

# Uncertainty Representation Explains and Helps Methodology of Physics and Science in General

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## 1 Introduction

In different sciences, there are different levels of certainty: e.g., a conclusion acceptable to a geologist will not be sufficient to convince a physicist (because physical laws are usually based on larger amounts of data and are, thus, more reliable).

A natural way to represent these different levels of uncertainty is to use different numbers from the interval  $[0,1]$ , i.e., to use fuzzy logic. The simplest possible operations of fuzzy logic, min and max, already enable us to explain why on each level we have a deductive system (i.e., a set of statements closed under deductions). The use of min and max is, of course, only a first approximation to the actual more adequate and- and or-operations: a sufficient number of repeated experiments (or other confirmations) can make a statement more convincing and even raise it to the next level of confidence.

What can we get from this representation? In this paper, we prove two theorems:

- First, we prove a *lifting* theorem: if some fuzzy statement is true then it can be upgraded to a more precise statement. (A similar notion is called *sharpening* of a fuzzy set in a 1986 paper [9] by the first author).
- Second, we prove that lifting is non-trivial in the sense that there is no feasible lifting algorithm.

These theorems explain why many scientific results are eventually upgraded from the lowest level of guesses and intuitions to the higher level of a physical law. It also explains that this transition cannot be done automatically, it requires the participation of human experts.

We provide several examples from science illustrating

these results. As an interesting side effect, we have a formalization of physicists belief that beauty of a theory indeed makes it more convincing. We also present a crisp analog of this result which uses a crisp notion of beauty formalized by the well-known mathematician G. Birkhoff in 1920s.

## 2 Fuzzy logic as a means to represent different levels of certainty

### In science, there are different levels of certainty.

In different sciences, there are different levels of certainty: e.g., a conclusion acceptable to a geologist will not be sufficient to convince a physicist (because physical laws are usually based on larger amounts of data and are, thus, more reliable).

### Fuzzy logic is a natural way of representing these levels of certainty.

A natural way to represent these different levels of uncertainty is to use different numbers from the interval  $[0,1]$ , i.e., to use fuzzy logic. In other words:

- To characterize our degree of certainty  $d(A)$  in a statement  $A$ , we will use a real number from the interval  $[0, 1]$ , so that:
  - $d(A) = 1$  means that we are absolutely certain in  $A$ ;
  - $d(A) = 0$  means that  $A$  is absolutely false; and
  - values  $d(A) \in (0, 1)$  describe intermediate degrees of certainty, so that the more certain we are in  $A$ , the large the degree of certainty.
- Similarly, we can characterize a *level of certainty* by the smallest value  $\alpha$  of the degree of certainty  $d(A)$  which makes a statement acceptable on this level.

Due to this definition of the degree of certainty of a certainty level, whether a statement  $A$  is considered acceptable on this level depends on the relation between the degree of certainty  $d(A)$  in this statement and the degree  $\alpha$  of the level:

- if  $d(A) > \alpha$ , then the statement  $A$  is *accepted* as valid on this level;
- if  $d(A) < \alpha$ , then the statement  $A$  is *not accepted* as valid on this level.

The simplest possible operations of fuzzy logic are min and max. With these operations, we can conclude that if two statements  $A$  and  $B$  are accepted on a certain level, then their Boolean combinations  $A \& B$  and  $A \vee B$  are also accepted as true on this level. In particular, this is true if  $A$  has the form  $p \rightarrow q$  and  $B$  is of the form  $p$ ; then, from  $p \rightarrow q$  and  $p$ , we can conclude that  $q$  belongs to the same certainty level. The transition from  $p \rightarrow q$  and  $p$  to  $q$  is called a *deduction*; therefore, we can say that on each level, we have a set of statements which is closed under deduction, i.e., which forms a *deductive system*.

### 3 Lifting: definitions and the main results

**Lifting: formulation of the problem.** Suppose that we have two different levels of certainty:

- a *lower level*, which is characterized by a smaller number  $\alpha$ , and
- a *higher level*, which is characterized by a higher degree of certainty  $\alpha' > \alpha$ .

If we have a statement which is accepted on a lower level, then a natural question is: *Is the given statement true on the higher level too?*

For example, if we have some heuristic arguments made by physicists, it is natural to ask whether these arguments can be “uplifted” to a precise mathematical result, etc.

If the answer to this question is “yes”, then the next natural question is: *How can we actually “uplift” this statement to the next level?*

In this paper, we will show that fuzzy logic helps us to formulate and analyze these problems.

**Towards formalizing the above questions.** A typical elementary piece of knowledge states that certain objects  $\mathbf{x}_1, \dots, \mathbf{x}_n$  satisfy a certain property  $\mathbf{P}$ ; in logical notations, the fact that the objects  $\mathbf{x}_i$  satisfy the property  $\mathbf{P}$  is usually written as  $\mathbf{P}(\mathbf{x}_1, \dots, \mathbf{x}_n)$ .

Since we are talking about a statement at a certain (low) level of certainty, in general, both the property  $\mathbf{P}$  and the objects  $\mathbf{x}_i$  are *fuzzy*.

When a physicist says that some effect is negligible, he does not usually give the exact quantitative definition of this negligibility; it is a rather fuzzy notion. Similarly, when a biologist talks about different species, she does not have a mathematically precise definition of a specie in mind; the notion of a specie is, also, to some extent fuzzy.

Therefore, we can use fuzzy set theory (see, e.g., [14, 23]) to describe fuzzy objects and properties:

- The notion of a fuzzy object is very straightforward to define. Let  $U$  be a universe of discourse; then, a *fuzzy object*  $\mathbf{x}_i$  is a fuzzy subset of  $U$ , i.e., a function  $\mu_i : U \rightarrow [0, 1]$  (called *membership function*).
- To define a *fuzzy property*, we must first describe the notion of a *crisp property*.
  - In 2-valued logic, properties on  $U^n$  are in 1-1-correspondence with subsets of  $U^n$  (every property is in correspondence with the set of all elements  $(u_1, \dots, u_n) \in U^n$  which satisfy this property). Thus, the Universe of discourse, i.e., the set of all crisp properties, is the set  $2^{U^n}$  of all subsets of the set  $U^n$ .
  - Now, we can define a *fuzzy property*  $\mathbf{P}$  as a fuzzy subset of the set  $2^{U^n}$ , i.e., as a membership function  $\mu_{\mathbf{P}} : 2^{U^n} \rightarrow [0, 1]$ .

So, we arrive at the following definition:

**Definition 1.** Let  $U$  be a set; it will be called a *universe of discourse*.

- By a *fuzzy object*  $\mathbf{x}_i$ , we mean a fuzzy subset of  $U$ , i.e., a membership function  $\mu_i : U \rightarrow [0, 1]$ ;
- Let  $n$  be a non-negative integer. By an *n-ary fuzzy property*  $\mathbf{P}$ , we mean a fuzzy subset of the power-set  $2^{U^n}$ , i.e., a membership function  $\mu_{\mathbf{P}} : 2^{U^n} \rightarrow [0, 1]$ .
- By a *it fuzzy statement*, we mean a tuple  $S = \langle n, \mathbf{P}, \mathbf{x}_1, \dots, \mathbf{x}_n \rangle$ , where  $n$  is a non-negative integer,  $\mathbf{P}$  is a *n-ary fuzzy property*, and  $\mathbf{x}_1, \dots, \mathbf{x}_n$  are *fuzzy objects*. This tuple will also be denoted as  $\mathbf{P}(\mathbf{x}_1, \dots, \mathbf{x}_n)$ .

Our goal is to describe whether the property  $\mathbf{P}$  holds for the objects  $\mathbf{x}_i$ , and if yes, to what extent. If the property and the objects were *crisp*, then we would either conclude that the property holds, or that it does not. Since both the property and the objects are *fuzzy*, the resulting statement is also fuzzy, so we can only talk about the *degree* with which the fuzzy objects  $\mathbf{x}_i$  satisfy the fuzzy property  $\mathbf{P}$ . To define this degree, we can use the general fuzzy idea of *extension principle*. This idea, in application to our specific problem, can be described as follows:

- The fact that  $\mathbf{P}$  is a fuzzy property means that we do not know exactly what property this is; it is possible that the notion expressed by the fuzzy property  $\mathbf{P}$  is described by different crisp properties  $P$  – each crisp property  $P$  is possible with the degree of possibility  $\mu_{\mathbf{P}}(P)$ .
- Similarly, for every  $i$  from 1 to  $n$ , the fact that  $\mathbf{x}_i$  is a fuzzy object means that we do not know exactly what object this is; it is possible that the object expressed by the fuzzy set  $\mathbf{x}_i$  is described by different crisp objects  $x_i$  – each crisp object  $x_i$  is possible with the degree of possibility  $\mu_i(x_i)$ .

What is the degree of possibility of  $\mathbf{P}(\mathbf{x}_1, \dots, \mathbf{x}_n)$ ? We have already mentioned that since we only specified *fuzzy* information about the property  $\mathbf{P}$  and the objects  $\mathbf{x}_i$ , there are many possibilities for the actual crisp property and objects, so that we have either the first one, or the second one, etc. In other words:

$$\mathbf{P}(\mathbf{x}_1, \dots, \mathbf{x}_n) \leftrightarrow \bigvee_{P, x_1, \dots, x_n} [(P \text{ is } P) \& (\mathbf{x}_1 \text{ is } x_1) \& \dots \& (\mathbf{x}_n \text{ is } x_n) \& P(x_1, \dots, x_n)].$$

We can determine the degree of belief in the right-hand side if we find the degrees of belief in the elementary statements which comprise this right-hand side, and then use the above-mentioned fuzzy logic operators  $\& = \min$  and  $\vee = \max$ . For the elementary statements, the degrees of belief are easy to get:

- the degree of belief in the statement “ $\mathbf{P}$  is  $P$ ” is equal to  $\mu_{\mathbf{P}}(P)$ ;
- the degree of belief in a statement “ $\mathbf{x}_i$  is  $x_i$ ” is equal to  $\mu_i(x_i)$ ;
- finally, the degree of belief in a statement  $P(x_1, \dots, x_n)$  is 1 or 0 depending on whether this statement is true or false.

Thus, we get the following formula for the desired degree of belief:

$$d(\mathbf{P}(\mathbf{x}_1, \dots, \mathbf{x}_n)) = \sup_{P, x_1, \dots, x_n} [\min(\mu_{\mathbf{P}}(P), \mu_1(x_1), \dots, \mu_n(x_n), P(x_1, \dots, x_n))].$$

This expression can be further simplified if we take into consideration that:

- for  $P(x_1, \dots, x_n) = 0$ ,  $\min = 0$ , and thus, the corresponding term does not affect the desired sup;
- for  $P(x_1, \dots, x_n) = 1$ ,  $\min(\dots, 1)$  is equal to the minimum of all the terms without this 1, so this term can be simply eliminated.

As a result, we get the following formula for the desired degree of certainty:

$$d(\mathbf{P}(\mathbf{x}_1, \dots, \mathbf{x}_n)) = \sup_{P, x_1, \dots, x_n: P(x_1, \dots, x_n)} (\min(\mu_{\mathbf{P}}(P), \mu_1(x_1), \dots, \mu_n(x_n))). \quad (1)$$

**Definition 2.** Let  $S = \langle n, \mathbf{P}, \mathbf{x}_1, \dots, \mathbf{x}_n \rangle$  be a fuzzy statement. By its degree of certainty  $d(S)$ , we mean the expression (1).

This expression can be described as a particular case of the general extension principle, if we treat  $P(x_1, \dots, x_n)$  as a function which maps a tuple  $\langle P, x_1, \dots, x_n \rangle$  into the set of binary truth values  $\{0, 1\}$ , and correspondingly describe  $\mathbf{P}(\mathbf{x}_1, \dots, \mathbf{x}_n)$  as the value of this function.

**First result: lifting is possible.** Let us first show that “lifting” is always possible, i.e., if a statement  $\mathbf{P}(\mathbf{x}_1, \dots, \mathbf{x}_n)$  is acceptable for a certain level  $\alpha$ , then it can be “uplifted” to a statement on a higher level of certainty.

**Definition 3.** Let  $U$  be a universe of discourse, and let  $\alpha \in (0, 1)$  be a real number.

- We say that a fuzzy statement  $S$  is acceptable at the level  $\alpha$  if  $d(S) > \alpha$ .
- We say that a crisp property  $P$  is a clarification of the fuzzy property  $\mathbf{P}$  (defined by a membership function  $\mu_{\mathbf{P}}$ ) if  $\mu_{\mathbf{P}}(P) > \alpha$ .
- We say that a crisp object  $x_i \in U$  is a clarification of the fuzzy object  $\mathbf{x}_i$  (defined by a membership function  $\mu_i(x)$ ) if  $\mu_i(x_i) > \alpha$ .

**Proposition 1.** For every level  $\alpha$ , if a fuzzy statement  $\mathbf{P}(\mathbf{x}_1, \dots, \mathbf{x}_n)$  is acceptable at this level, then there exists a clarification  $P$  of the fuzzy property  $\mathbf{P}$  and clarifications  $x_i$  of the fuzzy objects  $\mathbf{x}_i$  for which  $P(x_1, \dots, x_n)$  is true.

Thus, every fuzzy statement can be uplifted all the way to the crisp level ( $\alpha = 1$ ). (For reader’s convenience, all the proofs are placed in the last section.)

**Second result: lifting is difficult.** We will show that although lifting is always *possible*, no feasible universal algorithm can do it. We will show that this is true even in the simplest case, when:

- each of the variables  $x_i$  can only take two possible values 0 and 1 (i.e.,  $U = \{0, 1\}$ ); in this case, a fuzzy object, i.e., a membership function  $\mu_i : U \rightarrow [0, 1]$ , can be described by two numbers  $\mu_i(0), \mu_i(1) \in [0, 1]$ ; and
- we only consider crisp propositional properties  $P$ .

**Definition 4.** By a *propositional fuzzy variable*, we mean a pair of rational numbers  $(\mu_i(0), \mu_i(1))$  from the interval  $[0, 1]$ .

**Definition 5.** By a *simplified lifting problem*, we mean the following problem: Given:

- a rational number  $\alpha \in (0, 1)$ ,
- a non-negative integer  $n$ ,
- a propositional formula  $P$  with  $n$  variables, and
- $n$  propositional fuzzy variables  $\mathbf{x}_1, \dots, \mathbf{x}_n$  for which the fuzzy statement  $P(\mathbf{x}_1, \dots, \mathbf{x}_n)$  is acceptable at the given level  $\alpha$ ,

find the clarifications  $x_i$  of the fuzzy objects  $\mathbf{x}_i$  for which  $P(x_1, \dots, x_n)$  is true.

**Proposition 2.** The simplified lifting problem is NP-hard.

Crudely speaking, this means that no feasible algorithm can solve all particular instances of this problem (for detailed definitions, see, e.g., [12, 17]).

*Comment.* The use of min and max is, of course, only a first approximation to the actual more adequate and-or-operations: a sufficient number of repeated experiments (or other confirmations) can make a statement more convincing and even raise it to the next level of confidence.

#### 4 Applications of the lifting results

**Lifting is possible.** Our results explain why many scientific results are eventually upgraded from the lower level (e.g., of guesses and intuitions) to the higher level (e.g., of a physical law):

- ideas which were initially formulated on an intuitive or philosophical level become part of quantitative working science;
- heuristic methods developed in physics and biology eventually turn into precise mathematical ideas; this happened with Dirac's  $\delta$ -function in mathematical physics, etc.
- finally, mathematical theories which are first formulated in a non-constructive, non-algorithmic form, as complicated equations, are eventually *solved*, i.e., formulated in terms of algorithmic, constructive relations.

**Lifting is non-trivial.** Our results also explains that this transition cannot be done automatically, it requires the participation of human experts. For example:

- in spite of all the attempts of positivism and other science-oriented directions in philosophy (e.g., in philosophy of physics), and in spite of numerous successes of "uplifting" philosophical physical

ideas to working physical theories, philosophy of physics, has not been replaced by arguments on the physicist's level of certainty;

- in spite of all the attempts to find a formal system which would make all physicists reasoning formal and thus accurate, it has not been done, and the prevailing feeling is that there is no automatic way to formalize *all* physicist (and, more general, expert) reasoning;
- in spite of all the successes in finding algorithmic solutions to mathematical problems, there are still many problems for which no general algorithm is possible (see, e.g., [17]).

**The understanding that difference parts of knowledge have different degrees of certainty can improve multi-disciplinary research.** Scientists working in a certain area are accustomed to a certain level of certainty as acceptable. As a result:

- When they encounter results from a different discipline, in which a *lower* level of certainty is considered acceptable, they simply consider these results unacceptable and not scientific at all. Examples:
  - For physical problems which are formulated in terms of mathematical equations, physicists often have to invent heuristic methods of solving these equations, methods which do not have a precise mathematical justification. Mathematicians often tend to ignore these ideas and even make fun of physicists' statements.
  - Similarly, physicists, who are accustomed to a high level of what is considered to be an acceptable conclusion in their experimental work, sometimes ignore biological experimental results (e.g., on the relation between electromagnetic field and health) as not scientifically acceptable, not realizing that these rejected results are actually at the same level of confidence as many other biological experimental results which are routinely accepted by biologists.
  - People from natural sciences often consider social sciences and humanities not truly scientific, and all of them sometimes treat philosophy as mostly useless speculations.

Since, as we have shown, the knowledge at the lower level can be uplifted to a knowledge at the higher level of certainty, it is more productive not just to ignore this lower certainty knowledge as unacceptable, but rather to use it as a heuristic for the higher certainty level arguments. Later in this section, we will show that not only the lower certainty level knowledge can be uplifted to a higher level, but also that the existence of an

additional lower level confirmation can *increase* our degree of confidence in the higher-level statement.

- Similarly, when scientists encounter results from a different discipline, in which a *higher* level of certainty is considered acceptable, they do not understand the necessity of the extra work needed for achieving this higher level. Examples:
  - Physicists often do not see any reason in formalization of physics.
  - Philosophers often do not understand the necessity to, say, confirm, on a physical level, their conclusions (about space-time, or about the progress in science).

As a result of this misunderstanding, unnecessary conflict situations occur when researchers working in an area with lower acceptability threshold try to use their conclusions to falsify theories obtained on a higher level of certainty:

- Biologists sometimes try to use their experimental results to disprove physical theories, e.g., to prove that permutation of chemical elements occurs in a living cell, or that a new physical field is responsible for brain processes, etc. Physicists tend to simply dismiss these claims as “bad physics” and “bad science”, but these arguments do not sound convincing to biologists who counter-argue that their experiments are often on the same level of statistical confirmation as many other biological experiments which are routinely accepted.
- Some humanitarians try to dispute the validity of physical theories based on general sociological arguments, etc.

Multi-disciplinary collaboration definitely benefits when the collaborators replace their original “black-and-white” understanding of acceptability in science with a more accurate understanding which allows for a continuous scale of degrees of certainty.

**It helps when we have several pieces of knowledge, in particular, when we have consistent pieces of knowledge on several levels of certainty.** Suppose that we know a statement  $A$  at a certain level of certainty  $\alpha$ . It happens sometimes that, in addition to this knowledge, we also have some weaker arguments in favor of this same statement  $A$ , arguments which are at a much lower level.

For example, in addition to physical experiments confirming a certain theory, we may have intuitive or philosophical arguments in favor of this same theory.

The fact that these arguments are at a lower level means, in our terms, that they correspond to degree

of certainty  $\alpha' < \alpha$ . How does the existence of such arguments affect our degree of degree of belief in a statement  $A$ ?

This question can be reformulated in terms of fuzzy logic. Namely, we have two arguments in favor of  $A$ :

- an argument  $A_\alpha$  whose degree of certainty is

$$d(A_\alpha) \approx \alpha,$$

and

- an argument  $A_{\alpha'}$  of degree of certainty

$$d(A_{\alpha'}) \approx \alpha' < \alpha.$$

The statement  $A$  is true if at least one of these arguments is correct; in other words, the degree of belief in a statement  $A$  is equal to the degree of belief in a disjunction  $A_\alpha \vee A_{\alpha'}$ . Thus, the resulting degree of belief  $d(A)$  is equal to  $d(A) = f_\vee(d(A_\alpha), d(A_{\alpha'}))$ , where  $f_\vee$  is an or-operation (t-conorm).

- If we use  $f_\vee(a, b) = \max(a, b)$ , then we conclude that  $d(A) = d(A_\alpha)$ , i.e., that the existence of the additional confirmation on a *lower* level does not change our degree of confidence in the theory. For  $f_\vee(a, b) = \min(a, b)$ , we will also have to conclude that additional pieces of evidence on the *same* level do not increase our degree of certainty. While this conclusion is true on the level of mathematical certainty (where a second proof does not change the validity of a statement), it is not true on lower levels, like biology or physics, where additional confirmation definitely increases our degree of certainty.
- The above analysis shows that, although  $f_\vee(a, b) = \max(a, b)$  is a good first approximation to expert reasoning in different sciences, a more realistic description requires using *different* t-conorms, for which, as is well known,  $f_\vee(a, b) > \max(a, b)$ .

For such t-conorms, we have

$$d(A) = f_\vee(d(A_\alpha), d(A_{\alpha'})) > d(A_\alpha),$$

i.e., the existence of the additional confirmation on a *lower* level does increase our degree of confidence in the theory. This explains why, e.g., the fact that the theory is consistent with our general philosophical beliefs, or that the theory is simply *beautiful*, increases our degree of belief in this theory (see, e.g., [20, 21] and references therein).

In other words, we have a reasonable argument in favor of the statement that “beauty is a sign of truth in scientific theories”. In the following section, we show that we can increase our degree of certainty in this statement by providing additional (crisp) arguments in its favor.

## 5 Towards the use of aesthetics in decision making: Kolmogorov complexity formalizes Birkhoff's idea

### Traditional formalized engineering decision making does not take beauty into consideration.

Decision making is traditionally based on utilitarian criteria such as cost, efficiency, time, etc. These criteria are reasonably easy to formalize; hence, for such criteria, we can select the best decision by solving the corresponding well-defined optimization problem.

In many engineering projects, however, e.g., in designing cars, building, airplanes, etc., an important additional criterion which needs to be satisfied is that the designed object should be beautiful (or at least good looking). This additional criterion is difficult to formalize and, because of that, it is rarely taken into consideration in formal decision making.

**How can we formalize beauty? Birkhoff's formula.** In the 1930s, G. D. Birkhoff, one of the world leading mathematicians, has proposed a formula that described beauty in terms of "order"  $O$  and "complexity"  $C$  [2, 3, 4, 5, 6] (see also [8]). Namely, according to his formula, the beauty  $B$  of an object is equal to  $B = O/C$ . How can we describe "order" and "complexity"?

In the simplest cases, Birkhoff formalized these notions and showed that his formula is indeed working. Namely, he showed that the beauty of simple geometric patterns, of simple melodies, and even of simple verses can be well described by his formula. However, since there was no general notion of complexity, he was unable to formalize his idea in the general case. This is what we are planning to do in this paper.

**Towards formalization of Birkhoff's formula.** In our formalization, we will use the general computer-based notion of object complexity, which is widely used in computer science. For example, we can define the *complexity*  $C(x)$  of an object  $x$  as the length  $l(p)$  of the shortest program  $p$  (in a certain language) which generates this object. This notion of object complexity was originally proposed by G. Chaitin, A. Kolmogorov, and R. Solomonoff, and it is usually called *Kolmogorov complexity* (see, e.g., [10, 19]). Alternatively, we can use a *modification* of this notion which takes into consideration not only the *length*  $l(p)$  of the program  $p$  (i.e., the number of bits in its computer description), but also the *time*  $t(p)$  that the program  $p$  takes to generate the desired object  $x$ .

In order to choose an appropriate formalization, let us start with an informal discussion of Birkhoff's ideas.

**Informal motivations: the ideas behind Birkhoff's notions.** In Birkhoff's description, *complexity* of an object looks like time which is necessary to generate this object. For example, he defines the complexity of a polygon as the number of its vertices, etc. Intuitively, it is clear that beauty must be reasonably simple, so, all other characteristics being equal, the more over-complicated the object is, the less beautiful it is.

Similarly, Birkhoff's *order* looks like a simplicity of the description: if we can describe an object by using a shorter text, then its order is higher. If the only way of describing an object is to enumerate all its pixels (all its nodes for a melody, all its vertices for a polygon, etc.), then this object does not have much order in it. Intuitively, it seems reasonable that an object with some order in it should (all else being equal) look prettier than an object with less order. How can we formalize this notion of "order"?

By a *description*, we mean a *complete* description, i.e., a description which is detailed enough so that, given this description, we can uniquely reconstruct the object. In other words, the description must serve as a *program* for a computational device which, given this description, reconstructs the object. In these terms, the length of the description is the length  $l(p)$  of this program  $p$ . So, the smaller  $l(p)$ , the more order is there in the object.

Summarizing our discussion of complexity and order, we can conclude that the beauty  $B(x)$  of an object  $x$  depends on the *time*  $t(p)$  of the program  $p$  which generates  $x$ , and on the *length*  $l(p)$  of this program, i.e., that  $B(x) = f(t(p), l(p))$  for some function  $f(t, l)$ . The only thing we know about the function  $f(t, l)$  is that it should monotonically decrease with the increase of each of the variables  $t$  and  $l$ .

In these terms, the question of formalizing beauty can be reformulated in more mathematically-sounding terms: Which function  $f(t, l)$  should we choose?

**Which function  $f(t, l)$  should we choose?** It is well known in computer science that there is a *trade-off* between the program time and the program length. A short program usually uses only a few ideas of speeding up computations, and thus, takes a reasonable amount of time to run. If we want to speed up the computations, we must add some complicated ideas and modify the algorithm. As a result, to make the program faster, we must usually make it longer. Vice versa, we can often shorten the program by eliminating some of the time-saving parts and thus, by making its running time longer.

This trade-off is not only true for programs written in the same programming language, the same trade-off is true if we compare programs written on programming

languages of different level. For example, we can write a program in machine code (or in assembler language, which is close to the machine code).

- In a machine-code program, we have to spell out all necessary steps, so this program will be reasonably long. On the other hand, in a machine code program, every instruction will be immediately implemented, so running this program does not take too long.
- Alternatively, we can write our program in a high level programming language (e.g., in C++). In this case, the program is usually shorter, because we do not need to spell out all the details, it is sufficient to describe the construction that we want to use (like a loop or calling a function). However, when we run this short program, we first need to translate it into the machine code (i.e., *compile* it), and this compiling takes extra time.

Thus, we can get a shorter program which runs longer, or we can have a longer program which runs faster.

Our definition of the formalized beauty depended on the program  $p$ . It is reasonable to require that the “beauty” of an object  $x$  should not depend on which level we write this program  $p$ . Let us formalize this requirement.

By going to a different level of programming, we can cut a lot of bits from the length of the program. Let us describe this cut step-by-step and analyze what happens if we cut exactly one bit.

We are interested not in the abstract notion of beauty, but in the much more specific notion of the beauty of the engineering designs. When we talk about an engineering design, we mean that we have specifications which are usually relatively easy to check, and we want to find a design which meets these specifications. Checking is relatively easy, the most difficult part is finding the design.

If we cut a bit from the program that generates the design  $x$ , we get a new program  $p'$  which is exactly one bit shorter ( $l(p') = l(p) - 1$ ). To generate the desired design  $x$ , since we do not know whether the deleted bit was 0 or 1, we can try both possible values of this bit (i.e., run two programs  $p'0$  and  $p'1$ ) and find out which of the two designs meets the original specifications. Thus, if we delete a bit, then instead of running the original program  $p$  once, we run *two* programs  $p'0$  and  $p'1$ . Hence, crudely speaking, when we decrease the length of the program by 1, we thus get a double increase in the running time:  $t(p') = 2t(p)$ .

From this viewpoint, the fact that the beauty should not depend on the level means, in particular, that the values of  $B(x)$  computed as  $f(t(p), l(p))$  should stay the same if we replace the original program  $p$  by a

one-bit-shorter program  $p'$ . In other words, we should have  $f(t(p'), l(p')) = f(t(p), l(p))$ . Since we know that  $l(p') = l(p) - 1$  and  $t(p') = 2t(p)$ , we thus conclude that  $f(2t(p), l(p) - 1) = f(t(p), l(p))$  for every program  $p$ . In other words, the desired function  $f(t, l)$  must satisfy, for every two integers  $t$  and  $l$ , the following equation:

$$f(2t, l - 1) = f(t, l). \quad (2)$$

Functions which satisfy this equation can be explicitly described:

**Definition 6.** We say that a function  $f(t, l)$  is *invariant* if it satisfies the equation (2) for all positive integers  $t$  and  $l$ .

**Proposition 3.** A function  $f(t, l)$  is invariant if and only if  $f(t, l) = F(t \cdot 2^l)$  for some function  $F(z)$  of one variable.

**Our result justifies Birkhoff’s formula.** Let us show that this result justifies Birkhoff’s formula. Namely, we will show that the search for the “most beautiful” design is equivalent to looking for a design for which the ratio  $O/C$  the largest possible value for appropriately defined quantities  $O$  and  $C$ .

Indeed, since we assumed that the function  $f(t, l)$  is monotonically decreasing in both variables, we can conclude that the function  $F(z)$  is monotonically decreasing too. So, looking for the “most beautiful” design means looking for the design generated by a program  $p$  for which the product  $t(p) \cdot 2^{l(p)}$  takes the smallest possible value, or, equivalently, for which the inverse value  $2^{-l(p)}/t(p)$  takes the largest possible value. We have already mentioned that the running time  $t(p)$  is a natural formalization of Birkhoff’s complexity  $C$ , and that Birkhoff’s “order”  $O$  is a monotonically decreasing function of the program length  $l(p)$ . Thus, *looking for the most beautiful design means looking for a design for which the ratio  $O/C$  takes the largest possible value, where  $C = t(p)$  and  $O = 2^{-l(p)}$* . So, we indeed get a justification for Birkhoff’s formula.

**Our result makes perfect sense from the pragmatic viewpoint.** In the above formalization of the notion of the “most beautiful” design, we were using two things at the same time: the design  $x$  and the program  $p$  which generates this design. Let us separate the design from the program. Namely, for each possible design  $x$ , we can define its “beauty”  $a(x)$  as the smallest possible value of the product  $t(p) \cdot 2^{l(p)}$  for all possible programs  $p$  which generate this design. Then, finding the most beautiful design means finding the design  $x$  which satisfies all the requirements and for which thus defined quantity  $a(x)$  takes the smallest possible value.

This notion  $a(x)$  is known in the theory of Kolmogorov complexity: namely, it was introduced by Leonid A. Levin in [18] as one of the possible modifications of

Kolmogorov complexity which takes into consideration not only the length  $l(p)$  of the program, but its running time  $t(p)$  as well. Levin has proven that if we are looking for an optimal (asymptotically fastest) universal algorithm for solving different search problems (like the problems of design; see, e.g., [12]), then this optimal algorithm should check all possible designs in the increasing order of their Levin's complexity  $a(x)$  (see [1, 18, 19]).

Thus, our formalization of beauty makes perfect pragmatic sense: if we want to find the best design as fast as possible, we must first look among the prettiest designs (i.e., among the designs with the smallest possible value of  $a(x)$ ), then among the next prettiest designs, etc.

**This theoretical idea is not yet a practically working tool.** *Theoretically*, Birkhoff's idea seems to work well. However, *in practice*, there is a big obstacle to applying this idea, because Kolmogorov complexity is not algorithmically computable (see, e.g., [10, 19]). (Levin's modification of Kolmogorov complexity is actually computable but computing it requires too long time, so for all practical purposes, it is not computable at all.) What can we do to make this criterion practically useful?

**How to transform this theoretical idea into a practically working tool? Idea: let's use wavelets.** An approach towards making this notion practical is to take into consideration the fact that the Kolmogorov complexity is not computable because it is based on considering *all* possible algorithms. If we limit the class of algorithms, we get a computable version of Kolmogorov complexity. This idea was used, e.g., in [13], where a similarly modified version of Kolmogorov complexity was used to successfully predict the time required for a human to remember a geometric pattern. How can we come up with reasonable computable analogues of complexity and order (symmetry)?

Complexity of a computer object (string, image, etc.) can be measured by the ability of compressing programs to compress them. Thus, to get a computable estimate for complexity, we can use an advanced compression algorithm (e.g., an algorithm that underlies the widely used zip compression), and measure the complexity by the length of the compressed object: if the compressed text is short, the object was easy; if the compressed text still takes many bits, the compressed object was complex.

To measure the *order* (= *symmetry*, see, e.g., [25]) of an object, we can, similarly, use compression procedures, but this time, only procedures which use symmetry to compress. The most widely known symmetry-motivated compression techniques is the *wavelet* compression (see, e.g., [22, 24]). In view of this, we use the

length of the wavelet compression as an indication of the order: an image with a short wavelet compression has high order, while an image whose wavelet compression has approximately the same length as the original image has low order.

Our preliminary experiments show that these definitions indeed lead to a reasonable characterization of beauty.

**How to transform this theoretical idea into a practically working tool? Second idea: let's use fuzzy logic.** The *second* approach towards making the notion of beauty practically useful is to take into consideration that we humans have a good intuitive understanding of beauty, and thus, even when we cannot give a precise description of what exactly is beautiful and what is not, we can give a reasonable good description of beauty in terms of words from the natural language.

Thus, in order to formalize the notion of beauty, it is reasonable to use a methodology which successfully formalizes such statements – namely, the methodology of fuzzy logic (see, e.g., [14, 23]). Our preliminary analysis shows that we can indeed get a good description of beauty in this manner. Namely, we have shown that simple rules from natural language explain why golden proportion looks aesthetically pleasing [15, 16].

## 6 Proofs

**Proof of Proposition 1.** We assumed that the given property  $S = \mathbf{P}(\mathbf{x}_1, \dots, \mathbf{x}_n)$  is acceptable at the level  $\alpha$ . By definition of acceptability, this means that  $d(S) > \alpha$ . Thus, by definition (1) of the expression  $d(S)$ , means that  $\alpha$  is smaller than the supremum (least upper bound)  $d(S)$  of all the values

$$\min(\mu_{\mathbf{P}}(P), \mu_1(x_1), \dots, \mu_n(x_n))$$

for all tuples  $(P, x_1, \dots, x_n)$  for which  $P(x_1, \dots, x_n)$  is true. The fact that the least upper bound of an expression is larger than  $\alpha$  means that there exist values  $P, x_1, \dots, x_n$  for which  $P(x_1, \dots, x_n)$  is true and for which this expression is  $> \alpha$ , i.e., for which

$$\min(\mu_{\mathbf{P}}(P), \mu_1(x_1), \dots, \mu_n(x_n)) > \alpha.$$

The smallest of several numbers is  $> \alpha$  if and only if all of these numbers is  $> \alpha$ . This means that  $\mu_{\mathbf{P}}(P) > \alpha$ ,  $\mu_1(x_1) > \alpha, \dots, \mu_n(x_n) > \alpha$ , i.e., this means that:

- the property  $P$  clarifies the fuzzy property  $\mathbf{P}$ , and
- each of the objects  $x_i$  clarifies the corresponding fuzzy object  $\mathbf{x}_i$ .

The proposition is proven.



**Proof of Proposition 2.** In this proof, we will use notations, definitions, and results from [11]. In particular, in [11], it is shown that a problem of finding, for every satisfiable propositional formula  $P(x_1, \dots, x_n)$ , a satisfying vector, is NP-hard. Let us show that this problem can be reduced to the simplified lifting problem, and thus, that the simplified lifting problem is also NP-hard.

Indeed, for every such formula, we can take for each  $i$ , as  $\mathbf{x}_i$ , a propositional fuzzy variable for which  $\mu_i(0) = \mu_i(1) = 1$ . Since the formula is satisfiable, the resulting fuzzy statement  $P(\mathbf{x}_1, \dots, \mathbf{x}_n)$  has a degree of certainty  $1 > \alpha$ , but to find clarifications we would actually need to find the Boolean vector  $x_i$  which satisfies the formula  $P$ . Reduction is proven, and so is the proposition.

**Proof of Proposition 3.** From the equation (2), we conclude that  $f(t, l) = f(2t, l - 1)$ . Applying the same equation (2) to the right-hand side of the new equality, we conclude that  $f(2t, l - 1) = f(2^2t, l - 2)$ , and thus, that  $f(t, l) = f(2^2t, l - 2)$ . Similarly, we can prove that

$$f(t, l) = f(2^2t, l - 2) = f(2^3t, l - 3) = \dots = f(2^kt, l - k)$$

for an arbitrary  $k$ . In particular, for  $k = l$ , we conclude that  $f(k, l) = f(2^lt, 0)$ . Thus, Proposition 3 is true, for the function  $F(z) = f(z, 0)$ .

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