An operationalistic reformulation of Einstein's equivalence principle

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1 Introduction

Analyzing gravitational phenomena Einstein used the following postulate (which he called equivalence principle): what ever measurements we perform inside some spacetime region we cannot distinguish between the case when there is a homogeneous gravitational field and the case when all bodies in this region have constant acceleration with respect to some inertial frame. (And since any field can be considered homogeneous in a small enough region, this principle can be applied to a neighborhood of any point).

Einstein concluded from this principle that the spacetime metric is pseudo-Riemannian and in absence of all other fields but gravity the test particles are traveling along geodesics of this metric [1].

Yet V.A.Fock [2] noticed that this formulation is not exact enough: according to general relativity , the presence of gravitation means spacetime to be curved, i.e. curvature tensor is nonzero, $R_{ijkl} \neq 0$. This is valid in any frame, in particular in a uniformly accelerated one. Hence in presence of gravity $R_{ijkl} \neq 0$ while in uniformly accelerated frame $R_{ijkl} = 0$, and this can be distinguished experimentally by emitting a "cloud" of a particles endowed with clocks to various directions with various speeds. With a help of the clocks one can determine the proper time ds along every trajectory and then calculate the metric. By numerical differentiation of the metric we can obtain the values of R_{ijkl} and then compare all them with zero.

That is why most of authors do postulate the Riemannian metric within the strict mathematical account of general relativity. We would like to give here more profound and at the same time more strict mathematical grounds for this fact.

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The main drawback of traditional definition of Riemannian geometry of spacetime is that it is formulated in terms of length (*viz.* proper time) of idealized infinitely small intervals rather than real ones of finite size. Besides that, this definition demands length of space-like intervals to be determined which is not desirable from the operationalistic point of view. Some authors (see, *e.g.* [5]) give the equivalent definition including only proper time along finite parts of time-like curves. The postulate the metric to be Riemannian in the sense of this definition. This is more operationalistic but yet not motivated physically.

In our paper we show that one can reformulate the initial Einstein's equivalence principle in such a way that both Riemannian metric of spacetime and the geodesic motion of test particles will be obtained from it.

Considering only the gravitational field this result is of few interest, but the question becomes essential in presence of non gravitational fields - it stipulates the choice of covariant analogue of an equation. For example in [4] one asserts that conformally invariant scalar field equations $\Box\phi + 1/6\, R\phi = 0$ come into contradiction with equivalence principle since they contain scalar curvature R (a more detailed analysis of this issue can be found in [3]). Nevertheless, such reasonings does not seem convincing: for example, usual Maxwell equations in curved space contain explicitly the curvature, but they undoubted by agree with equivalence principle. However if we twice differentiate both parts of the equation and exchange $F_{kl;ij}$ by $F_{kl;ji} + R_{ij}{}^{p}{}_{l} F_{kp} + R_{ij}{}^{p}{}_{k} F_{pl}$ we obtain equations containing the curvature explicitly. Thus the presence of curvature tensor in an equation does not mean at all the violation of equivalence principle. The formulation of equivalence principle proposed below allows to solve this problem in a physically meaningful and mathematically strict way.

The idea of our reformulation is the following. The Fock's experiment with the cloud of particles described above is idealized since all real measurements have their errors. Therefore all the values calculated via these measurements, in particular, the curvature, have their errors too. Thus if the error of so calculated curvature tensor will be great enough (of the order of the curvature itself) it would not be possible to determine whether the genuine value of curvature tensor is equal to zero or not. In the meantime, the Einstein's principle claims that any *real* (rather than exact) measurement performed in small enough region will not allow us to distinguish real (possibly curved) space from flat one.

2 Mathematical formulation

Begin with a formalization of basic notions.

Definition 1 *A spacetime region M is called ϵ-SMALL with respect to some fixed frame iff for any i and any $a, b \in M$*

$$|x^i(a) - x^i(b)| < \epsilon$$

This definition depends on coordinate frame; however the reformulation of equivalence principle based on this definition turns out not to depend on frame.

The spacetime properties determine the relative movement of uncharged particles. There are devices to measure coordinates and other kinematic features of the motion: time, velocity (e.g. using the Doppler effect), acceleration etc. However one mostly measure time or length (e.g. Doppler measurement of velocity contains determining frequency - the time interval between neighbor maxima). So, further we shall consider only time and length measurement. Clearly the measuring of small intervals of time and length can be performed with smaller absolute error. Let us denote by $\lambda(\epsilon)$ the error of our measurements in the ϵ-small region.

Definition 2 *A* SPACETIME *is a triple* (M, Γ, τ) *with* M *– a smooth manifold,* Γ *– a family of smooth curves on* M *(trajectories of test particles) and for any* $\gamma \in \Gamma$ *a smooth function* $\tau : \gamma \times \gamma \to \mathbf{R}$ *(the proper time along* γ*) is determined such that*

$$\tau(a, c) = \tau(a, b) + \tau(b, c)$$

whenever

$$\gamma^{-1}(a) < \gamma^{-1}(b) < \gamma^{-1}(c)$$

The FLAT spacetime is a triple (M_0, Γ_0, τ_0) where $M_0 = \mathbf{R}^4$, Γ_0 is set of all time-like straight lines, τ_0 is Minkowski metric.

Definition 3 *A spacetime* (M, Γ, τ) *is called* λ*-flat if for any point* $m \in M$ *there exists such a frame that for a sufficiently small* $\epsilon > 0$ *all coordinate and time measurements in any* ϵ*-small region of* M *coincide (up to an error* $\leq \lambda(\epsilon)$*) with the analogous result in the flat spacetime.*

The final formulation of the equivalence principle must not of course depend on accessible devices (*i.e.* the kind of the function λ). Thus, instead of a single function we must operate with a class of such functions $\Lambda = \{\lambda\}$. We shall assume possible refinement of any measurement, namely Λ together with every λ is assumed to contain also the function $k\lambda$ for every $0 < k < 1$.

Definition 4 *A spacetime is siad to be* Λ*-flat if it is* λ*-flat for all* $\lambda \in \Lambda$*.*

The formulation of the equivalence principle we propose is the following:

the spacetime is λ*-flat*

If we exclude degenerate cases (Λ is too great and Fock's reasoning is valid or Λ is too small so that Λ-flatness implies nothing) the proposed formulation yields us a basis for Riemannian metrics and geodesic motion.

3 Main results

Theorem 1 *For any class of functions $\Lambda : \mathbf{R}^+ \to \mathbf{R}^+$ such that $\lambda \in \Lambda$ implies $k\lambda \in \Lambda$ for any positive $k \leq 1$ one of the following statements is valid:*

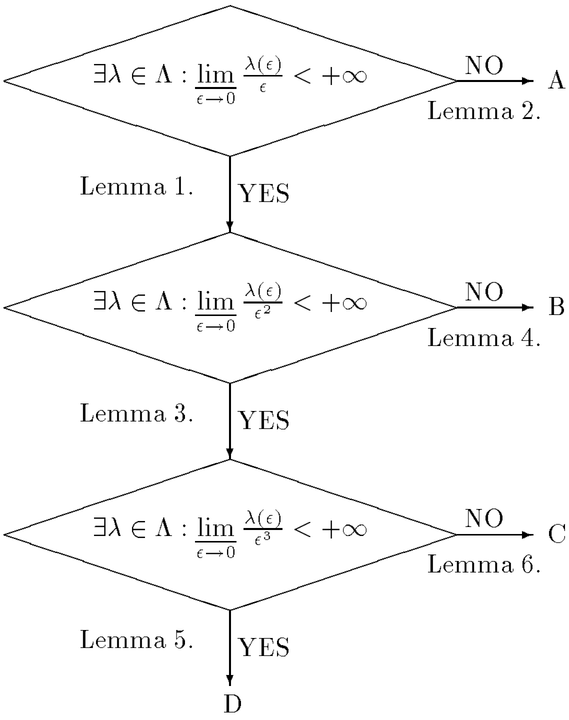
A. Any spacetime is Λ-flat.

B. Only pseudo-Riemannian spacetime are Λ-flat and the class of curves Γ is arbitrary.

C. Only pseudo-Riemannian spacetimes are Λ-flat and Γ is the set of timelike geodesics.

D. Only flat spacetime is Λ-flat.

The proof will be organized according to the following plan:



Lemma 1 *If there exists $\lambda \in \Lambda$ for which*

$$\lim_{\epsilon \to 0} \frac{\lambda(\epsilon)}{\epsilon} = K < +\infty \tag{1}$$

then any Λ-flat spacetime is pseudo-Riemannian.

Proof. Consider a curve $\gamma \in \Gamma$ and a point $a \in m$ in a coordinate frame $\{x^i\}$. Let a have the coordinates $\{x^i_0\}$ in this frame.

In accordance with the definition of $\underline{\lim}$ there exists such a sequence ϵ_n that $\lambda(\epsilon_n)/\epsilon_n$ tends to K. Let all ϵ_n be small enough (it can be assumed with no loss of generality) then for any n all measurements in the region

$$|x^i - x^i_0| < \frac{\epsilon_n}{2}$$

with the error not greater than $\lambda(\epsilon_n)$ coincide with same measurements in flat spacetime, in particular

$$|\delta\tau - \delta\tau_0| \le \lambda(\epsilon_n)$$

where

$$\begin{aligned} \delta\tau &= \tau(x^i_0 + \delta x^i, x^i_0) \\ \delta\tau_0 &= \tau_0(x^i_0 + \delta x^i, x^i_0) \end{aligned}$$

and τ, τ_0 are metrics along two geodesics both passing through the region described above.

If $x_0 + \delta x$ lies on the frontier of the region then $|\delta x^i \ge C\epsilon_n$ for some $C = \mathrm{const}$ thus

$$\left| \frac{\delta\tau}{\delta x^i} - \frac{\delta\tau_0}{\delta x^i} \right| \le \frac{\lambda(\epsilon_n)}{C\epsilon_n}$$

therefore

$$\frac{\delta\tau_0}{\delta x^i} - \frac{\lambda(\epsilon_n)}{C\epsilon_n} \le \frac{\delta\tau}{\delta x^i} \le \frac{\delta\tau}{\delta x^i} + \frac{\lambda(\epsilon_n)}{C\epsilon_n}$$

So when $n \to \infty$ (and $\epsilon_n \to 0$)

$$\frac{\delta\tau_0}{\delta x^i} - \frac{K}{C} \le \underline{\lim}\frac{\delta\tau}{\delta x^i} \le \overline{\lim}\frac{\delta\tau}{\delta x^i} \le \frac{d\tau_0}{dx^i} + \frac{K}{C}$$

All the above reasonings are valid for any $k\lambda$ with $k < 1$ hence

$$\frac{d\tau_0}{dx^i} + \frac{kK}{C} \le \underline{\lim}\frac{\delta\tau}{\delta x^i} \le \overline{\lim}\frac{\delta\tau}{\delta x^i} \le \frac{d\tau_0}{dx^i} + \frac{kK}{C}$$

Since k can be taken arbitrary small, we have

$$\lim \frac{\delta\tau}{\delta x^i} = \frac{d\tau_0}{dx^i}$$

i.e. in any point in some coordinate frame the metric of our spacetime coincides with Minkowskian one, that is why it is pseudo-Riemannian. □

Lemma 2 *If for any $\lambda \in \Lambda$*

$$\underline{\lim}\frac{\lambda(\epsilon)}{\epsilon} = +\infty \tag{2}$$

then any spacetime is Λ-flat.

Proof. (2) implies $\lim \epsilon \rightarrow 0 \lambda(\epsilon)/\epsilon = +\infty$. Hence for arbitrary N we have $\lambda(\epsilon) > N\epsilon$ beginning from some ϵ. In particular, it is valid for $N > \sup |d\tau/dx^i|$, hence

$$\delta\tau < N\delta x^i \leq N\epsilon < \lambda(\epsilon)$$

thus $|\delta\tau_0 - \delta\tau| < \lambda(\epsilon)$ for any $\lambda \in \Lambda$. □

Lemma 3 *If there exists $\lambda \in \Lambda$ such that*

$$\underline{\lim} \frac{\lambda(\epsilon)}{\epsilon^2} < +\infty \qquad (3)$$

then in any Λ-flat spacetime the set Γ is a set of geodesics.

Proof is similar to that of Lemma 1, but uses the second derivatives of τ:

$$\left| \frac{d^2 x^i}{d\tau^2} - \frac{d^2 x^i}{d\tau_0^2} \right| \leq \frac{\lambda(\epsilon)}{\epsilon^2}$$

hence is some coordinate frame $d^2x^i/d\tau^2 = 0$ thus $D^2x^i/ds^2 = 0$. □

Lemma 4 *If for any $\lambda \in \Lambda$*

$$\underline{\lim} \frac{\lambda(\epsilon)}{\epsilon^2} = +\infty \qquad (4)$$

then any pseudo-Riemannian space with any set of trajectories Γ is Λ-flat.

Proof. Is similar to that of Lemma 2. We obtain that

$$\lambda(\epsilon) > |D^2x^i/ds^2| = |D^2x^i/ds^2 - 0| = |D^2x^i/ds^2 - D^2x_0^i/ds^2|$$

□

Lemma 5 *If for some $\lambda \in \Lambda$*

$$\underline{\lim} \frac{\lambda(\epsilon)}{\epsilon^3} < +\infty \qquad (5)$$

then any Λ-flat spacetime is flat.

Proof. In this case the Fock's reasoning is valid: in terms of $d^3\tau/dx_i^3$ we can define the curvature tensor with arbitrary small error. $\square$

Lemma 6 *If for any* $\lambda \in \Lambda$

$$\underline{\lim}\frac{\lambda(\epsilon)}{\epsilon^3} = +\infty \tag{6}$$

and for some $\lambda \in \Lambda$

$$\underline{\lim}\frac{\lambda(\epsilon)}{\epsilon^2} < +\infty \tag{7}$$

then any pseudo-Riemannian space with the set of geodesics Γ *is* Λ*- flat.*

Proof is carried out likewise. This competes the proof of the main theorem. $\square$

The physical meaning of the results obtained is the following: $\lim\frac{\lambda(\epsilon)}{\epsilon} < +\infty$ means the possibility of arbitrary exact measuring of velocities, $\lim\frac{\lambda(\epsilon)}{\epsilon^2} < +\infty$ means the possibility of arbitrary exact measuring of accelerations, and $\lim\frac{\lambda(\epsilon)}{\epsilon^3} < +\infty$ means the possibility of arbitrary exact measuring of derivatives of accelerations. So the physical meaning of the result we obtained is the following: the equivalence principle is valid only for measuring velocities and accelerations (in any point they can be turned to zero by corresponding choice of the coordinate frame) but not valid for derivatives of accelerations (which correspond to invariant tidal forces).

4 Some remarks on other fields

We require the results of all measurements (including trajectories of particles traveling under other fields) performed in ϵ-small region (see Def.1) to coincide up to $\lambda(\epsilon)$ with the results of analogous experiments in flat spacetime. We also require it to be so for all functions λ of a class Λ containing a function $\underline{\lim}\lambda(\epsilon)/\epsilon^2 < +\infty$ satisfying and containing no function λ with $\underline{\lim}\lambda(\epsilon)/\epsilon^3 < +\infty$

Let us demonstrate that the equivalence principle in this formulation holds for equation of radiating charged particle and theories with conformally invariant scalar field and does not hold for scalar-tensor Brans-Dicke theory.

The equation describing the motion of a *radiating charged particle* contains second derivatives of its velocity $\ddot{u}_i = d^2u_i/ds^2$ viz. third derivatives of coordinates measured up to $\lambda(\epsilon)/\epsilon^3$. However $\lambda(\epsilon)/\epsilon^3$ tends to infinity for any $\lambda \in \Lambda$, hence the smaller is the region, the greater is vagueness in determining $\ddot{u}_i$, thus any equation containing $\ddot{u}_i$ does not contradict our equivalence principle (including equations containing the curvature explicitly).

On *conformally-invariant scalar field.* As we already mentioned, the only way to measure the curvature is by exploring its influence on trajectories of particles in accordance with the formula $\ddot{x}^i = \varphi_{,i}$. Since $\ddot{x}^i$ is measured up to $\lambda(\epsilon)/\epsilon^3$, the measurement of $\varphi_{,i}$ has the same value, hence $\Box\varphi = (\varphi_{,i}^{;i}$ is measured up to $\lambda(\epsilon)/\epsilon^3$ and in accordance with the preceding reasoning any equation containing $\Box\varphi$ do not contradict the operational equivalence principle.

In *Brans-Dicke theory* the principle does not hold since micro-black holes do not move along geodesics there.

5 Locally almost isotropic space is almost uniform: on the physical meaning of Schur's theorem

The Schur's theorem asserts that if a space is locally isotropic (i.e. in any point the curvature tensor has no directions chosen by some properties)

$$R_{ijkl} = K(x)(g_{ik}g_{jl} - g_{il}g_{jk}) \tag{8}$$

then the space is homogeneous.

Since all the real measurements are not exact their results can provide only local almost isotropicity. To what extent we can consider it to be homogeneous? It was the problem arised in 1960-s by Yu.A.Volkov.

The main difficulty in solving the problem is that the usual proof of Schur's theorem is based on Bianchi identities applied to (8) where after summing we obtain $K_{,i} = 0$ hence $K =$ const. However the fact that curvatures along different directions are almost equal does not provide $K_{;i}$ to be small enough. So, if we consider a space with almost equal curvatures along all the directions as locally isotropic one, we cannot obtain any isotropy [?]. The goal of this section is to answer this question assuming the local almost isotropicity to be the closeness of results of all the measurements along any direction.

Definition 5 *A spacetime region M is called ϵ-small if for any $a, b \in M$*

$$|x^i(a) - x^i(b)| \le \epsilon$$
$$\rho(a,b) \le \epsilon$$

where ρ is a metric on M.

Definition 6 *By an (x^i)-distance between points $a, b \in M$ we shall call* $\max_i\{|x^i(a) - x^i(b)|, \rho(a,b)\}$. *The ϵ-neighborhood of a point a is the set of all points m of M such that the (x^i)-distance between m and a does not exceed ϵ. All geometric and kinematic measurements, as it was already mentioned in section 2 are reduced to the measurement of the metric, the distances and proper time interval along trajectories of particles.*

Definition 7 *A spacetime region M is ϵ-locally isotropic if for any $a \in M$ one can define the action of an apropriate rotation group so that a is invariant under this action and all results obtained in the ϵ-neighborhood of a coincide up to $\lambda(\epsilon)$ with the results obtained after the action of any element of the group.*

Definition 8 *In an analogous way we shall call a region δ-uniform if one can define an action of a transition group on this region so that the results of any measurement on a system of N particles coincide with the precision $\lambda(\epsilon)$ with analogous measurement on the system, obtained from the first one after applying to it any element of the group.*

A region is called ϵ-locally δ-uniform if all the above is valid for the measurement in ϵ-neighborhoods of any two points of the region. Further we shall consider only geodesically connected domains.

The problem now is to find the least δ (over ϵ, δ and L) such that any ϵ-locally λ-isotropic region of the size L is δ-uniform.

Theorem 2 *Any ϵ-locally λ-isotropic region of the size L is:*

1. *$C\epsilon$-locally $\frac{CL\lambda}{\epsilon}$-uniform*
2. *$C(L/\epsilon)^2\lambda$-uniform*

where $C =$ const, and there are spaces for which this evaluations cannot be diminished.

Sketch of the Proof. 1). Since the local metric of Riemannian space is close to Euclidean there exists some c of order 1 such that into the intersection of two ϵ-neighborhoods of two points on the distance ϵ one can inscribe a $C\epsilon$-neighborhood.

2). Consider two $C\epsilon$-neighborhoods of arbitrary points $x, y \in M$. Let x^1, y^1 be points on their frontiers (in pseudo-Riemannian case we choose these points so that xx^1 and yy^1 would be of the same kind). Since M is geodesically connected, x^1 and y^1 can be connected with a geodesic whose (x^i)-length does not exceed L (see Definition 6). It can be divided into L/ϵ parts of (x^i)-length of order ϵ. Then we approximate the geodesic by a broken line so that the intervals $x^1z^1, z^1z^2, \dots, z^ny^1$ have the (x^i)-length ϵ.

Now let us rotate $C\epsilon$-neighborhood of x in order to make it appears all inside the intersection of ϵ-neighborhoods of x^1 and z^1.

Then we rotate it around z^1, so that it gets to the ϵ-neighborhood of z_2 and so on until the ϵ-neighborhood of the point y. The results of all measurements do not differ more than $\delta y\lambda$, hence the results in neighborhoods of x^1 and y^1 do not differ by more than $\lambda L/\epsilon$.

Here is an example when the evaluation cannot be refined: when all differences of measurements are of the same sign i.e. curvature monotonically varies from x^1 to y^1.

3). In a similar way we obtain the result for the global uniformity. An interval of curve of length $\sim L$ is composed of $\sim CL/\epsilon$ intervals of length $\sim C\epsilon$. The results of measurements along this intervals are indistinguishable up to $L\lambda/\epsilon$, hence the error in time or length measurement along all the curve does not exceed $(CL/\epsilon)\cdot(L\lambda/\epsilon) = C(L/\epsilon)^2\lambda$. This completes the proof of Theorem 2.

6 Physical interpretation of the results and possible applications

If we perform in an ϵ-small (see Definition 5) region a measurement with error λ we know the metric $g_{ij} \sim \delta\tau/\delta x^j$ up to λ/ϵ, the Cristoffel symbols $\Gamma^i_{jk} \sim \partial g_{ik}/\partial x^j \sim \delta^2\tau/\delta x^{j\,2}$ up to λ/ϵ^2, and the curvature up to λ/ϵ^3. In any point we can choose a coordinate frame making Γ^i_{jk} zero, therefore measurements with error $\lambda \sim \epsilon^2$ do not allow us to distinguish non-isotropic case from a locally isotropic (and even from flat one, i.e. any Riemannian space is ϵ-locally $C\epsilon^2$-isotropic). For the sake of such distinction the error of the curvature must not exceed its value K, so the relation $\lambda/K\epsilon^2$ characterize the relative error of measurement of local *isotropy*.

The relative error of measuring the local isotropy is the same. Therefore local isotropy implies local uniformly with ϵ/λ times greater error. Hence if $\lambda \sim \epsilon^3$ we obtain (for small enough ϵ) the ϵ^2-uniformity obtained above for any Riemannian metric. To obtain non-trivial information on local uniformity one must have $\lambda \leq \epsilon^4$ which corresponds to the possibility of exact enough measurement of curvature tensor and its derivatives, viz. tidal forces and their spacetime gradients. Mathematically it means that if both and R_{ijkl} and $R_{ijkl;m}$ are almost isotropic then the space is almost uniform.

Thus the physical result is the following: if accelerations and tidal forces an locally isotropic then nothing can be said on uniformity of the region. However if gradients of tidal forces are also isotropic then the space is locally uniform.

Imagine we verify the isotropy in a few points (e.g. close to the Earth) - in general the same points as other points of space. If it happens that in these points the space is ϵ-locally λ-isotropic then it is naturally to assume ϵ-local λ-isotropy everywhere and the results obtained give us the evaluation of its homogeneity.

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