On the Possibility of Using Complex Values in Fuzzy Logic For Representing Inconsistencies

Hung T. Nguyen[1], Vladik Kreinovich[2], and Valery Shekhter[3]

[1]Department of Mathematical Sciences
New Mexico State University, Las Cruces, NM 88003, USA
email hunguyen@nmsu.edu

[2]Department of Computer Science and
[3]Department of Languages and Linguistics
University of Texas at El Paso, El Paso, TX 79968, USA
email vladik@cs.utep.edu.

Abstract

In science and engineering, there are "paradoxical" cases when we have some arguments in favor of some statement A (so, the degree to which A is known to be true is positive (non-zero)), and we also have some arguments in favor of its negation $\neg A$, and we do not have enough information to tell which of these two statements is correct. Traditional fuzzy logic, in which "truth values" are described by numbers from the interval $[0,1]$, easily describes such "paradoxical" situations: the degree a to which the statement A is true and the degree $1-a$ to which its negation $\neg A$ is true can be both positive. In this case, if we use traditional fuzzy $\&-$operations (min or product), the "truth value" $a\&(1-a)$ of the statement $A\&\neg A$ is positive, indicating that there is some degree of inconsistency in the initial beliefs.

When we try to use fuzzy logic to formalize expert reasoning in humanities, we encounter the problem that in humanities, in addition to the above-described paradoxical situations caused by the incompleteness of our knowledge, there are also *true paradoxes*, i.e., statements that are perceived as true and false at the same time. For such statements, $A\&\neg A =$"true". The corresponding equality $a\&(1-a)=1$ is impossible in traditional fuzzy logic (where $a\&(1-a)$ is always ≤ 0.5), so, in order to formalize such true paradoxes, we must *extend* the set of truth values from the interval $[0,1]$.

In this paper, we show that such an extension can be achieved if we allow truth values to be complex numbers.

1 Introduction

1.1 Fuzzy Logic Was Initially Proposed to Describe Humanities As Well

It is well known that commonsense reasoning and reasoning in human sciences is not always adequately described by a classical (two-valued) logic, in which each statement A is either true or false. To describe such reasoning, Zadeh has proposed [50] a new formalism that he called *fuzzy set theory*; its logical part was later called *fuzzy logic*. In fuzzy logic, the "truth value" (also sometimes called degree of certainty or degree of belief) of every statement A is a number from the interval $[0, 1]$: 0 corresponds to "false", 1 to "true", and values from the open interval $(0, 1)$ correspond to intermediate truth values.

Since we have enlarged the set of possible "truth values", we must somehow extend logical operations $(\&, \vee, \neg)$ (or, equivalently, corresponding operations with fuzzy sets) from the classical two-valued set set $\{0, 1\}$ $(= \{$"true", "false"$\})$ to the entire interval $[0, 1]$. In [50], Zadeh proposed two possible extensions:

- $a \& b = \min(a, b)$ (called *minimum*), $a \vee b = \max(a, b)$ (called *maximum*), $\neg(a) = 1 - a$;
- $a \& b = a \cdot b$ (called *algebraic product*), $a \vee b = a + b - a \cdot b$ (called *algebraic sum*), $\neg(a) = 1 - a$.

He also mentioned that other extensions were possible.

1.2 At Present, Contrary to Zadeh's Expectations, the Main Application Area of Fuzzy Logic is Fuzzy Control

In his first papers, Zadeh suggested that his formalism will be actively used to describe informal notions from human sciences such as philosophy, linguistics, etc.; he also had other, more technical applications in mind, including the idea of applying fuzzy logic to *control*. In surprising contrast to Zadeh's initial predictions, fuzzy control has become the most successful area of applications of fuzzy logic (see, e.g., [20]), while numerous attempts to apply fuzzy logic to human sciences has not yet led to a decisive and convincing breakthrough.

1.3 Maybe Fuzzy Logic in its Present Form is not Sufficient for Humanities?

How can we explain that fuzzy logic is currently more successful in control applications than in formalizing common sense and humanities?

Fuzzy logic was proposed to describe features of human reasoning that are not covered by the classical two-valued logic.

The success of fuzzy control shows that *fuzzy logic, in its present form, is* pretty much *adequate for* describing uncertainty in statements about *control* situations and control strategies (like "if an obstacle is close, and you are driving fast, break fast").

Similarly, the limited success of fuzzy applications in humanities can indicate that *fuzzy logic in its present form is not* always *adequate for* describing uncertainties in the *humanities*.

1.4 One Area Where Fuzzy Logic May Fail in Describing Humanities: True Paradoxes

What exactly is not covered by the current fuzzy logic?

There are many differences between reasoning in humanities and reasoning in science and engineering; from the logical viewpoint, the main and the most striking difference is the attitude to *paradoxes*, i.e., statements A for which we have $A\&\neg A$.

In science and engineering, if two different arguments lead to opposite conclusions, the resulting "paradox" is viewed as a problem, as a challenge indicating that something is wrong. At any given moment of time, it is possible to have such "paradoxical" situations, in which we have both arguments in favor of some statement A and in favor of its negation $\neg A$. In science and engineering, these situations are caused by our *uncertainty*, i.e., by the *incompleteness* of our knowledge; in these situations, both the degree of certainty a of a statement A and the degree of certainty $1-a$ of the negation $\neg A$ of the statement A are < 1 (i.e., correspond to incomplete knowledge). Therefore, the degree of certainty of $A\&\neg A$ is also < 1.

In humanities and in commonsense reasoning, there are areas where *true* paradoxes are inevitable, in which we are absolutely 100% sure that both a statement A and its negation $\neg A$ are true, i.e., that $A\&\neg A$ is true. Let us give a few examples:

Paradoxes in Human Attitude: Ambivalent Emotions. When psychologists describe human emotions, they often encounter the situations of *ambivalent* emotions in which a person X loves a person Y and at the same time hates Y (i.e., does not love Y); see, e.g., [10, 39].

We can, of course, dismiss these situations by arguing that a person thinks and acts illogically. Such a "dismissal" would have been a reasonable idea in control applications: we definitely do not want a schizophrenic robotic controller, and our objective in control is not so much to *simulate* the human control, but to generate the *best possible* control, improving over human control if possible (fuzzy control is indeed often better than the human control that it is based on; see, e.g., [25]). In humanities (and in particular, in psychology), our goal is not to describe how a person *should* behave, but to describe, explain, and ideally, control the way a person actually behaves. Since ambivalent feelings are normal, we must have a formalism for describing these paradoxical feelings.

Moreover, the inconsistent feelings are not only *present*, they turn out to be an essential part of several useful *psychotherapy* methods [33, 46, 35, 38]. So, if the goal of our formalization is not only to *describe* how people think and feel, but also to *help* people, it is very important to be able to describe true paradoxes.

True Paradoxes in Philosophy and Theology. Many true paradoxes occur in philosophy [48], theology [19], etc.

In contrast to natural sciences and engineering, these paradoxes are not viewed as something shameful and temporary; starting, probably, from Hegel, they are viewed as an inherent feature of our understanding of the Universe, and as a driving force for the evolution of science. This view is convincingly argued, e.g.:

- in [47, 48] for knowledge in general;
- in [18], for *natural science*;
- in [27], for *logic* (as description of human reasoning);
- in [41, 34], for *sociology*;
- in [22, 8] for *political sciences*.

Again, one can make fun of this viewpoint and, following positivists, reject such arguments as pure nonsense. However, many researchers agree that philosophers' research is of (at least some) help and use. Therefore, it makes sense to formalize the way they actually think (and not only the way we, from precise sciences, would like them to think). Hence, in particular, we must be able to describe true paradoxes.

Similarly to paradoxes in psychology, paradoxes in philosophy are used not only to *describe* human reasoning, but to find the ways to *influence* human relations (see, e.g., [41, 34]). In order to formalize the corresponding methods of management and organization, we must be able to formalize true paradoxes.

1.5 What We Are Planning to Do

In this paper, we describe a new algebraic approach for describing true paradoxes. It turns out that in this approach, adding true paradoxes to fuzzy logic makes it *complex-valued*, i.e., leads to a formalism in which truth values are not only real numbers from the interval $[0, 1]$, but also *arbitrary* complex numbers.

We also show that the same set of truth values will appear if we add to traditional fuzzy logic not true paradoxes, but some other types of logical statements that traditional fuzzy logic logic cannot cover.

1.6 We Were Inspired By:

True Paradoxes Can Be Formalized: Belnap's 4-Valued Logic. The idea that true paradoxes can be formalized in a meaningful way can be illustrated by the work of Belnap [7], who developed a 4-valued logic for describing knowledge bases with possible inconsistencies; these 4 values are "true", "false", "unknown", and "true and false".

Belnap's logic was the first applied example of the so-called *paraconsistent* logics, i.e., logics in which, in contrast to the traditional Boolean logic, the fact that some statement is at the same time true and false does not necessarily imply that everything can be deduced (for the current survey of paraconsistent logics, see, e.g., [11]).

Belnap's logic is, by itself, insufficient for our purposes: it does describe true paradoxes, but it does not have any means to describe different truth values that are very important in formalizing humanities.

Comment. It should be noted that a useful generalization of Belnap's logic was proposed by M. Ginsberg [14] under the name of *bilattices*.

Interval-Valued Fuzzy Logic Leads to 4-Valued Belnap's Logic. A natural generalization of traditional fuzzy logic is *lattice-valued* logic, in which truth values form an arbitrary *lattice*, in which the order may be only partial. Lattice-valued fuzzy logic were proposed by Goguen in [16, 17] under the name of *L-fuzzy logics*.

The most widely used L-fuzzy logic are *interval-valued* fuzzy logic, which were originally proposed by J. A. Goguen and which have been actively developed by I. B. Türkşen and L. Kohout. In interval-valued fuzzy logic, truth values are *intervals*: namely, subintervals of the interval $[0, 1]$ (see, e.g., interval sections of NAFIPS 1994 [28], or surveys [30, 32]).

In the interval-valued fuzzy logic, it is natural to ask questions like: *when are given logical formulas* P *and* Q (i.e., formulas built from variables A, B, ..., by using logical connectives, e.g., $A\&(\neg A \vee B)$), *equivalent?* (i.e., when do values $d(P)$ and $d(Q)$, that estimate to what extent P and Q are true, coincide for all possible degrees of belief $d(A)$, ..., in A, B, ...?) *When does* P *imply* Q*?* (i.e., when $d(P) \geq d(Q)$ for all $d(A)$, ...?).

It turns out that for fuzzy logics in which $A\&A = A$ and $A \vee A = A$ (i.e., in which $f_\& = \min$ and $f_\vee = \max$), formulas P and Q are equivalent (i.e., $\mathbf{d}(P) = \mathbf{d}(Q)$ for all possible intervals $\mathbf{d}(A)$, ...) iff $\mathbf{d}(P) = \mathbf{d}(Q)$ for all cases in which each variable is assigned one of the following four intervals: $0 (= [0, 0])$, 1, 0.5, and $[0, 1]$. In other words, P and Q are equivalent in interval-valued logic iff they are equivalent in a 4-valued logic (for details and proof, see [30, 29]). A similar result holds for "P implies Q".

This 4-valued logic is isomorphic to the logic proposed by Belnap in [7]: 0 stands for Belnap's "false", 1 for "true", $[0, 1]$ for "unknown", and 0.5 for Belnap's "true and false".

Comment. Since we are, in this paper, especially interested in paradoxes and negation, it is worth mentioning that a similar interval-valued formalism can be obtained if we follow the consequences of the following natural negation-related idea. Namely, in traditional fuzzy logic, the degree $d(\neg A)$ to which a statement A is false is uniquely determined by the degree $d(A)$ to which A is true (usually, as $d(\neg A) = 1 - d(A)$). In some circumstances, it makes sense to no longer tie the degree $d(\neg A)$ to the degree $d(A)$. In such situations, to describe the degrees to which a statement is true and to which it is false, it is no longer sufficient to have a *single* real number, we need *two* different numbers. Usually, it is natural to impose a restriction $d(A) + d(\neg A) \le 1$. This inequality is equivalent to $d(A) \le 1 - (\neg A)$. Thus, the set of all possible pairs $\langle d(A), d(\neg A)\rangle$ which satisfy this inequality is in 1-1-correspondence with the set of all subintervals $[a^-, a^+]$ of the interval $[0, 1]$:

- to each pair $\langle d(A), d(\neg A)\rangle$, we can put into correspondence an interval $[d(A), 1 - d(\neg A)] \subset [0, 1]$; and
- to each subinterval $[a^-, a^+]$, we can put into correspondence a pair $\langle d(A), d(\neg A)\rangle$ with $d(A) = a^-$ and $d(\neg A) = 1 - a^+$.

Intuitively, if the difference between $d(A)$ and $1 - d(\neg A)$ is caused by the incompleteness of our knowledge, then the interval $[d(A), 1 - d(\neg A)]$ describes the interval of possible degrees to which A will be known to be true when we get additional knowledge. This interesting formalism was proposed and developed by K. Atanassov *et al.* under the name of *intuitionistic fuzzy logic* (for references, see, e.g., [1, 2, 3, 4, 5, 6, 32]).

The Use of Complex Numbers: Quantum Mechanics. The idea of using complex numbers for constructing models of physical situations with seemingly paradoxical descriptions can be clearly traced to quantum mechanics. The seemingly paradoxical description of quantum mechanics originated from the fact that in classical (pre-quantum) physics, elementary objects were either particles or waves, but never both. Therefore, in classical physics, when we observe that some object behaves like a particle, we can thus be sure that this object does not exhibit any wave-like behavior. Vice versa, if we observe a wave-like behavior (e.g., interference or diffraction), we conclude that this object cannot be a particle.

However, experiments with micro-objects like electrons has shown that, e.g., electron behaves like a particle in some experimental conditions and like a wave in some other experiments. Therefore, if we analyze these experiments from the viewpoint of classical physics, we conclude that electron is, on one hand, a particle, and, on the other hand, a wave (i.e., not a particle). In other words, if we apply formalisms based on classical physics to the analysis of micro-object experimental data, we get paradoxes. To resolve these paradoxes, the authors of quantum mechanics proposed to use formalisms which use complex numbers.

Let us briefly describe how and why complex numbers are used in quantum physics.

Comment. The main objective of our description is to explain how *complex numbers* appear in physics. With this objective in mind, we emphasize the relevant features of quantum mechanics and omit its other features which are, from the physical viewpoint, crucial and vitally important. Because of this, the following text is rather cursory and should not be viewed as an introduction to quantum mechanics. Readers who are interested to learn more about quantum mechanics, are advised to see, e.g., [12].

Quantum Physics Can, Generally Speaking, Only Make Probabilistic Predictions. One of the main differences between quantum mechanics and traditional mechanics is as follows:

- In traditional mechanics, if we know the state of the system at a certain moment of time t_0, we can uniquely predict the state of this system (and thus, the results of all possible measurements) at any future moment of time $t > t_0$. In this sense, traditional physics is *deterministic*.
- In quantum mechanics, even when we know the exact state of the system at the moment t_0, we cannot predict its future behavior and the results of future measurements; we can, at best, predict the *probabilities* of different measurement results. In this sense, quantum physics is often called a *non-deterministic* theory.

The Main Difference Between Quantum Physics and Statistical Physics: No Joint Probability Distribution for Quantum Systems. This *probabilistic* (statistical) character of quantum mechanics is not its only difference from classical mechanics. There is a special "branch" of classical physics dealing with systems under statistical uncertainty: statistical physics. In statistical physics, unlike Newtonian mechanics, we do not know the exact coordinates $\vec{x}$ and velocities $\vec{v}$ of the particles; instead, we know the probabilities of different values, or, to be more precise, we know the *probability density* $\rho(\vec{x}, \vec{v})$ of different values. From this probability density, we can compute the so-called *marginal* probability densities $\rho_x(\vec{x})$ and $\rho_v(\vec{v})$ that correspond to coordinates and velocities, and for each measurement procedure, we can compute the probabilities of different measurement results.

Since based on the probability density $\rho(\vec{x}, \vec{v})$, we can predict the probabilities of all possible experiments with the system, this probability density can be viewed as describing the *state* of the system.

In quantum physics, the situation is somewhat more complicated. As we have already mentioned, in quantum physics, an elementary particle has different properties which would be incompatible in classical physics:

- On one hand, a quantum particle has some features typical for *particles* of classical physics. For example, we can measure its coordinates with better and better accuracy.

- On the other hand, a quantum particle has some other features that are typical for *waves* as described in quantum physics: quantum particles exhibit such typical wave behavior as diffraction, wave superposition, etc. Based on these observable effects, we can measure (with better and better accuracy) such wave characteristics as its wavelength, frequency ω, phase, etc.

These two features are incompatible in classical physics, because, e.g., a wave is, be definition, a process in motion, and it makes no sense to talk about the current coordinate of a *wave* (to be more precise, we *can* talk about the coordinates of different *points* on a wave, but not of the wave itself).

This Unusual Statistical Feature of Quantum Physics is an Essential Part of Quantum Applications. This combination of classically incompatible features is not just a mental paradox, it is the basis of quantum mechanics effects and quantum mechanics applications: e.g., for electrons:

- Some experiments (and the majority of applications) are based on the fact that electrons are particles.

- Some other experiments and applications are based on the *wave* character of electrons. For example, in classical physics, a particle cannot penetrate the potential barrier if the energy of this barrier exceeds the initial energy of the particle; in quantum physics, however, when we take into consideration that an electron is also a wave, we can conclude that it is quite possible (although not highly probable) that a particle "tunnels" through this barrier and end up on the other side of it. This is not simply a theoretical conclusion, this "tunnel effect" is the basis of special electronic devices called *tunnel diodes* that are extensively used in nowadays electronics.

How Can We Describe the State of the Quantum System? Heisenberg's Approach. This dual character of quantum objects leads to the difficulty in describing them. Namely:

- When we measure coordinates $\vec{x}$, we can talk about the probability density $\rho_x(\vec{x})$.

- When we measure some wave characteristics (e.g., frequency ω), we can talk about the probability density $\rho_\omega(\vec{\omega})$.

However, unlike statistical physics, we cannot describe the state of the system by some *joint* probability distribution $\rho(\vec{x}, \vec{\omega})$, for two reasons:

- *Physical reason:* since we cannot measure $\vec{x}$ and ω at the same time, there is no physical sense in talking about the "joint probabilities".

- *Mathematical reason:* in some situations, the desired expression for "joint distribution" not only does not make sense physically, but it is also impossible mathematically: Namely, quantum mechanics predicts the probability distributions ρ_z for different *observables* z, i.e., for the basic physical variables like coordinates x_1, x_2, x_3, velocities v_1, v_2, v_3, and for (physically meaningful) functions of these basic variables, such as the *energy*

$$E = \frac{m\vec{v}^2}{2} - V(\vec{x}),$$

  where $V(\vec{x})$ is a potential energy. In statistical physics, we also have distributions ρ_z that correspond to different observables z, but in statistical physics, there exists a joint distribution $\rho(\vec{x}, \vec{v}, \ldots)$ from which all distributions ρ_z can be obtained as *marginal* distributions. In terms of probability theory, the fact that we can say that in statistical physics, the marginal distributions ρ_z are *consistent* in the sense of Kolmogorov [40]. In quantum physics, we can also compute the "marginal" distributions ρ_z; however, these distributions are sometimes *inconsistent* (in the sense of mathematical statistics [40]), i.e., there is simply *no* joint distribution ρ that has all these distributions $(\rho_x, \rho_v, \rho_E, \ldots)$ as its marginals.

So, we cannot describe a state of the quantum system by a joint distribution. How can we describe it? The first natural idea is as follows: since we cannot describe our system by a single probability distribution, let us keep *all* probability distributions that are needed to describe the result of different measurements. In computer terms, instead of a single probability distribution, we need an *array* of probability distributions. If we can characterize a distribution by a single parameter, then we have a one-dimensional array, or, in mathematical terms, a *vector*. In many physically meaningful situations, it is more natural to characterize a measurement by *two* different parameters; in this case, we have a 2-dimensional array, or, in mathematical terms, a *matrix*. Matrices and vectors were indeed the basis of historically the first formalism for quantum mechanics proposed by W. Heisenberg.

The Main Drawback of Heisenberg's Approach. The main drawback of this formalism was that it drastically increased the amount of data that we need to represent a state: crudely speaking, instead of a single probability distribution, we need a while bunch of them.

This problem was resolved by E. Schrödinger who came up with an alternative formalism, that is mathematically equivalent to Heisenberg's matrix mechanics, but that, unlike Heisenberg's mechanics, does not require such a drastic increase in the amount of description data.

Schrödinger's Formalism: Enter Complex Numbers. Schrödinger's formalism had some reasonable physical motivations which are not directly related to our problem of interest (representing uncertainty in case of inconsistency), but from our prospective, the main *mathematical* idea of Schrödinger's formalism can be best explained if we recall the mathematical reason why the joint probability distribution is impossible: we can have marginal probabilities that correspond to $\vec{x}$, ω, and to several other physical quantities that we can measure, but we cannot have the joint probability distribution that would lead to all these marginal probabilities.

In (classical) probability theory, there are formulas that transform given marginal distributions into a joint probability density from which these marginal distributions can be reconstructed. If we literally apply these formulas to the marginal distributions computed by formulas of quantum mechanics, we will get a function from which all marginal (experimental) probability density functions can be reconstructed. From this function, we can reconstruct the probabilities of all measurement results, and therefore, this function can be used to describe the state of the quantum system.

There is only one problem with this function: If for all quantum systems, the values of this function were non-negative, this function would be a joint probability distribution that (as we have mentioned above) is not always possible. Therefore, for some quantum systems, this function takes *negative* or even *complex* values.

Schrödinger's main idea was to describe the state of a quantum system by a *complex-valued* function ψ from which we can reconstruct the probabilities of different measurements. He called this function a *wave function* because it is necessary if we want a consistent description of quantum particles that describe not only their particle features, but their *wave* features as well.

As we have already mentioned, this idea worked, and it is still the basis of quantum formalisms.

How We Are Planning to Use Schrödinger's Idea. Schrödinger's approach can be summarized as follows: To describe quantum uncertainty that cannot be consistently described by traditional statistical physics, Schrödinger proposed to use formulas of probability theory and statistics even in the situations when these formulas are not physically justified. The "probabilities" computed by these formulas turn out to be often negative or even complex-valued. As a result, he came up with *complex-valued* "probabilities" to describe quantum uncertainty.

How we are planning to use this idea: the limitation in describing physical uncertainty that Schrödinger had to overcome was that the probability should be between 0 and 1. In describing expert uncertainty, we face a similar problem: that the truth values which describe the expert's uncertainty should belong to the interval $[0, 1]$. Such truth values cannot describe "true paradoxes". As a solution to this problem, we will formally apply the formulas of fuzzy logic to describe the truth values in these paradoxical situations. Due to the impossibility result mentioned above, we will not get the values from the interval [0,1]; these values will be negative or sometimes complex. As a result, we will arrive at the use of complex numbers to describe uncertainty.

Comments.

- The use of *negative* values to represent truth values or degrees of belief might help in the Dempster-Shafer in the *theory of evidence* [49]. Dempster-Shafer theory of evidence can be viewed as a natural generalization of the probability approach to uncertainty. Both probability and Dempster-Shafer approach describe the case when we know the finite set S of possible alternatives, but we do not know which of them is actually happening.

  In *probability* approach, for each alternative $s \in S$, we describe the *probability* $p(s)$ that s is the actual alternative. Each of these probabilities must be from the interval $[0, 1]$, and the sum of these probabilities must be equal to 1.

  In *Dempster-Shafer* approach, we get probabilities assigned to different *opinions*; however, in contrast to probability approach, an opinion does not necessarily mean that we know the *exact* alternative; an opinion may mean a belief that an alternative belongs to a certain *subset* B of the set S of all alternatives. So, from a mathematical viewpoint, assigning probabilities to opinions means assigning probabilities to *sets* B (to be more precise, to *subsets* $B \subseteq S$). To make a clear distinction between these probabilities of *opinion* and probabilities of *alternatives*, probabilities of opinions are usually called *masses* and denoted by $m(B)$. Since they are probabilities, we have $m(B) \in [0, 1]$, and $\sum m(B) = 1$.

  From the mathematical viewpoint, masses are a very convenient way to describe this uncertainty. However, it is often difficult to extract these masses from the experts whose opinions we describe. Instead, we can describe *beliefs*, where for every set $A \subseteq S$, the belief $Bel(A)$ in A is the total expert's degree of belief (= subjective probability) that the real alternative belongs to A. In terms of masses, belief can be expressed as

  $$Bel(A) = \sum_{B: B \subseteq A} m(B).$$

  If we know the beliefs, then we can reconstruct the masses by using the so-called *Möbius transformation*

  $$m(B) = \sum_{A: B \subseteq A} (-1)^{|A-B|} Bel(A),$$

where $|A - B|$ denotes the number of elements in the set $A - B$.

So far, we have described Dempster-Shafer formalism for the case when it is based on real probabilities. In reality, probabilities may not be the best way to describe how experts really think. In such situations, we can still determine the beliefs *Bel* by interviewing experts. A natural question is: can the resulting belief function (that is *not* necessarily coming from probabilities $m(B)$) still be described by Dempster-Shafer formalism?

In principle, we can always apply Möbius transformation to the belief function, and get some "masses". If the resulting masses all belong to the interval $[0, 1]$, then we are in the realm of Dempster-Shafer theory.

Conditions that *guarantee* the resulting masses to be in the interval $[0, 1]$ are known as *monotonicity of infinite order.* Some researchers consider is desirable to *weaken* these conditions, and thus, to *enlarge* the class of functions *Bel* that can be used to describe beliefs. In particular, it was suggested that a weaker condition on uncertainty measure $Bel(A)$ such as *2-monotonicity* (also known as *convexity*) may be useful. However, if we take a Möbius transformation of such convex (2-monotone) belief function, we may get *negative* values of the mass function. Traditionally, such mass functions are not considered because they cannot be interpreted in probabilistic terms.

The use of negative (and complex) values in quantum physics may suggest a justification of the use of negative values in mass for such belief functions.

- The connection between complex numbers and true paradoxes in quantum mechanics was made explicit in a recent philosophical book [21]. This book lacks a formalism for describing this relation (to be more precise, the formalism presented in this book does not seem adequate for our purpose, which is *not* formalization of such a well-defined and well-mathematized discipline as quantum mechanics, but of paradoxes that have not yet been formalized at all). So, we had to design a new formalism.

- In designing our formalism, we were also inspired by the papers [36, 37, 24], in which an algebro-logical interpretation is given to different truth values (from the interval $[0, 1]$) used in fuzzy logic description of quantum systems.

Complex-valued logic of Spencer-Brown. The idea of using "complex numbers" as truth values was first proposed by G. Spencer-Brown, a student of Bertrand Russell, in [43]; this idea was later described in detail in his best-selling book [44] (this book even got a book blurb from Bertrand Russell himself).

Spencer-Brown came up with this idea when he analyzed digital electrical circuits (i.e., electronic circuits in which each signal can only take two possible values). Digital circuits are in many ways similar to more traditional *analog* circuits, in which the value of a signal can be any real number.

In analog circuits, complex numbers turn out to be very useful to describe circuits with *dynamic* elements. Namely:

- The simplest element of an electric circuit is a *resistance*, for which the voltage $V(t)$ at a given moment of time is dependent only on the values of the current $I(t)$ at this same moment of time, and this dependency is described by the well-known Ohm's law $V(t) = I(t)R$, where R is the resistance of the given resistor. Therefore, the analysis of simple circuits that only contain resistors, can be dine by solving a system of algebraic equations.

- For other components of analog circuits, such as capacitors and inductors, the voltage $V(t)$ depends not only on the current $I(t)$ at this same moment of time, but also on the currents in the previous moments of time. For example, for the capacitor with capacity C, $V(t) = (1/C)\int I(s)\,ds$; for the inductor with inductivity L, $V(t) = L \cdot dI(t)/dt$. As a result, to analyze the general electric circuits (that contain not only resistors, but also capacitors and inductors), we need to solve systems of integro-differential equations. Computationally, this is a drastically more complicated problem that solving systems of algebraic equations that correspond to resistors only.

Here is where complex numbers help. It turns out that if we restrict ourselves to *harmonic* signals (i.e., signals in which the dependency on time is described by a sine or cosine function), then we can reduce the system of integro-differential equations to a (much simpler) system of algebraic equations in *complex numbers*. Namely, for each of the above-mentioned components of the electric circuit, we can describe the dependency of the voltage on the current as $\breve{V}(t) = \breve{I}(t) \cdot \breve{R}(t)$, where $\breve{I}(t)$, $\breve{V}(t)$, and $\breve{R}(t)$ are complex numbers. Here, e.g., $\breve{I}(t)$ is a complex number whose real part is the actual current at the moment t, and the imaginary part describes, crudely speaking, the *derivative* of the current.

For digital circuits, we can make a similar distinction between:

- the simplest circuits in which the output at time t depends on the inputs at this same moment of time (i.e., in which time delay is negligible), and

- more complicated circuits in which time delay has to be taken into consideration.

Simple digital circuits are usually described by formulas of Boolean logic, and Boolean logic also provides a way to design a circuit computing a given Boolean function. For more complicated circuits, traditional Boolean logic cannot be directly applied, so we have to write difference equations to describe such circuits.

Spencer-Brown proposed to enlarge the extend traditional Boolean logic the same way complex numbers extend real numbers in the description of analog circuits. Namely, to describe a state of the digital component in a circuit with a possible delay, he suggested, instead of using a single truth value ε (0 or 1), to use a *pair* of Boolean values $(\varepsilon_0, \varepsilon_1)$, where ε_0 is the truth value at a given moment of time, and ε_1 describes whether the value has changed since the previous moment of time. In this description, the second component ε_1 describes the *rate of change* of the signal and is, therefore, a direct analog of the imaginary part of the complex current that describes analog circuits. He even called this part *imaginary*.

As a result, instead of *two* possible truth values 0 and 1, we get *four* possible values:

- two old values: $(0, 0)$ that corresponds to classical "false" and $(1, 0)$ (that corresponds to classical "true"), and
- two new values: $(0, 1)$ that describes the signal that was 1 and is now 0, and $(1, 1)$ that describes the signal that was 0 and is now 1.

Since the new truth values can be described by a moment-by-moment description of the state, we can apply propositional operations like "and" and "or" to these moment-by-moment descriptions and thus get the extension of these traditional propositional operators to the new truth values. In particular, we get $(0, 1) \vee (1, 1) = (1, 0)$ and $(0, 1) \,\&\, (1, 1) = (0, 0)$.

This is a very useful formalism, that is similar but, strictly speaking, different from the formalism that we are proposing in this paper. Indeed:

- In Spencer-Brown's formalism, there exist two values $z_1 = (0, 1)$ and $z_2 = (1, 1)$ neither of which is equal to "false" ($= (0, 0)$), but for which $z_1 \& z_2 = 0$.
- In our formalism, truth values are described by complex numbers, and for complex numbers, the only way to have $z_1 \& z_2 = z_1 \cdot z_2 = 0$ is when one of the truth values z_1, z_2 is equal to 0.

2 Main Idea

2.1 What is a True Paradox?

A true paradox A is a statement for which $A \& \neg A$ is true. In terms of truth values, this means that the truth value a of the statement A must satisfy the equality $a \& \neg a = 1$.

2.2 Traditional Fuzzy Logic Cannot Adequately Describe True Paradoxes

For traditional fuzzy logic, in which a can take values from the interval $[0,1]$, this equality is not possible: neither for any of the two sets of operations initially proposed by Zadeh (and described above), nor for other sets of operations introduced later. Therefore, we must *extend* the set of truth values by including some other values.

The main thing is to extend the set of values; both sets of operations given above use functions that are defined on *arbitrary* numbers, not necessarily on the numbers from the interval $[0,1]$, so the operations can easily be extended to new values.

Let us analyze the above two sets of operations one by one.

2.3 Minimum and Maximum Cannot Formalize True Paradoxes

If we use min as & and $1-a$ as $\neg$, then $a\&\neg a = 1$ means $\min(a, 1-a) = 1$. This means that either $a = 1$, or $1-a = 1$. In the first case, we have $1-a = 0$, and $\min(a, 1-a) = \min(1,0) = 0 \neq 1$. In the second case, we have $a = 0$, so, $\min(a, 1-a) = \min(0,1) = 0 \neq 1$. So, by using these two operations, we cannot describe a true paradox (for which $a\&\neg a = 1$).

2.4 Other Arguments in Favor of Non-Idempotent Operations With Truth Values

This argument shows that for our problem, it is better to use a *non-idempotent* version of fuzzy logic (for which $a\&b \neq \min(a,b)$ and $a \vee b \neq \max(a,b)$). It is worth noticing that this is not the only case when non-idempotent operations are preferable to idempotent ones:

Non-Idempotent Operations in Fuzzy Control. In *fuzzy control*, non-idempotent operations are better for several objectives, including smoothness and stability [25, 26, 42].

Non-Idempotent Operations Describe Human Reasoning. Non-idempotent operations are also often better in describing human reasoning: see [31] and references therein. Indeed, if we have two independent statements A and A' with equal degrees of certainty $d(A) = d(A') = a$, then the degree of certainty that *both* statements are true is smaller than the degree of certainty a that *one* of them is true: $a\&a < a$; hence, such formalisms adequately describe human reasoning.

Linear Logic: A Logic Based on Non-Idempotent Operations

The Main Idea of Linear Logic. Non-idempotent operations (i.e., operations for which $A\&A \neq A$) are also used in a special logic called *linear logic* [15, 45]. The following example explains its idea: let A stands for "I have a dollar", B for "I can buy a coke", C for "I can buy a cookie", and let coke and cookie cost \$1 each. Then, $A \to B$ and $A \to C$. In 2-valued logic, we can conclude that $A \to (B\&C)$, but, since our resources are limited, with \$1, we cannot buy both a coke *and* a cookie. We need two dollars to buy both, which can be expressed as $(A\&A) \to (B\&C)$. Here, clearly, $A\&A \neq A$.

Why is Linear Logic Called Linear? In traditional 2-valued logic, since $A\&A$ mean the same thing as A, in a proof, we can use any premise A as many times as we want. As a result, if we represent the proof graphically, the original premise A may *branch* into several possible branches that correspond to uses of A in different parts of the proof (e.g., in the above example, A is used to prove B and to prove C in the proof of $B\&C$). As a result, we get a *tree.* In resource-bounded logic, we cannot use a premise twice, so, we cannot branch, and the proofs become *linear* (hence the name of this logic).

Linear Logic Can Describe Truth Values. Linear logic was originally proposed to describe such bounded resources as the number of available processors, but degree of certainty can also be viewed as a resource [9]: For example, in 2-valued logic, from the statement "all birds fly" $(\forall b\, F(b))$, we can conclude that $F(b_1)\&\ldots\&F(b_n)$ for any number of birds n. If we only have some degree of belief in $\forall b\, F(b)$, then we may be able to believe that $F(b_1)$ for a "randomly" picked bird b_1, but our degree of belief that, say, 10^6 birds located in some area all fly will be much smaller, almost at the level of disbelief.

Two Different "And" Operations in Linear Logic May Clarify the Switching Between Different "And" Operations in Fuzzy Control. An interesting feature of linear logic is that it has *two* different connectives to describe the commonsense statement "B and C":

- "both", meaning that we can have both conclusions, and
- "and", meaning that we can have B, and we can have C, but not necessarily both.

For "both", we have A"both"$A \neq A$; for "and", we have A"and"$A = A$. Crudely speaking, "and" correspond to $\min(a, b)$, while "both" looks more like $a \cdot b$. This fine distinction may explain the necessity to switch between several different $\&-$operations on different stages of fuzzy control [42]: as the control situation changes, we are *not* changing the way we think (that would be impossible), we are just changing the meaning of the word *and.*

2.5 Algebraic Product and Algebraic Sum Can Be Used to Formalize True Paradoxes

In the previous subsections, we explained that the *idempotent* operations $a\&b = \min(a, b)$ and $a \vee b = \max(a, b)$ are not always the best ones; in particular, these operations cannot be used to explain true paradoxes.

The situation changes favorably if we consider the *algebraic* operations $a\&b = a \cdot b$ and $a \vee b = a + b - a \cdot b$. In terms of these operations, the equality $a\&\neg a = 1$ that describes a true paradox leads to $a(1 - a) = 1$, or, equivalently, to $a^2 - a + 1 = 0$. This quadratic equation does not have any solutions in the interval $[0, 1]$; moreover, it does not have any solutions in the real line at all. However, if we consider *complex* numbers, then this equation has two solutions:

$$a = \frac{1 \pm \sqrt{3}i}{2}. \quad (1)$$

So, the only way to describe true paradoxes is to ascribe to them *complex* truth values.

As a result, we get a *complex-valued* fuzzy logic, in which truth values of different statements can be not only *real* numbers from the interval $[0, 1]$, but *complex* numbers as well.

Comments.

- By enlarging the set of truth values, we got the ability to describe true paradoxes, but this ability does not come for free: to gain this ability, we had to drop many important (and often useful) properties of traditional fuzzy logic, such the totality of order.

- From the mathematical viewpoint, what we are really doing is a logic where truth is vector-valued (since complex numbers are isomorphic to the real plane). As we have already mentioned, the idea of vector-valued logic is not new: e.g., the above-mentioned interval-valued logic is a good (and practically useful) example of a vector-valued logic (interestingly, both complex-valued and interval-valued logics use 2D vectors).

- Why complex numbers and not any other generalization of fuzzy logic? Traditional fuzzy logic is based on using the numbers from the interval $[0, 1]$. This interval, as part of the set of real numbers, has a rich algebraic structure which is actively used in applications of fuzzy logic: namely, we have the *order* between different values from the interval $[0, 1]$; we have natural *operations* such as product, average, etc. We have already mentioned that there are many possible generalizations of fuzzy logic, including the generalization to general *lattices* in which the order between truth values may only be partially defined. In all these generalizations, we still an *order* but we may not have other useful algebraic operations. Complex numbers are the set in which we have, practically, all algebraic operations defined for real numbers. This is the main reason why we believe that complex numbers are preferred to using any other general lattice L.

2.6 With Complex Values That Describe Paradoxical Statements, The Order Between Truth Values Becomes Partial

True Paradoxes Lead to Incomparable Truth Values. In traditional fuzzy logic, truth values are *linearly ordered* in the sense that for every two truth values $a = d(A)$ and $b = d(B)$ of some statements A and B, we have one of the following three possibilities:

- $a > b$, meaning that the degree to which A is true is *larger* than the degree to which B is true;
- $a = b$, meaning that the degrees to which A and B are true are *identical*;
- $a < b$, meaning that the degree to which A is true is *smaller* than the degree to which B is true.

For true paradoxes, we have to add the *fourth* possibility: that the degrees to which A and B are true cannot be compared. Indeed, let A describe an ambivalent feeling (great love and at the same time, great hatred), and let B describe a strong love with no ambivalent feelings present. Which of these two feelings corresponds to the greater degree of love? (i.e., for which the degree to which they represent "love" is the greatest?).

- On one hand, since A represents great love, and B only strong love, A is a greater degree of love than B.
- On the other hand, since A also represents a great hatred, while B represents strong love, B can be viewed as representing a greater degree of love than A.

These two degrees are clearly *incomparable*. As a result, if we include such degrees, the order on the set of all degrees (truth values) that corresponds to the notion of *greater* degree becomes a *partial* order.

Incomparable Truth Valued (Degrees) Are Not a New Idea: Such Degrees Already Exist in Belnap's Logic and in Interval-Valued Fuzzy Logic. In Belnap's 4-valued logic [7], the four possible truth values "true", "false", "unknown", and "true and false" ("paradoxical") are only partially ordered:

- "true" corresponds to the greater degree of truth than the other three;
- "false" corresponds to the smaller degree of truth than the other three; and
- "unknown" and "true and false" are incomparable.

Geometrically, if we draw a graph that describes this ordering, we will get a *diamond*.

As we have already mentioned, Belnap's logic can be viewed as a particular case of the *interval-valued* fuzzy logic. Hence, from the fact that some truth values in Belnap's logic are not comparable, we can conclude that incomparable truth values also exist in interval-valued logic. Actually, in this logic, truth values $\mathbf{a} = [a^-, a^+]$ and $\mathbf{b} = [b^-, b^+]$ are only comparable in two cases:

- if $a^+ < b^-$, then $\mathbf{a} < \mathbf{b}$;
- if $b^+ < a^-$, then $\mathbf{b} < \mathbf{a}$.

In all other case, the truth values represented by intervals **a** and **b** are incomparable.

Incomparable Truth Values Are in Good Accordance with the Spirit of Fuzziness. The fact that some truth values are incomparable is in good accordance with the spirit of fuzzy logic: crudely speaking, if we are formalizing fuzzy notions, why should the ordering between truth values be crisp?

If We Use Complex Numbers, Incomparability is Natural. The existence of incomparable truth values is also in good accordance with the proposed complex-number representation of true paradoxes, because, in contrast to the interval $[0, 1]$ on which there is a natural linear ordering $<$, there is no natural linear ordering on the set of all complex numbers.

3 Pragmatic Aspects of Complex Values in Fuzzy Logic: What Do We Gain by Using them?

3.1 A (simplified) case study: a person with ambivalent (inconsistent) feelings

What follows will be a very simplified example. From the practical viewpoint, what do we gain by allowing complex values in fuzzy logic? As we have mentioned before, we gain a formalism that enables us to handle "true paradoxes".

We are still very far from our ultimate goal "to formalize humanities", this is a very difficult area, in which we only made one small step. Therefore, in this section, we cannot yet give any examples of important real-life applications; instead, we will give a toy example (from abnormal psychology) that illustrates the type of possible future applications.

This example will describe a behavior of a person with ambivalent (inconsistent) feelings. From the psychological viewpoint, this example is an oversimplification. It should not be taken as a ready-to-use model of such behavior, but rather as a feasibility study that shows that our formalism seems to make such models possible.

The behavior of an ideally rational person. Traditionally, a behavior of a rationally minded person whose beliefs contain no inconsistencies are described by rational models. These models predict the decisions of such rational persons pretty well.

Real-life persons may be inconsistent, but these inconsistencies are correctable. Of course, real-life people's ability to logically analyze their own beliefs is limited, and as a result, people may hold beliefs that they think to be consistent but which, after a logical analysis, turn out to be inconsistent.

> For example, a member of the Ku Klux Klan may believe his preacher that God has created all ethnic groups equal, and at the same time, believe his Klan's mentors that his own ethnic group is somehow genetically superior to all the others.

However, usually, when a person is confronted with this inconsistency, he tries to "patch" his system of beliefs so as to make them consistent.

There exist cases of true inconsistencies. There are cases, that are borderline on abnormal psychology, in which a person's system of beliefs is persistently inconsistent. As an example, we have already given an ambivalent strong love – strong hate relationship.

When faced with a behavior of such a person, a specialist in decision making and computer simulations of decision would often simply dismiss such a person as "irrational". This term kind of pre-supposes that for such a person, no behavior predictions are possible. In reality, as psychologists know, the fact that a person has a single inconsistency, does not automatically mean that he or she is completely unpredictable; usually, most of such people behave quite predictably and reasonably in most social situations. They have their own "logic" that explains their behavior.

Our goal is to describe and formalize this "logic", and thus, to predict and ideally, to correct their behavior.

An example that will be formalized. We will give a simple example related to ambivalent feelings:

- Usually, a person feels comfortable and even good about his or her own strong feelings of love or hatred. These feelings are often socially praised and even enviable (especially if love and hatred are directed to, correspondingly, good and evil).

- However, a person with ambivalent feelings of *both* strong live and strong hatred often feels quite unhappy about these strong feelings.

How to explain this fact?

From the viewpoint of a modeler of rational behavior, this is just one more example of this ambivalent person's irrationality. However, we will show that the use of complex-valued fuzzy logic enables to explain this unhappiness quite naturally, without involving any additional inconsistencies.

3.2 Complex-valued fuzzy logic explains a frequent negative self-attitude of a person with ambivalent feelings towards others

Our plan. To achieve the desired explanation, we will do the following:

- First, we will formulate the above statements (that a person is normally happy about his or her own strong feelings) as fuzzy rules of the type used in fuzzy control.

- Second, we will show that for complex-valued degrees that represent ambivalent feelings, these same rules predict exactly the very (seemingly inconsistent) behavior that we try to explain.

Thus, we do not need any additional "irrationalities" to describe the person's unhappiness with his own feelings: the original inconsistency of these feelings, as captured by our complex-valued formalism, is quite enough to explain this unhappiness.

Step 1: Description in terms of fuzzy rules. Let us first describe the above statement (that a person is normally happy about his or her strong feelings) by fuzzy rules similar to the ones used in fuzzy control. Here, the input is a love-hate feeling x, and the output is a person's feeling u about him/herself. Let us take a positive number $u = 1$ to indicate good (happy) feelings that a person has about him/herself (then, negative numbers will describe bad feelings of unhappiness). In these notations, the above statements can be expressed by the following three fuzzy rules:

$$u = 1 \leftarrow VeryStrongLove(x);$$
$$u = 1 \leftarrow VeryStrongHatred(x);$$
$$u = 0 \text{ otherwise.}$$

Step 2: Formalization of fuzzy rules. How can we formalize these rules?

Let L denote love. Then its negation $\neg L$ indicates hatred; in fuzzy logic, $\neg L$ is usually described as $1 - L$. In fuzzy logic, "very a" is usually interpreted a^2, so it seems natural to describe "very strong", which sounds somewhat stronger than simply "very", as a^3. Under these assumptions, $VeryStrongLove(x)$ is interpreted as L^3, $VeryStrongHatred(x)$ is interpreted as $(1 - L)^3$, and the above three rules take the following form:

$$u = 1 \leftarrow L^3;$$
$$u = 1 \leftarrow (1 - L)^3;$$
$$u = 0 \text{ otherwise.}$$

Step 3: the resulting behavior. We can now apply the traditional fuzzy control methodology to determine the resulting value of u. In general, if we have a system of rules of the type

$$u = c_i \leftarrow A_i(x),$$

then this methodology prescribes the resulting value

$$\bar{u} = \frac{\sum c_i \cdot \mu_{A_i}(x)}{\sum \mu_{A_i}(x)}.$$

In particular, if rules have an "otherwise" clause with 0 action involved, it makes sense to assume that all cases are covered, and thus, $\sum \mu_{A_i}(x) = 1$, and

$$\bar{u} = \sum c_i \cdot \mu_{A_i}(x).$$

In particular, for our simple system of rules, we get

$$\bar{u} = 1 \cdot L^3 + 1 \cdot (1 - L)^3 = L^3 + (1 - L)^3.$$

Normal feelings: there is no reason to be unhappy. As expected, the resulting function $\bar{u} = L^3 + (1 - L)^3$ is equal to 1 for $L = 1$ (strong love) and for $L = 0$ (strong hatred), and it is (reasonably) positive for all intermediate values $L \in (0, 1)$, achieving its smallest value of 0.25 for $L = 0.5$. This fact can be easily checked if we re-formulate the above expression as

$$\bar{u} = L^3 + (1 - L)^3 = L^3 + (1 - 3L + 3L^2 - L^3) = 1 - 3L + 3L^2.$$

In other words, no matter how strongly you feel this or that way, there seems to be no reason to be unhappy about it.

Ambivalent feelings: desired explanation. If the feelings are ambivalent, i.e., if $L\&\neg L = 1$ and therefore,

$$L = \frac{1 + \sqrt{3}i}{2},$$

then

$$1 - L = \frac{1 - \sqrt{3}i}{2},$$

and the above formula leads to

$$\bar{u} = \left(\frac{1 + \sqrt{3}i}{2}\right)^3 + \left(\frac{1 - \sqrt{3}i}{2}\right)^3 = -2,$$

i.e., to a negative value that describes deep unhappiness.

Thus, our model does explain this observed unhappiness.

3.3 Complex-valued fuzzy logic also explain how these bad feeling can be cured

The use of complex numbers also explains the use of rationalization as a method of *cure* (rationalization is used, e.g., in psychoanalysis). Indeed, how can we describe rationalization in fuzzy terms?

Our goal is to change the person's feelings from ambivalent to normal (or at least to more normal). How can we, in general, change someone's degree of feeling?

To answer this question, let us first consider a normal person's belief. Let us assume that a person has some degree of belief d_{init} in a certain statement A, and we want to increase this degree of belief.

- If we are dealing with a very rational person, then this person may actually know why exactly he has his current degree of belief. Since his current degree of belief is smaller than 1, this means that he may have some doubts about the situation. We may then want to discuss these doubts with him and try to convince him that these doubts are unfounded.

- Often, however, the person himself may have forgotten about the reasons for his current degree of belief. In this situation, we cannot meaningfully argue about his beliefs, the only thing we can do is to present extra arguments in favor of the statement A. Let us denote the degree of belief in A caused by these extra reasons by d_{extra}. As a result, for the person whom we are trying to influence, A is true if either the original reasons were true or if the new ones were true. Hence, his new degree of belief in A is $d_{\text{new}} = d_{\text{init}} \vee d_{\text{extra}}$.

Similarly, if a person had some ambivalent feeling $L_{\rm init} = (1+\sqrt{3}i)/2$ that described his/her love, and we give this person extra reasons for love, then his resulting live feeling will be equal to

$$L_{\rm new} = L_{\rm init} \vee L_{\rm extra} = L_{\rm init} + L_{\rm extra} - L_{\rm init} \cdot L_{\rm extra}.$$

When $L_{\rm extra}$ tends to 1, the resulting value $L_{\rm new}$ tends to

$$L_{\rm new} \to L_{\rm init} + 1 - L_{\rm init} \cdot 1 = L_{\rm init} + 1 - L_{\rm init} = 1.$$

Thus, as $L_{\rm extra} \to 1$, the patient indeed gets closer to normal.

4 Mathematical Aspects of Using Complex Values in Fuzzy Logic: Natural Questions

From the mathematical viewpoint, the complex-number description of true paradoxes leads to the following two questions that will be answered in the following two sections.

Can Other "And" And "Or" Operations Describe True Paradoxes? In our description, we used algebraic product and algebraic sum to describe paradoxes. Can we use other $\&-$ and $\vee-$ operations?

Are All Complex Numbers Necessary? We know that in order to describe true paradoxes, the set D of truth values must include at least one of the two complex numbers described by the formula (1). The question is: *do we need all complex numbers*?

Since we want to use $\&$, $\vee$, and $\neg$ for the new truth values, we must require that the set D be closed under these operations, i.e., that whenever d and d' are possible complex truth values, then the values $d \cdot d'$, $d + d' - d \cdot d'$, and $1 - d$ are also possible (or, if we use other $\&-$ and $\vee-$operations $a\&b$ and $a \vee b$, the values $d\&d'$ and $d \vee d'$). What can we then say about the set D of truth values?

In the following text, we will prove that D *must* coincide with the set C of all complex numbers. Moreover, we will show that D must coincide with C not only if we add a complex number that corresponds to a true paradox (for which $a\&\neg a = 1$), but also if we add a complex number that corresponds to any other type of logical statements not covered by traditional fuzzy logic.

5 Can Other "and" and "or" Operations Describe True Paradoxes?

5.1 "And" Operations

The General Description of "And" Operations on [0,1]: A Brief Reminder. It is known (see, e.g., [23] and references therein) and under certain reasonable conditions, every $\&-$operation $\& : [0,1]\times[0,1] \to [0,1]$ can be represented as follows: the interval $[0,1]$ is divided into several subintervals, on each of which & is isomorphic to either $a \cdot b$, or to $\max(a+b-1,0)$, or to $\min(a,b)$, and when a and b are from different subintervals, then $a\&b = \min(a,b)$.

"Isomorphic" means that there exists a strictly increasing mapping ("isomorphism") ψ that maps our operation & into one of the three standard operations, i.e., for which, on the corresponding sub-intervals of the interval $[0,1]$, we have $\psi(a\&b) = \psi(a)\cdot\psi(b)$, or $\psi(a\&b) = \max(\psi(a)+\psi(b)-1,0)$, or $\psi(a\&b) = \min(\psi(a),\psi(b))$. If we apply the inverse function ψ^{-1} to both sides of these equalities, we get the expressions for the operations themselves: $a\&b = \psi^{-1}(\psi(a)\cdot\psi(b))$, $a\&b = \psi^{-1}(\max(\psi(a)+\psi(b)-1,0))$, or $a\&b = \psi^{-1}(\min(\psi(a),\psi(b))$ (since ψ is an increasing function, the last expression is simply equal to $a\&b = \min(a,b)$).

Not All "And" Operations Can Be Extended to Complex Numbers: Only Strictly Archimedean Ones. We want to describe operations that can be extended to complex numbers. Extension to complex numbers is a well-known operation for real-valued functions, known also as *analytical continuation.* It is known that complex-valued analytical functions have the following *uniqueness property* (see, e.g., [13]): that an analytical complex function can be uniquely determined if we know its values on an interval (even on a small interval). Therefore, if a complex-valued analytical operation & is described by a certain expression on some sub-interval of the interval $[0,1]$, it must be described by this same expression for all a and b.

This conclusion excludes all $\&-$operations that are described by different operations on different sub-intervals. So, if an $\&-$operation $a\&b$ can be generalized to complex numbers, we have three possibilities:

- & is isomorphic to $a \cdot b$;
- & is isomorphic to $\max(a+b-1,0)$;
- & is isomorphic to $\min(a,b)$.

We have already mentioned that $\min(a,b)$ cannot represent true paradoxes, so we can exclude the third possibility.

The second possibility can also be excluded, because in this case, for $a,b \in [0,1/2]$, this operation coincides with 0, and therefore, due to uniqueness theorem, its analytical continuation will result in identical 0.

Hence, we conclude that *if an $\&-$operation can be extended to complex numbers, it must be isomorphic to $a \cdot b$*, i.e., it must have the form

$$a \& b = \psi^{-1}(\psi(a) \cdot \psi(b)) \tag{2}$$

for some one-to-one (strictly increasing) function $\psi(a)$ for which $\psi(0) = 0$ and $\psi(1) = 1$. Such operations are called *strictly Archimedean* [23].

Not All Strictly Archimedean Operations Can be Extended to Complex Numbers. Not all strictly Archimedean $\&-$operations can be extended to complex numbers. For example:

- Operations (like Hamacher's operations) whose definition involve division by $1 + kab$ for some $k > 0$ are well defined for $a, b \in [0, 1]$, but become undefined (due to the necessity to divide by 0) if we take such complex numbers for which $ab = -1/k$.
- Operations that involve taking $p-$th order root for some $p > 0$ are well-defined for real numbers from the interval $[0, 1]$, but are ill defined for complex numbers, because in the filed of complex numbers, $p-$th order root is not a uniquely defined operation: for each $z \neq 0$, there are p different values z' for which $(z')^p = z$ and that can, therefore, qualify as values of the $p-$th root.

Summarizing: for the above formula (2) to be applicable to arbitrary complex numbers, we must require two things:

- First, that it is possible to extend the function $\psi(z)$ to complex numbers z, i.e., that $\psi(z)$ is an *analytical* function defined for all complex values z (such analytical functions are called *entire* functions [13]).
- Second, that the inverse function ψ^{-1} is defined for all complex numbers z (i.e., in mathematical terms, that the mapping ψ is one-to-one).

Let us describe operations that satisfy these two properties:

Definition 1. *By a complex-valued $\&-$operations, we mean an operation of the type $z \& z' = \psi^{-1}(\psi(z) \cdot \psi(z'))$ for some analytical function $\psi(z)$ that is a one-to-one mapping from the set of all complex numbers to itself, and for which $\psi(0) = 0$ and $\psi(1) = 1$.*

PROPOSITION 1. *$z \& z' = z \cdot z'$ is the only complex-valued $\&-$operation.*

Comment. For reader's convenience, all proofs are placed in the last section.

5.2 "Or" Operations

The description of $\vee-$operations easily follows from the description of $\&-$operations if we take into consideration that, as a corollary of natural properties of $\&-$ and $\vee-$operations, we can conclude that $a \vee b$ is $\vee-$operation iff its *dual* $a\&b = 1-(1-a)\vee(1-b)$ is an $\&-$operation. From this dual $\&-$operation, we can reconstruct the original $\vee-$operation as $a \vee b = 1-(1-a)\&(1-b)$.

Since we have already shown that the only possible complex-valued $\&-$operation is $a \cdot b$, we can thus conclude that *the only possible complex-valued* $\vee-$*operation is* $a \vee b = 1-(1-a)\cdot(1-b) = a+b-a\cdot b$.

5.3 Conclusion

The only possible complex-valued operations are $a\&b = a\cdot b$ and $a\vee b = a+b-a\cdot b$.

6 Are All Complex Numbers Necessary?

Due to the result described in the previous section, we will only consider the operations $a\&b = a\cdot b$ and $a \vee b = a+b-a\cdot b$.

6.1 The Main Result of This Section

Definition 2. *By a complex-valued fuzzy logic, we mean a set D of complex numbers for which $[0,1] \subseteq D \subseteq C$ and which satisfies the following three properties:*

i) *D is closed under $\&$, i.e., if $d \in D$ and $d' \in D$, then $d \cdot d' \in D$;*

ii) *D is closed under $\vee$, i.e., if $d \in D$ and $d' \in D$, then $d + d' - d \cdot d' \in D$;*

iii) *D is closed under $\neg$, i.e., if $d \in D$, then $1 - d \in D$.*

Comments.

- From the mathematical viewpoint, the condition ii) follows from the conditions i) and iii): indeed, if $d \in D$ and $d' \in D$, then:
  - due to iii), $1-d \in D$ and $1-d' \in D$;
  - due to i), $(1-d)(1-d') \in D$;
  - finally, applying iii) again, we conclude that $1-(1-d)(1-d') \in D$, and $1-(1-d)(1-d') = d+d'-d\cdot d'$.
- Sets that are closed under multiplication (i.e., satisfy the condition i)) are called *multiplicative semigroups*. So, in purely algebraic terms, the problem of classifying all complex-valued fuzzy logics can be reformulated as follows: to characterize all multiplicative subsemigroups of the semigroup

of complex numbers that contain the interval $[0,1]$ and that are closed under the operation $d \to 1-d$.

MAIN THEOREM. *There are three and only three different complex-valued fuzzy logics:*

- *D equal to the interval $[0,1]$;*
- *D equal to the set R of all real numbers;*
- *D equal to the set C of all complex numbers.*

Comment. So, if for some reason we need at least *one* complex number with non-zero imaginary part, we thus need *all* complex numbers as possible truth values.

6.2 The Auxiliary Result

The main result was an answer to the problem that the interval $[0,1]$ is not sufficient. It is natural to ask a similar question: are all values from the interval $[0,1]$ really necessary? A similar technique can help answer this question:

Definition 3. *By a $[0,1]$-logic, we mean a closed set D of real numbers for which $\{0,1\} \subseteq D \subseteq [0,1]$ and which satisfies the properties i)–iii).*

Comment. In purely algebraic terms, the problem of classifying all $[0,1]$-logics can be reformulated as follows: to characterize all closed multiplicative sub-semigroups of the semigroup $[0,1]$ of real numbers that are closed under the operation $d \to 1-d$.

AUXILIARY THEOREM. *There are two and only two different $[0,1]$-logics:*

- *D equal to the set $\{0,1\}$;*
- *D equal to the interval $[0,1]$.*

Comments.

- So, if we allow at least one value different from 0 and 1, we must use *all* values from the interval $[0,1]$.
- The condition that D is closed is necessary: without it, we would have other sets different from $\{0,1\}$ and $[0,1]$: e.g., the set of all *rational* numbers from the interval $[0,1]$ satisfies all three conditions i)–iii). Similarly, we can take as D the set of all numbers that belong both to the interval $[0,1]$ and to some sub-field of the field of all real numbers.

7 Proofs

7.1 Proof of Proposition 1

In our Definition 1, we have required that the function $\psi(z)$ be analytical and one-to-one. It is known (see, e.g., [13], Section 8.1.4, Problem 5) that every one-to-one analytical function $\psi(z)$ is linear, i.e., $\psi(z) = a \cdot z + b$.

From the condition that $\psi(0) = 0$ and $\psi(1) = 1$, we can conclude that $a = 1$ and $b = 0$, i.e., that $\psi(z) = z$. Hence, $z \& z' = \psi^{-1}(\psi(z) \cdot \psi(z')) = z \cdot z'$. The proposition is proven.

7.2 Proof of the Main Theorem

First, it is easy to see that all three sets described in the formulation of the Main Theorem are closed under &, $\vee$, and $\neg$. So, the only thing that we need to prove is that if D is a complex fuzzy logic (i.e., if D satisfies the conditions i)–iii)), then D coincides either with the interval $[0,1]$, or with the set R of all real numbers, or with the set C of all complex numbers.

This proof will consists of three lemmas.

Case When D Contains a Real Number from Outside the Interval $[0,1]$. Let us first prove the following lemma:

LEMMA 1. *If a complex-valued fuzzy logic D contains at least one real number from outside the interval $[0,1]$ (i.e., a number α that is either negative, or > 1), then every real number $d \in R$ belongs to D.*

Proof. Let us first show that every non-negative real number $d \geq 0$ belongs to D. Let us consider a real number $d \geq 0$.

From the fact that D contains a number $\alpha \notin [0,1]$, we can conclude that D contains a real number $\beta > 1$; indeed:

- If $\alpha \in D$ and $\alpha < 0$, then, due to iii), $1 - \alpha \in D$, and $1 - \alpha > 1$. So, we can take $\beta = 1 - \alpha$.
- If $\alpha \in D$ and $\alpha > 1$, then we can take $\beta = \alpha$.

From $\beta \in D$ and the condition i), we can conclude that $\beta^n \in D$ for all $n = 1, 2, \ldots$. For $\beta > 1$, we have $\beta^n \to \infty$ as $n \to \infty$. In particular, there exists an n for which $\beta^n > d \geq 0$ and hence, $0 \leq d/\beta^n < 1$. From $d/\beta^n \in [0,1]$ and $[0,1] \subseteq D$, we conclude that $d/\beta^n \in D$. Hence, $d = \beta^n \cdot (d/\beta^n)$ is a product of two elements of D and therefore, due to property i), d itself belongs to the set D.

So, under the condition of the lemma, every *non-negative* real number d belongs to the set D. To complete the proof of this lemma, we must also prove that every *negative* real number d belongs to D. Indeed, if $d < 0$, then $1 - d > 0$

and therefore, as we have just proven, $1 - d \in D$. Hence, from iii), we conclude that $d = 1 - (1 - d) \in D$. The lemma is proven.

Case When D Contains a Complex Number α With $\Im(\alpha) \neq 0$. Part 1. The previous lemma showed that if D contains at least one non-standard *real* number, then D coincides with the set R of all real numbers. Let us now consider the case when D contains at least one non-real number α, i.e., a complex number with a non-zero imaginary part $\Im(\alpha) \neq 0$.

For this case, we will prove the following Lemma:

LEMMA 2. *If a complex-valued fuzzy logic D contains an element α with $\Im(\alpha) \neq 0$, then there exists a real number $r > 0$ such that D contains all complex numbers z with $|z| \leq r$.*

Proof. Without loss of generality, we can assume that $\Im(\alpha) < 0$: indeed, if $\Im(\alpha) > 0$, then we can take as α a number $\alpha' = 1 - \alpha$, which also belongs to D (due to iii)) and for which, $\Im(\alpha') = -\Im(\alpha) < 0$.

Due to i), and to the fact that $[0, 1] \subseteq D$, from $\alpha \in D$, we can conclude that the entire segment $S = \{t \cdot \alpha \,|\, t \in [0, 1]\}$ belongs to the set D. In a standard complex plane, this segment is a straight-line segment connecting the points 0 and α. Since $\Im(\alpha) \neq 0$, this segment's only intersection with the real axis is the point 0; in particular, we can conclude that this segment does not contain the value 1. Since $\Im(\alpha) < 0$, the entire segment S lies in the lower complex semiplane (in which $\Im(z) \leq 0$).

Due to ii), for each element s from this segment S, the number $1 - s$ also belongs to the set D. Geometrically, going to $-s$ means inversion relative to 0, and going from s to $1 - s$ means a shift by 1 in the direction of the real axis. Both operations leave straight lines straight, so, the set $1 - S$ of all points $\{1 - s \,|\, s \in S\}$ is a straight-line segment that connects the point 1 on the real axis with a point $1 - \alpha$ (with $\Im(1 - \alpha) > 0$) in the upper semi-plane. Let us denote the phase of the complex number $1 - \alpha$ by φ.

For numbers $s \neq 0$, the phase continuously depends on the number; hence, as the number $1 - s$ goes from 1 to $1 - \alpha$, the phase continuously changes from 0 to φ; therefore, it takes all intermediate values.

Since the segment S did not contain the value 1, the segment $1 - s$ does not contain the point 0; hence, the distance between 0 and this segment is positive. Let us denote this distance by γ.

Let us now define r as γ^k, where $k = \lceil (2\pi/\varphi) \rceil$, and let us prove that for every complex number d for which $|d| \leq r$, we have $d \in D$.

To prove it, let us represent a number d in the exponential form: $d = |d| \exp(i\theta)$, where $0 \leq |d| \leq r$ and $0 \leq \theta \leq 2\pi$. From the definition of k, it follows that $k \geq (2\pi/\varphi)$, and, hence, $0 \leq (2\pi)/k \leq \varphi$. Hence, $\theta/k \leq (2\pi)/k \leq \varphi$, i.e., $\theta/k \in [0, \varphi]$. We have already proven that every phase from the interval $[0, \varphi]$ can be represented as a phase of some number from the segment $1 - S$; hence, there exists a number $z \in 1 - S \subseteq D$ for which $z = |z| \exp(i(\theta/k))$. Since the distance between 0 and $1 - S$ is γ, we have $|z| \geq \gamma$.

From $z \in D$ and the property i), we can conclude that $z^k \in D$, i.e., $|z|^k \exp(i\theta) \in D$, where $|z|^k \geq \gamma^k = r$. Since $|d| \leq r$, we have $|d| \leq r \leq |z|^k$; hence, $|d|/|z|^k \in [0, 1]$. But all elements from $[0, 1]$ are at the same time elements of D; therefore, $d = |d| \exp(i\theta)$ can be represented as a product of two elements of the set D: an element $z^k = |z|^k \exp(i\theta)$ and an element $|d|/|z|^k$. Hence, due to i), $d \in D$. The lemma is proven.

Case When D Contains a Complex Number α With $\Im(\alpha) \neq 0$. Part 2. To conclude the proof of the Main Theorem, we need to prove the following lemma:

LEMMA 3. *If a complex-valued fuzzy logic D contains an element α with $\Im(\alpha) \neq 0$, then all complex numbers belong to the set D.*

Proof. Indeed, from the previous lemma, we can conclude that a negative real number $-r$ belongs to the set D. From this fact and from the first lemma, we can now conclude that *all* real numbers belongs to the set D.

Let us now prove that every *complex* number $d = |d| \exp(i\theta)$ belongs to the set D. Indeed, every complex number d can be represented as a product of two numbers $d = d_1 \cdot d_2$, where $d_1 = r \exp(i\theta)$ belongs to D due to the previous lemma, and $d_2 = |d|/r$ is a real number, and hence, as we have just proven, also belongs to the set D. So, due to i), the product d also belongs to D. The lemma is proven.

Proof of the Theorem Itself. The three lemmas proven above enables us to prove the Main Theorem: indeed, if $D \supseteq [0, 1]$ contains only real numbers, then either D coincides with the interval $[0, 1]$, or the set D contains at least one element from outside this interval, in which case, according to the first lemma, D coincides with the set of all real numbers.

If D contains at least one non-real number, then, due to the third lemma, D coincides with the set of all complex numbers. The theorem is proven.

7.3 Proof of the Auxiliary Theorem

We need to show that if $\alpha \in D$ for some $\alpha \in (0, 1)$, then every element $d \in (0, 1)$ belongs to D. Since the set D is assumed to be closed, it is sufficient to construct a sequence of elements $d_n \in D$ that converges to d; this will prove that $d = \lim d_n \in D$. Let us fix an arbitrary $d \in (0, 1)$ and find such a sequence.

From $\alpha \in D$ and from condition i), we can conclude that $\alpha^n \in D$ for all $n = 1, 2, 3, \ldots$. Due to iii), we can conclude that $1 - \alpha^n \in D$. When $n \to \infty$, then $\alpha^n \to 0$ and therefore, $1 - \alpha^n \to 1$. In particular, starting from some N, we have $1 - \alpha^n > d$. In the remaining part of the proof, we will only consider $n \geq N$, so, we will assume that $1 - \alpha^n > d$.

Applying i) once again, we can conclude that for every p, we have

$$(1-\alpha^n)^p \in D.$$

For every n, the sequence of elements $(1-\alpha^n)^p$ for $p=1,2,\ldots$, is monotonically decreasing, and tends to 0 as $p\to\infty$. This means that for sufficiently large p, we have $(1-\alpha^n)^p < d$. Since for $p=1$, we have $1-\alpha^n > d$, and since the sequence is monotonically decreasing, we can conclude that there exists the value p for which $(1-\alpha^n)^p$ goes from above d to below d, i.e., for which

$$(1-\alpha^n)^{p+1} < d \le (1-\alpha^n)^p. \tag{3}$$

Let us take $(1-\alpha^n)^p$ for this p as d_n. This element clearly belongs to D. Then, dividing all sides of the inequality (3) by its right-hand side (that is equal to d_n), we conclude that

$$1-\alpha^n < \frac{d}{d_n} \le 1. \tag{4}$$

When $n\to\infty$, we have $\alpha^n\to 0$, $1-\alpha^n\to 1$, and therefore, $d/d_n\to 1$, i.e., d is a limit of some sequence d_n of elements of D. Since we assumed that the set D is closed, we can conclude that $d\in D$. The theorem is proven.

Acknowledgments

This work was supported in part by NSF Grants No. EEC-9322370 and DUE-9750858, by NASA Research Grants No. 9-757 and NCCW-0089, and by the Future Aerospace Science and Technology Program (FAST) Center for Structural Integrity of Aerospace Systems, effort sponsored by the Air Force Office of Scientific Research, Air Force Materiel Command, USAF, under grant number F49620-95-1-0518.

The authors are greatly thankful to Bart Kosko for the encouragement and for important comments, to anonymous referees for valuable suggestions, and to Ron Yager for his support.

References

[1] Atanassov, K. T., Intuitionistic fuzzy sets, *Fuzzy Sets and Systems*, 1986, Vol. 20, No. 1, pp. 87–96.

[2] Atanassov, K. T., More on intuitionistic fuzzy sets, *Fuzzy Sets and Systems*, 1989, Vol. 33, No. 1, pp. 37–45.

[3] Atanassov, K. T., New operations defined over the intuitionistic fuzzy sets, *Fuzzy Sets and Systems*, 1993, Vol. 61, No. 2, pp. 131–142.

[4] Atanassov, K. T., Operations over interval valued intuitionistic fuzzy sets, *Fuzzy Sets and Systems*, 1994, Vol. 64, pp. 159–174.

[5] Atanassov, K. T., Editor-in-Chief, *Notes on Intuitionistic Fuzzy Sets* (*NIFS*), International Journal, 1995–.

[6] Atanassov, K. T., and Gargov, G., Interval valued intuitionistic fuzzy sets, *Fuzzy Sets and Systems*, 1989, Vol. 31, No. 3, pp. 343–349.

[7] Belnap, N. D., *The logic of questions and answers*, Yale Univ. Press, New Haven, CT, 1976.

[8] Brams, S. J., *Paradoxes in politics: an introduction to the nonobvious in political science*, Free Press, NY, 1976.

[9] Cooper, L. A., and Kreinovich, V., *Using Linear Logic to represent uncertainty of our knowledge*, University of Texas at El Paso, Computer Science Department, Technical report UTEP-CS-90-2, May 1990.

[10] Corballis, M. C., *The ambivalent mind: the neuropsychology of left and right*, Nelson-Hall, Chicago, 1983.

[11] Da Costa, N. C. A., Beziau, J.-Y., Bueno, O. A. S., Aspects of paraconsistent logic, *Bull. IGPL*, 1995, Vol. 3, No. 4, pp. 597–614.

[12] Feynman, R. P., Leighton, R. B., and Sands, M. L., *The Feynman Lectures On Physics*, Addison-Wesley, Redwood City, CA, 1989.

[13] Flanigan, F. J., *Complex variables: harmonic and analytic functions*, N.Y., Dover, 1983.

[14] Ginsberg, M., Multiple-valued logics, *Combinatorial Intelligence*, 1988.

[15] Girard, J.-Y., Linear logic, *Theoretical Computer Science*, 1987, Vol. 50, pp. 1–102.

[16] Goguen, J. A., $L-$fuzzy sets, *Journal of Mathematical Analysis and Applications*, 1967, Vol. 18, pp. 145–174.

[17] Goguen, J. A., The logic of inexact reasoning, *Synthese*, 1969, Vol. 19, pp. 325–373 (reprinted in Dubois, D., Prade, H., and Yager, R. (eds.), *Reading in Fuzzy Sets for Intelligent Systems*, Morgan Kaufmann, San Mateo, CA, 1994, pp. 417–441).

[18] Grunbaum, A., *Modern science and Zeno's paradox*, Wesleyan University Press, Middletown, CT, 1967.

[19] Herbert, R. T., *Paradox and identity in theology*, Cornell University Press, Ithaca, NY, 1979.

[20] Hirota, K., and Sugeno, M., *Industrial Applications of Fuzzy Technology in the World*, World Scientific, Singapore, 1996.

[21] Honig, W. M., *Nonstandard logics and nonstandard metrics in physics*, World Scientific, Singapore, 1995.

[22] Hook, S., *The paradoxes of freedom*, University of California Press, Berkeley, CA, 1962.

[23] Klir, G., and Yuan, B., *Fuzzy sets and fuzzy logic: theory and applications*, Prentice Hall, Upper Saddle River, NJ, 1995.

[24] Koshelev, M., and Kreinovich, V., Fuzzy interpretation of quantum mechanics made more convincing: every statement with real numbers can be reformulated in logical terms, *Fuzzy Logic and the Management of Complexity (Proceedings of the 1996 International Discourse)* (V. Dimitrov and J. Dimitrov, Eds.), UTS Publ., Sydney, Australia, 1996, Vol. 3, pp. 296–299.

[25] Kreinovich, V., Lea, R., Fuentes, O., and Lokshin, A., Fuzzy control is often better than manual control of the very experts whose knowledge it uses: an explanation, In: *Proceedings of 1992 IEEE International Conference on Tools with Artificial Intelligence, Arlington, VA, 1992*, IEEE Computer Society Press, Los Alamitos, CA, 1992, pp. 180–185.

[26] Kreinovich, V. *et al.*, What non-linearity to choose? Mathematical foundations of fuzzy control, *Proceedings of the 1992 International Conference on Fuzzy Systems and Intelligent Control*, Louisville, KY, 1992, pp. 349–412.

[27] Mackie, J. L., *Truth, probability, and paradox*, Claredon Press, Oxford, 1973.

[28] *NAFIPS/IFIS/NASA'94, San Antonio, December 1994*, IEEE, Piscataway, NJ, 1994.

[29] Nguyen, H. T., Kosheleva, O. M., and Kreinovich, V. Is the success of fuzzy logic really paradoxical? Or: Towards the actual logic behind expert systems, *Int'l J. of Intelligent Systems*, 1996, Vol. 11, No. 5, pp. 295–326.

[30] Nguyen, H. T. and Kreinovich, V., Towards theoretical foundations of soft computing applications, *Int'l J. on Uncertainty, Fuzziness, and Knowledge-Based Systems,* 1995, Vol. 3, No. 3, pp. 341–373.

[31] Nguyen, H. T., and Kreinovich, V., Fuzzy logic, logic programming, and linear logic: towards a new understanding of common sense, *Proceedings of NAFIPS'96, Biennial Conference of the North American Fuzzy Information Processing Society*, Berkeley, CA, June 20-22, 1996, pp. 546–550.

[32] Nguyen, H. T., Kreinovich, V., and Zuo, Q., Interval-valued degrees of belief: applications of interval computations to expert systems and intelligent control, *International Journal of Uncertainty, Fuzziness, and Knowledge-Based Systems (IJUFKS)*, 1997, Vol. 5, No. 3, pp. 317–358.

[33] *Paradox and counterparadox: a new model in the therapy of family in schizophrenic transactions*, Jason Aronson, N.Y., 1978.

[34] *Paradox and transformation: towards a theory of change in organization and management*, Ballinger Pub., Cambridge, MA, 1988.

[35] *Promoting change through paradoxical therapy*, Dow Jones-Irwin, Homewood, IL, 1985.

[36] Pykacz, J., Quantum logics as families of fuzzy subsets of the set of physical states, *Proc. Second IFSA Congress,* 1987, Vol. 2, pp. 437–440.

[37] Pykacz, J., Fuzzy quantum logics and infinite-valued Lukasiewicz logic, *Int'l J. Theor. Physics,* 1994, Vol. 33, No. 7, pp. 1403–1416.

[38] Seltzer, L. F., *Paradoxical strategies in psychotherapy: a comprehensive overview and guidebook*, Wiley, N.Y., 1986.

[39] Shapiro, K. J., *Bodily reflective modes: a phenomenological method for psychology*, Duke University Press, Durham, NC, 1985.

[40] Shiriaev, A. N., *Probability*, Springer-Verlag, N.Y., 1984.

[41] Smith, K. K., *Paradoxes of group life: understanding conflict, paralysis, and movement in group dynamics*, Jossey-Bass, San Francisco, 1987.

[42] Smith, M. H., and Kreinovich, V., Optimal strategy of switching reasoning methods in fuzzy control, Chapter 6 in *Theoretical aspects of fuzzy control* (H. T. Nguyen, M. Sugeno, R. Tong, and R. Yager, Eds.), J. Wiley, N.Y., 1995, pp. 117–146.

[43] Spencer-Brown, G., *British patent specifications 1006018 and 1006019*, London, U.K., 1965.

[44] Spencer-Brown, G., *Laws of form*, George Allen and Unwin, London, 1969.

[45] Troelstra, A. S., *Lectures on linear logic*, CSLI, Stanford, 1992.

[46] Weeks, G. R., *Paradoxical psychotherapy: theory and practice with individuals, couples, and family*, Brunner/Mazel, N.Y., 1982.

[47] Wisdom, J., *Paradox and discovery*, Philosophical Library, N.Y., 1965.

[48] Wolgast, E. H., *Paradoxes of knowledge*, Cornell University Press, Ithaca, NY, 1977.

[49] Yager, R. R., J. Kacprzyk, J. and Pedrizzi, M., Eds., *Advances in the Dempster-Shafer Theory of Evidence*, Wiley, N.Y., 1994

[50] Zadeh, L., Fuzzy sets, *Information and Control,* 1965, Vol. 8, pp. 338–353.