.

Comment. We are formulating and proving these two Propositions for functions that are defined for all possible inputs (i.e., defined on the entire line $R$, or on the entire space $R^n$). It is easy to show that similar results are true for functions $f$ defined on an connected open domain of $R$ (cor., of $R^n$). (Of course, for $R$, "connected open domain" simply means an open interval, finite or semi-infinite.)

3 Proofs

As we have mentioned, the proofs are very simple, and many readers will be definitely able to prove our statements themselves. However, for the sake of completeness, we will describe these proofs in all the detail.
3.1 Proof of Proposition 1

1) → 2). If $f$ is strictly increasing, then $f^u([a, b]) = [f(a), f(b)]$; therefore, as one can easily check, if $a < b$, then $f^u(a) < f^u(b)$. Similarly, the property 2) is true for strictly decreasing functions.

2) → 1). Let $f$ be a continuous function that satisfies the property 2). Since $f$ is continuous function, on each interval $[a, b]$, it achieves its supremum at some point $a_0 \in [a, b]$. i.e., formally, there exists a point $a_0$ such that $f(a_0) \geq f(a)$ for all $a \in [a, b]$.

Let us first show, by reduction to a contradiction, that this point $a_0$ cannot be an inner point of the interval. Indeed, suppose it is. This means that the two range intervals $f^u([a, a_0])$ and $f^u([a_0, b])$ both share the same upper endpoint $f(a_0)$. Let us compare the lower endpoints of these intervals. Without loss of generality, we can assume that the first interval has a lower endpoint that is lower than or equal to the lower endpoint of the second interval. In terms of range intervals, this means that $f^u([a, a_0]) \supseteq f^u([a_0, b])$. Hence, $f^u([a, a_0]) = f^u([a, a_0] \cup [a_0, b]) = f^u([a_0, b])$. In other words, we have $[a, a_0] \subseteq [a, b]$ and $f^u([a, a_0]) = f^u([a, b])$, which contradicts the property 2). This contradiction shows that our assumption that $a_0$ is an inner point is false, and $a_0$ has to be one of the endpoints.

Similarly, on each interval, the function $f$ must attain its minimum only in an endpoint of this interval.

The function $f$ cannot attain its maximum in both endpoints, because then, the minimum of $f$ on this interval (which is also attained in one of the endpoints) must coincide with the maximum, and therefore, the function $f$ would be a constant, but for constant functions, contrary to what we have just proved, maximum is attained in any inner point.

Hence, for each interval $[a, b]$, minimum is attained in one of the endpoints and maximum in the other one. Let us show that for every two intervals $[a, b]$ and $[c, d]$, maximum and minimum are attained at exactly the same endpoints (i.e., either maximum is always at the upper endpoint, or always at the lower endpoint). To prove this fact, let us consider $\xi = \min(a, b)$ and $\tau = \max(c, d)$. Without loss of generality, let us assume that on the interval $[a, a_0]$, maximum is attained for $\tau$ and minimum for $a$. Let us show that the same will be true for the interval $[\xi, \tau]$.

We will show it in two steps; first, we will show it for the interval $[a, \tau]$, and then, for the interval $[\xi, \tau]$. Let us start with $[a, \tau]$. If the maximum of $f$ on this interval is attained on $a$, then we have $f(a) > f(a)$ for all other $a \in [a, \tau]$, in particular, for $a = \tau$; in other words, we conclude that $f(a) > f(\tau)$. But we know that $a$ was the minimum point of $f$ on the interval $[a, \tau]$, hence, we had $f(a) < f(\tau)$. This contradiction shows that $a$ cannot be the maximum point, and hence, the location of minimum and maximum of $f$ on the interval $[\xi, \tau]$ is similar to the one for $[a, \tau]$.

Similarly, the location of minimum and maximum of $f$ on $[\xi, \tau]$ is similar to
the one for \([\mathbf{a}, \mathbf{c}]\) and therefore, to the one on \([\mathbf{a}, \mathbf{c}]\).

Similarly, we can prove that location of minimum and maximum of \(f\) on \([\mathbf{c}, \mathbf{d}]\)
is similar to the one for \([\mathbf{b}, \mathbf{d}]\). Hence, the locations of minimum and maximum on
\([\mathbf{a}, \mathbf{c}]\) and on \([\mathbf{b}, \mathbf{d}]\) is indeed the same for all pairs of intervals.

If the maximum is always on the right, this means that for every \(\mathbf{a} < \mathbf{c}\), we have
\(f(\mathbf{a}) < f(\mathbf{c})\) and therefore, \(f\) is strictly increasing. If the maximum is to
the left, the function is strictly decreasing. Q.E.D.

3.2 Proof of Proposition 2

1)\(\rightarrow\)2). If the property 2) is true for a function \(f\), then, in particular,
for every vector \(\mathbf{x} = (x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n) \in R^{n-1}\), the property 2)
is true for (degenerate) boxes of the type \([x_1, x_1] \times \ldots \times [x_{i-1}, x_{i-1}] \times a \times \[x_{i+1}, x_{i+1}] \times \ldots \times [x_n, x_n]\). Hence, due to Proposition 1, the function \(f_\mathbf{x}(\mathbf{a}) = f(x_1, \ldots, x_{i-1}, a, x_{i+1}, \ldots, x_n)\) is either strictly increasing, or strictly decreasing.

To complete the proof, we need to show that if this function \(f_\mathbf{x}(\mathbf{a})\) is strictly increasing for some values \(\mathbf{x}\), then it will be strictly increasing for every other set of values \(\mathbf{y} = (y_1, \ldots, y_{i-1}, y_{i+1}, \ldots, y_n)\).

To prove this result, let us denote by \(I\) the set of all vectors \(\mathbf{x} \in R^{n-1}\) for
which the function \(f_\mathbf{x}(\mathbf{a})\) is strictly increasing, and by \(D\), the set of all vectors
\(\mathbf{x}\) for which \(f_\mathbf{x}(\mathbf{a})\) is strictly decreasing. Clearly, we have \(I \cap D = \emptyset\), and we have just proved that \(I \cup D = R^{n-1}\). In terms of these sets, we must prove that
either \(I = \emptyset\), or \(D = \emptyset\). We will prove that by showing that both sets \(I\) and \(D\)
are closed; then, from connectedness of \(R^{n-1}\), it follows that one of these sets
is empty.

We will show that the set \(I\) is closed (for \(D\), the prove is similar). Closeness
of \(I\) means the following: if \(\mathbf{x}\) is a limit of the sequence of elements \(\mathbf{x}^{(n)} \in I\),
then \(\mathbf{x} \in I\). The desired conclusion means that if \(a < a'\), then \(f_\mathbf{x}(a) < f_\mathbf{x}(a')\).
Indeed, from \(\mathbf{x}^{(n)} \in I\), we conclude that \(f_\mathbf{x}^{(n)}(a) < f_\mathbf{x}^{(n)}(a')\). Since the function
\(f\) is continuous, we have \(f_\mathbf{x}^{(n)}(a) \rightarrow f_\mathbf{x}(a)\) and \(f_\mathbf{x}^{(n)}(a') \rightarrow f_\mathbf{x}(a')\). Hence, in the
limit \(n \rightarrow \infty\), \(f_\mathbf{x}(a) \leq f_\mathbf{x}(a')\). Since \(R^{n-1} = I \cup D\), we have only two options:
\(\mathbf{x} \in I\) and \(\mathbf{x} \in D\). If \(\mathbf{x} \in D\), we would have \(f_\mathbf{x}(a) > f_\mathbf{x}(a')\), which contradicts to
the inequality that we have just proved. Hence, \(\mathbf{x} \in I\), i.e., \(I\) is closed. Q.E.D.

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