From Fuzzification and Intervalization to Anglification: a New 5D Geometric Formalism for Physics and Data Processing

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Data Processing: Geometric Interpretation is Needed. Our interest originated from the needs of data processing; the data to be processed consists of values of several measured quantities, i.e., usually, several real numbers. In the computer, the sequence of real numbers $(x_1, \ldots, x_n)$ is represented as an array, i.e., in mathematical terms, as an n-dimensional vector (an element of an n-dimensional space).

This reformulation in terms of n-dimensional geometry often enables us to reformulate non-visual notions from data processing in more visual geometric terms. For example:

- When we use Gaussian distribution, then the level sets of this distribution forms ellipsoids in this n-dimensional space.
- An a priori relation between the variables, if linear, describes a plane in this n-dimensional space.

For small n ($\leq$ 3 or 4), this geometric interpretation is often helpful for data processing, because we can then use: our geometric intuition and geometric results developed in geometry and physics to simplify data processing methods.

It would be nice to have similar geometric techniques for larger n, but for larger n, we do not have ready-made geometric results to apply:

- our geometric intuition does not work that well in multi-dimensional spaces, and
- physical results are no longer applicable, because physics is mainly interested in 3D space and 4D space-time.

To get such results, we turned to the areas of physics where multi-dimensional geometries are currently used.

Physics: 5D Geometry is Useful. After the 1916 success of Einstein, who explained gravitation by combining space and time into a 4D space, there have been many efforts to explain other physical fields by adding other physical dimensions.

The first successful attempt was made by Kaluza and Klein in 1921. They showed that if we formally consider the equations of general relativity theory in a 5D space, then the equations for the normal 4×4 components g_{ij} of the metric tensor still describe gravitation, while the new components g_{5i} of the metric tensor satisfy Maxwell's equations. Thus, if we go to 5D space, we get a geometric interpretation of electrodynamics.

The only problem with this interpretation is that it is formal: change in first 4 dimensions makes perfect physical sense, while there seemed to be no physical effects corresponding to change in 5th dimension. To solve this problem, Einstein and Bergmann proposed, in 1938, that the 5th dimension forms a tiny circle, so that only micro-particles "see" it, while for us, the world is 4D. (This is a standard view now in particle physics: space is 10- or 11-dimensional, all dimensions except the first four are tiny).

The Physical Model is Unusual, But This Un-Usualness is Appropriate for Data Processing. The standard multi-D physical model is unusual geometrically: the space is a cylinder, not a plane anymore. This feature is, however, interestingly related to data processing: some measured data are angles, and angles do form a circle, so these geometric ideas can be directly applied to data processing.

Formulas from Physical 5D Theories that Need to Be Explained in Purely Geometric Terms. First, we would like to explain the fact that the observed values of physical fields do not depend on the fifth coordinate x^5, e.g., that $\partial g_{ij}/\partial x^5 = 0$ (this condition is called *cylindricity*).

Several other formulas came from the attempts to give the fifth dimension a physical interpretation. Namely, in early 1940s, Yu. Rumer showed that if we interpret x^5 as action $S = \int L\,dx\,dt$ (i.e., the quantity whose extrema define the field's dynamics), then the fact that x^5 is defined on a circle is consistent with the fact that in quantum physics (e.g., in its Feynman integral formulation), action is used only as part of the expression $\exp(iS/h)$, whose value is not changed if we add a constant $2\pi \cdot h$ to S. (For a H atom, this idea leads to the original Bohr's quantization rules.)

Action is defined modulo arbitrary transformation $S \to S + f(x^i)$; thus, the corresponding transformation $x^5 \to x^5 + f(x^i)$ should be geometrically meaningful. Similar transformations stem from the electrodynamic interpretation of g_{5i} as A_i: gauge transformations $A_i \to A_i - \partial f/\partial x_i$.

Natural Idea and Its Problems. The main difference between a standard 4D space and Einstein-Bergmann's 5D model is that we have a cylinder $W = R^4 \times S^1$ instead of a linear space. It is, therefore, desirable to modify standard geometry by substituting W instead of R^4 into all definitions.

The problem with this idea is that the corresponding formalisms of differential geometry use the underlying linear space structure, i.e., addition and multiplication by a scalar. We still have addition in W, but multiplication is not uniquely defined for angle-valued variables: we can always interpret an angle as a real number modulo the circumference, but then, e.g., $0 \sim 2\pi$ while $0.6 \cdot 0 \not\sim 0.6 \cdot 2\pi$.

What We Suggest. We do need a real-number representation of an angle variable. A more natural representation of this variable is not as a single real number, but as a *set* $\{\alpha + n \cdot 2\pi\}$ of all possible real numbers that correspond to the given angle.

Similarly to interval arithmetic, we can naturally define element-wise arithmetic operations on such sets, e.g., $A + B = \{a + b \mid a \in A, b \in B\}$. We can then define tensors as linear mappings that preserve the structure of such sets, and we can define a *differentiable* tensor field as a field for which the set of all possible values of the corresponding partial derivatives is also consistent with the basic structure.

Resulting Formalism. In mathematical terms, the resulting formalism is equivalent to the following: We start with the space W which is not a vector space (only an Abelian group). We reformulate standard definitions of vector and tensor algebra and tensor analysis and apply them to W: K-vectors are defined as elements of W; K-covectors as elements of the dual group, etc. All physically motivated conditions turn out to be natural consequences of this formalism.

Potential Applications to Data Processing. For example, a natural analog of Gaussian distribution is $\exp(-\sum a_{ij} x^i x^j)$ for a K-tensor a_{ij}.

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References

[1] Th. Kaluza, *Sitzungsberichte der K. Prussischen Akademie der Wiseenschaften zu Berlin*, 1921, p. 966 (in German); Engl. translation "On the unification problem in physics" in [3], pp. 1–9.

[2] O. Klein, *Zeitschrift für Physik*, 1926, Vol. 37, p. 895 (in German); Engl. translation "Quantum theory and five-dimensional relativity" in [3], pp. 10–23.

[3] H. C. Lee (ed.), *An introduction to Kaluza-Klein theories*, World Scientific, Singapore, 1984.