Axiomatic Description of Implication Leads to a Classical Formula with Logical Modifiers: (In Particular, Mamdani's Choice of "and" as Implication is not so Weird After All)

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Abstract

There are many ways to define fuzzy logic, i.e., to extend logical operations from the set $\{0, 1\}$ of truth values of classical logic to the set $[0, 1]$ of truth values of fuzzy logic in such a way that natural properties of these logical operations are still true. For "and", "or", and "not", we get quite reasonable results. However, for implication, the extensions that are obtained in this manner do not usually include Mamdani's choice of "and" as implication, the choice that forms the basis of fuzzy control — the most successful application of fuzzy logic.

In the present paper, we develop a new set of reasonable properties that leads to a new description of fuzzy implication. We get a general description of all operations that satisfy these properties.

This description includes Mamdani's operation as a natural particular case. Thus, Mamdani's choice of "and" as implication is not so strange after all.

1 Formulation of the Problem

1.1 Logical Operations Need to be Defined for the Fuzzy Case

In classical two-valued logic, every formula is either true or false; in other words, there are only two possible truth values: "true" and "false". In the computers, "true" is usually represented by 1, and "false" by 0. The main idea of fuzzy logic is to add new "degrees of belief" ("truth values") that are *intermediate* between "true"($= 1$) and "false" ($= 0$). Traditionally, real numbers from the interval $[0, 1]$ are used to represent these degrees of belief.

Since we are extending the set of truth values, it is necessary to also extend logical operations such as "and", "or", "not", "implies", etc., that are normally defined on these truth values. Of course, there are many ways to extend a function defined on a 2-element set $\{0, 1\}$ to the entire interval $[0, 1]$, so usually, extensions are chosen that preserve as many important properties of the original operations as possible.

1.2 Implication is the Only Logical Operation for Which the Most Successful Fuzzy Extension is Counter-Intuitive

For "and", "or", and "not" operations, this methodology works perfectly well, resulting in a well-accepted well-recognized class of fuzzy logical operations (t-norms, t-conorms, etc.); see, e.g., [7, 10]. Unfortunately, for implication, the situation is quite different: there are well-defined and nicely described classes of implication operations, but the most widely used implication operation is the one proposed (and used) by Mamdani [9], in which implication is simply identified with "and".

From the logical point of view, this choice is counter-intuitive, because from the common sense viewpoint, "implication" and "and" are two different logical operations. Mamdani's implication operation, however, was introduced not in the *logical* context (where it really looks weird), but in the context of *fuzzy control*, as a way to compute the degree of validity of a fuzzy rule "if $A(x)$ then $B(u)$" as the truth value of conjunction $A(x)\&B(u)$. This idea of Mamdani leads to a very successful control [6], one of the main success stories of fuzzy logic. This practical success clearly indicates that Mamdani's operation was a very good choice.

1.3 Formulation of the problem

It is, therefore, desirable to *explain* Mamdani's choice in logical terms, so that this choice of a fuzzy implication operation would be as logically acceptable as the choices of fuzzy "and", "or", and "not" operations.

1.4 What Was Done Before

The main problem with Mamdani's operation is that there is a big gap between this operation and more traditional fuzzy implication operations (operations that have intuitively clear explanations). This gap was felt for quite some time, and several results were proposed to make this gap narrower:

- In [3, 1, 2, 4, 5], it was shown that the use of logical modifiers (hedges) such as "very", "slightly", etc., helps to bridge the gap; namely, these papers introduced a new family of implication operations in which there is a continuous transformation between the traditional and Mamdani's implications. These results, however, did not completely close the gap, because this new family was not justified by simple axioms.

- In [8], an axiomatic explanation for Mamdani's implication was proposed. The existence of such an explanation shows that Mamdani's choice is not as weird as it may seem. However, this explanation is based on the modern non-monotonic logic and as such, is far from being intuitive.

1.5 What We are Planning to Do

The main goal of the present paper is to explain Mamdani's implication in clear intuitive terms.

To make the situation completely intuitive, we will give this explanation for the simplest possible choice of "and" and "not" operations: $a\&b = \min(a, b)$ and $\neg a = 1 - a$.

2 Definitions and the Main Result

Definition 1. *Let & be a t-norm [7, 10]. By a fuzzy implication, we mean a continuous function* $\to$ *from* $[0, 1] \times [0, 1]$ *to* $[0, 1]$ *for which for all* a*,* b*, and* c*, the following two equalities hold:*

- $a \to (b\&c) = (a \to b)\&(a \to c)$;

- $a \to (b \to c) = (a\&b) \to c$.

Comment 1. Let us explain why these conditions are intuitively reasonable:

- Intuitively, if A implies "B and C", it means that A implies B and A implies C. Vice versa, if A implies B and A implies C, then A implies $B\&C$. So, it is natural to require that the truth values of $A \to (B\&C)$ and $(A \to B)\&(A \to C)$ coincide.

- If from A we can conclude that, given B, C is true, this means that C is true if both A and B are true. This explains the second requirement.

Comment 2. To formulate the result, we need one more definition.

Definition 2. *By a modifier, we mean a continuous function* $h : [0,1] \to [0,1]$.

THEOREM. *If* $\& = \min$ *and* $\neg(a) = 1 - a$, *then for every function* $a \to b$ *from* $[0,1] \times [0,1]$ *to* $[0,1]$, *the following two conditions are equivalent to each other:*

- $a \to b$ *is a fuzzy implication (in the sense of Definition 1);*

- $a \to b$ *has the form*

$$a \to b = \neg h(a) \vee (h'(a)\&b)$$

  for some modifiers h *and* h' *for which* $h(x) + h'(x) \geq 1$ *for all* x.

Comments.

- In two-valued logic, implication $a \to b$ is equivalent to $\neg a \vee (a\&b)$, i.e., it means that either the condition a is false ($\neg a$), or else the condition a is true and the conclusion b is true as well. Our theorem shows that in fuzzy logic, fuzzy implication means "either to some extent a is false, or to a certain extent a is true, and then b is also true." The condition $h(x)+h'(x) \geq 1$ (equivalent to $1-h(x) \geq h'(x)$) can be, crudely speaking, interpreted as follows: if, due to the modifier, the first case is not covered, then the second case must be covered.

- As particular cases of this general formula, we get several known fuzzy implication operations:

  - For $h(x) = x$ and $h'(x) = 1$, we get Kleene-Dienes' implication

$$a \to b = \max(1 - a, b).$$

  - For $h(x) = x$ and $h'(x) = x$, we get Willmott's implication

$$a \to b = \max(1 - a, \min(a, b)).$$

  - For $h(x) = 1$ and for $h'(x) = x$, we get Mamdani's implication

$$a \to b = \min(a, b).$$

- For other choices of modifiers, we get new operations, e.g., for $h(x) = 1$ and $h'(x) = x^2$, we get $a \to b = \max(1 - a^2, b)$.
- In general, the theorem shows that operations implicitly described in [3, 1, 2, 4, 5] are the most general.
- Our derivation only works for $\& = \min$. For operations & different from min, the conditions of the theorem become too restrictive: instead of a reasonably general expression for fuzzy implication, we end up with a generalization $a \to b = b^{(a^p)}$ of Yager's implication operation $a \to b = b^a$ [12] (this result was proven in [11]):

PROPOSITION. [11] *If $a \& b = a \cdot b$ and $\neg(a) = 1 - a$, then for every function $a \to b$ from $[0,1] \times [0,1]$ to $[0,1]$, the following two conditions are equivalent to each other:*

- *$a \to b$ is a fuzzy implication (in the sense of Definition 1);*
- *$a \to b$ has the form*
$$a \to b = b^{(a^p)}$$
for some real number $p \geq 0$.

3 Proof of the Theorem

1. Since $\& = \min$, the first requirement $a \to (b \& c) = (a \to b) \& (a \to c)$ means that $a \to \min(b, c) = \min(a \to b, a \to c)$, i.e., it means that if $b \leq c$, then $a \to b = \min(a \to b, a \to c)$. In other words, it means that if $b \leq c$, then $a \to b \leq a \to c$.

In mathematical terms, this means that the function $a \to b$ is non-decreasing as a function of b.

2. Let us now prove the desired formula. For that purpose, let us denote $a \to 0$ by $f^-(a)$ and $a \to 1$ by $f^+(a)$. Since $a \to b$ is non-decreasing as a function of b, we have $f^-(a) \leq f^+(a)$ for all a. We will prove the following:

- If $b \leq f^-(a)$, then $a \to b = f^-(a)$.
- If $f^-(a) \leq b \leq f^+(a)$, then $a \to b = b$.
- If $f^+(a) \leq b$, then $a \to b = f^+(b)$.

2.1. For $b = f^-(a) = a \to 0$, due to the second requirement from Definition 1, we have
$$a \to (a \to 0) = (a \& a) \to 0 = a \to 0 = f^-(a).$$

Thus, $a \to 0 = a \to f^-(a) = f^-(a)$. Hence, from $0 \le b \le f^-(a)$, due to monotonicity of fuzzy implication (see Part 1 of this proof), we can conclude that

$$f^-(a) = a \to 0 \le a \to b \le a \to f^-(a) = f^-(a),$$

and hence, that $a \to b = f^-(a)$.

2.2. If $f^-(a) = a \to 0 \le b \le a \to 1 = f^+(a)$, then, since $\to$ is a continuous function, we can apply the Intermediate Value Theorem and conclude that there exists a c for which $b = a \to c$. Due to the second requirement of Definition 1, we have $a \to b = a \to (a \to c) = (a\&a) \to c = a \to c = b$.

2.3. For $b = f^+(a) = a \to 1$, due to the second requirement, we have

$$a \to (a \to 1) = (a\&a) \to 1 = a \to 1 = f^+(a).$$

Thus, $a \to 1 = a \to f^+(a) = f^+(a)$. Hence, from $f^+(a) \le b \le 1$, due to the monotonicity of fuzzy implication (see Part 1 of this proof), we can conclude that

$$f^+(a) = a \to f^+(a) \le a \to b \le a \to 1 = f^+(a),$$

and hence, that $a \to b = f^+(a)$.

2.4. Now, we can take $h(x) = 1 - f^-(x)$ and $h'(x) = f^+(x)$. In terms of h and h', the inequality $f^-(x) \le f^+(x)$ takes the desired form $h(x) + h'(x) \ge 1$.

3. The fact that every expression of the desired type satisfies the two conditions of Definition 1 is easy to check. The theorem is proven. Q.E.D.

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