Zeros of Riemann's Zeta Function Are Uniformly Distributed, But Not Random: An Answer to Calude's Open Problem

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Abstract

It is known that the imaginary parts of the roots of the Riemann's $\zeta-$function are uniformly distributed. This fact led Calude to a natural question: is the corresponding sequence (of binary expansions merged together) random in the sense of Martin-Löf? Our answer is negative: this sequence is not random.

1 Problem

One of the main open problems of mathematics is the famous Riemann's Hypothesis according to which all complex roots (zeros) $s_1 = \mathrm{Re}(s_1)+i\,\mathrm{Im}(s_1)$, $s_2 = \mathrm{Re}(s_2)+i\,\mathrm{Im}(s_2), \ldots,$ of the Riemann's zeta-function

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}$$

(i.e., the values for which $\zeta(s_k) = 0$) are located on the straight line $\mathrm{Re}(s_k) = 1/2$ in the complex plane (except for the known zeros which are negative integers). This hypothesis, very important for prime numbers, was included by Hilbert as Problem No. 8 in his famous 1900 list of 23 problems that the 19 century mathematical community considered the most important for the 20 century to solve (for an English translation of the original Hilbert's paper, see [5]; for further developments, see [3] and [7]).

Although Riemann's Hypothesis is still an open question, it has actually been proven that the real parts of zeros s_k are close to 1/2; so we can say that these real parts are highly organized. As for the *imaginary* parts $\mathrm{Im}(s_k)$ of the same zeros s_k, they are far from organized: namely, it was shown (see [6]) that for every real number $t \neq 0$, the fractional parts $f_n(t)$ of the sequence

$$\frac{t}{2\pi} \cdot \mathrm{Im}(s_k)$$

are uniformly distributed (in the sense that for every subinterval $[a, b] \subseteq [0, 1]$, the portion of points $f_1(t), \ldots, f_N(t)$ that belong to this subinterval tends to $b - a$ as $N \to \infty$).

Uniform distribution is one of the most natural properties of *random* sequences in the sense of Martin-Löf ([10]; for an up-to-date survey, see [8]). So, a natural question is: *do zeros of Riemann's zeta function form a random sequence?* To be more precise: for a computable real number t, is the sequence formed by merging the binary digits of the fractional parts $f_1(t), f_2(t), \ldots$, random? This question was formulated by C. Calude in [4] as one of 14 open problems about Kolmogorov complexity and algorithmic randomness (this problem is No. 14).

2 Solution

In this short note, we want to show that the answer to Calude's question is *No*: this sequence is not random.

Indeed, Riemann's zeta-function is a *constructive* (algorithmic) analytical function (for precise definitions, see, e.g., [1, 2]). Therefore, according to a theorem proven by Orevkov [11] (see also [9]), its zeros can be computed algorithmically with any given accuracy. Thus, the sequence of zeros is algorithmic and hence, it cannot be Martin-Löf random.

References

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