How to Describe Partially Ordered Preferences: Mathematical Foundations

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1 Introduction

Traditional decision theory (see, e.g., [1, 2]) is based on the assumption that a person whose preferences we want to describe can always (linearly) order his preferences, i.e., that for every two alternatives a and a', he can decide:

- whether a is better than a' (we will denote it by $a' \prec a$);
- or whether a' is better than a ($a \prec a'$);
- or whether a and a' are (for this person) of the same quality (we will denote it by $a \sim a'$).

A similar assumption (often implicit) underlies the traditional description of degrees of belief ("subjective probabilities") by numbers from the interval [0,1].

In reality, a person is often unsure about which of the alternatives is the best. In mathematical terms, the person's preferences form only a *partial* order. There exist several heuristic formalisms for describing such partial orders, but due to their heuristic character, we cannot guarantee that these formalisms are *general* enough to describe the preferences of all reasonable decision makers, and at the same time, *narrow* enough not to include partial ordering relations that do not satisfy natural reasonable conditions.

In this paper, we:

- modify the reasonability conditions normally considered in decision theory so that these conditions allow partial ordering; and
- develop a formalism that describes all the partial ordered preferences and degrees of belief that satisfy these conditions.

These results were partially published in [5].

2 Traditional utility theory: a brief reminder

In this section, we will mainly follow standard definitions (see, e.g., [1, 2]), but we will not always follow them exactly: in some cases, we will slightly rephrase these definitions (without changing their mathematical contents) so as to make the following transition to partially ordered preferences as clear as possible.

Definition 1. *Let $\mathcal{A}$ be a set; this set will be called the set of alternatives (or the set of pure alternatives). By a lottery on $\mathcal{A}$ we understand a a probability measure on $\mathcal{A}$ with finite support.*

In other words, a lottery is a pair $\langle A, p\rangle$, where $A = \{a_1, \ldots, a_n\} \subseteq \mathcal{A}$ is a finite subset of $\mathcal{A}$, and p is a mapping $p : A \to [0, 1]$ for which $p(a_i) \ge 0$ and $\sum p(a_i) = 1$. A lottery will also denoted as

$$p(a_1) \cdot a_1 + \ldots + p(a_n) \cdot a_n.$$

We do not consider lotteries with infinite numbers of alternatives, because every real-life randomizing device, be it a dice or a computer-based random number generator, produces only finitely many possibilities.

The set of lotteries will be denoted by L. On this set L, we can naturally define an operation of *probability combination*as a convex combination of the corresponding probability measures: namely, if we have m values $q_1, \ldots, q_m \in [0, 1]$ with $\sum q_j = 1$, and m lotteries $\ell_j = \langle A_j, p_j\rangle$, then we can define the probability combination $\ell = q_1 \cdot \ell_1 + \ldots + q_m \cdot \ell_m$ as a lottery $\ell = \langle A, p\rangle$ with $A = \cup A_j$ and $p(a) = \sum q_j \cdot p_j(a)$, where the sum is taken over all j for which $a \in A_j$.

Definition 2. *Let $\mathcal{A}$ be a set, and let L be the set of all lotteries over $\mathcal{A}$. By a preference relation, we mean a pair $\langle \prec, \sim \rangle$, where $\prec$ is a (strict) order on L, $\sim$ is an equivalence relation on L, and for every $\ell, \ell', \ell'' \in L$ and every $p \in (0, 1)$, the following conditions hold:*

1) *if $\ell \sim \ell'$ and $\ell' \prec \ell''$, then $\ell \prec \ell''$;*
2) *if $\ell \prec \ell'$ and $\ell' \sim \ell''$, then $\ell \prec \ell''$;*
3) *if $\ell \prec \ell'$, then $p \cdot \ell + (1-p) \cdot \ell'' \prec p \cdot \ell' + (1-p) \cdot \ell''$;*
4) *if $p \cdot \ell + (1-p) \cdot \ell'' \prec p \cdot \ell' + (1-p) \cdot \ell''$, then $\ell \prec \ell'$;*
5) *if $\ell \sim \ell'$, then $p \cdot \ell + (1-p) \cdot \ell'' \sim p \cdot \ell' + (1-p) \cdot \ell''$;*
6) *if $p \cdot \ell + (1-p) \cdot \ell'' \sim p \cdot \ell' + (1-p) \cdot \ell''$, then $\ell \sim \ell'$.*

Definition 3. *A preference relation is called linearly ordered (or linear, for short) if for every $\ell, \ell' \in L$, either $\ell \preceq \ell'$, or $\ell' \preceq \ell$ (where $\ell \preceq \ell'$ means that either $\ell \prec \ell'$ or $\ell \sim \ell'$).*

It is known that linearly ordered preference relations can be characterized in terms of special functions called *utility functions*:

Definition 4. *A function u from the set L of all lotteries to an ordered set V is called a utility function. For each $\ell \in L$, $u(\ell)$ will be called a value of the utility function. We say that a utility function u describes the preference relation if for every $\ell, \ell' \in L$, the following two conditions hold:*

- *$\ell \prec \ell'$ if and only if $u(\ell) < u(\ell')$;*
- *$\ell \sim \ell'$ if and only if $u(\ell) = u(\ell')$.*

Definition 5. *A utility function $u : L \to V$ is called convexity-preserving if on the set V, convex combination $p_1 \cdot v_1 + \ldots + p_n \cdot v_n$ is defined for all $p_i \geq 0$, $\sum p_i = 1$, and if for every p_i and ℓ_i, we have $u(p_1 \cdot \ell_1 + \ldots + p_m \cdot \ell_m) = p_1 \cdot u(\ell_1) + \ldots + p_m \cdot u(\ell_m)$.*

To describe linearly ordered preference relations, we use *scalar* utility functions, i.e., convexity-preserving utility functions for which $V = R$. It is known that for every convexity-preserving function $u : L \to R$, the relations $u(\ell) < u(\ell')$ and $u(\ell) = u(\ell')$ define a linearly ordered preference relation. It is also known that this utility function is determined uniquely modulo a linear transformation, i.e.:

- If two different scalar utility functions $u : L \to R$ and $u' : L \to R$ describe the same preference relation, then there exists a linear function $T(z) = k \cdot z + m$, with $k > 0$, such that for every lottery ℓ, $u'(\ell) = T(u(\ell))$.
- Vice versa, if a scalar utility function $u : L \to R$ describes a preference relation, and $k > 0$ and m are real numbers, then the function $u'(\ell) = T(u(\ell))$ (where $T(z) = k \cdot z + m$) is also a scalar utility function which describes the same preference relation.

One can also show that every *Archimedean* (in some reasonable sense) linearly ordered preference relation $\langle \prec, \sim \rangle$ can be described by an appropriate scalar utility function.

In other words, each (Archimedean) *linearly* ordered preference relation can be described by a utility function, and this utility function is determined uniquely modulo a linear transformation. This is not necessarily true for *non-Archimedean* preference relations, e.g., for a lexicographic ordering $(x_1, x_2) > (y_1, y_2)$ iff either $x_1 > y_1$ or ($x_1 = y_1$ and $x_2 > y_2$). It turns out that non-Archimedean linearly ordered preferences can be described by utilities with values in linearly ordered affine spaces (for a general introduction into ordered algebraic structures, see, e.g., [3]):

3 Utilities with values in linearly ordered affine spaces: brief reminder

An *affine space* (see, e.g., [4] and references therein) is "almost" a vector space, the main difference between them is that in the linear space, there is a fixed starting point (0), while in the affine space, there is no fixed point. More formally:

- A *linear space* is defined as a set V with two operations: addition $v + v'$ and multiplication $\lambda \cdot v$ of elements from V by real numbers $\lambda \in R$ (operations which must satisfy some natural properties). With this two basic operations, we can define an arbitrary linear combination $\lambda_1 \cdot v_1 + \ldots + \lambda_n \cdot v_n$ of elements $v_1, \ldots, v_n \in V$.
- In the *affine space*, we can only define those linear combination which are shift-invariant, i.e., linear combinations with $\sum \lambda_i = 1$.

The relation between a linear space and an affine space is rather straightforward:

- if we have an affine space V, then we can pick an arbitrary point $v_0 \in V$, are define a linear space in which this point is 0. Namely, we can define $v + v'$ as $1 \cdot v + 1 \cdot v' - 1 \cdot v_0$: since we took v_0 as 0, this linear combination will be exactly $v + v'$.

- Vice versa, if we have a hyperplane H in a linear space, then (unless this hyperplane goes through 0) this hyperplane is *not* a linear space, but it is *always* an affine space.

Definition 6. *A vector space V with a strict order $<$ is called an ordered vector space if for every $v, v', v'' \in V$, and for every real number $\lambda > 0$ the following two properties are true:*

1) *if $v < v'$, then $v + v'' < v' + v''$;*
2) *if $v < v'$, then $\lambda \cdot v < \lambda \cdot v'$.*

Since this ordering does not change under shift, it, in effect, defines an ordering on the affine space.

Definition 7. *By a vector utility function, we mean a convexity-preserving utility function with values in an ordered affine space V.*

To analyze uniqueness of vector utility functions, we must consider isomorphisms. A mapping T between two affine spaces is called *affine* if it preserves the affine structure, i.e., if $T(\sum \lambda_i \cdot v_i) = \sum \lambda_i \cdot T(v_i)$ whenever $\sum \lambda_i = 1$. For finite-dimensional affine spaces, affine mappings are just linear transformations $(x_1, \ldots, x_n) \to (y_1, \ldots, y_m)$, i.e., transformations in which each resulting coordinate y_i is determined by a linear function $y_i = a_i + \sum b_{ij} \cdot x_j$.

Definition 8. *A one-to-one affine transformation $T : V \to V'$ of two ordered affine spaces is called an isomorphism if for every $v_1, v_2 \in V$, $v < v'$ if and only if $T(v) < T(v')$.*

Recall that for every subset $S \subseteq V$ of an affine space, its *affine hull* $A(S)$ can be defined as the smallest affine subspace containing S, i.e., equivalently, as the set of all affine combinations $\sum \lambda_i \cdot s_i$ ($\sum \lambda_i = 1$) of elements from S.

Theorem. *Let $\mathcal{A}$ be a set, and let L be the set of all lotteries over $\mathcal{A}$.*

- *(consistency) For every convexity-preserving function $u : L \to V$ from L to a linearly ordered affine space V, the relations $u(\ell) < u(\ell')$ and $u(\ell) = u(\ell')$ define a linearly ordered preference relation.*
- *(existence) For every linearly ordered preference relation $\langle \prec, \sim \rangle$, there exists a vector utility function (with values in a linearly ordered affine space) which describes this preference.*
- *(uniqueness) The utility function is determined uniquely modulo an isomorphism:*
  - *If two different vector utility functions $u : L \to V$ and $u' : L \to V'$ describe the same linearly ordered preference relation, then there exists an isomorphism $T : A(u(L)) \to A(u'(L))$ between the affine hulls of the images of the functions, such that for every lottery ℓ, $u'(\ell) = T(u(\ell))$.*
  - *Vice versa, if a vector utility function $u : L \to V$ describes a preference relation, and $T : A(u(L)) \to V'$ is an isomorphism of ordered affine spaces, then the function $u'(\ell) = T(u(\ell))$ is also a vector utility function, and it describes the same preference relation.*

4 New result: utility theory for partially ordered preferences

It turns out that a similar result holds for *partially* ordered references as well:

Theorem 1. *Let $\mathcal{A}$ be a set, and let L be the set of all lotteries over $\mathcal{A}$.*

- *(consistency) For every convexity-preserving function $u : L \to V$ from L to an ordered affine space, the relations $u(\ell) < u(\ell')$ and $u(\ell) = u(\ell')$ define a preference relation.*
- *(existence) For every preference relation $\langle \prec, \sim \rangle$, there exists a vector utility function which describes this preference.*
- *(uniqueness) The utility function is determined uniquely modulo an isomorphism:*
  - *If two different vector utility functions $u : L \to V$ and $u' : L \to V'$ describe the same preference relation, then there exists an isomorphism $T : A(u(L)) \to A(u'(L))$ between the affine hulls of the images of the functions, such that for every lottery ℓ, $u'(\ell) = T(u(\ell))$.*
  - *Vice versa, if a vector utility function $u : L \to V$ describes a preference relation, and $T : A(u(L)) \to V'$ is an isomorphism of ordered affine spaces, then the function $u'(\ell) = T(u(\ell))$ is also a vector utility function, and it describes the same preference relation.*

(For reader's convenience, all the proofs are moved into the last section).

Alternatively, we can describe partially ordered preferences not by a *single* utility function with a value in a *partially* ordered affine space, but by *several* utility functions with values in *linearly* ordered affine spaces:

Definition 9. *We say that a family U of utility functions describes the preference relation if for every $\ell, \ell' \in L$, the following two conditions hold:*

$$\ell \prec \ell' \text{ if and only if } u(\ell) < u(\ell') \text{ for all } u \in U; \quad (1)$$

$$\ell \sim \ell' \text{ if and only if } u(\ell) = u(\ell') \text{ for all } u \in U. \quad (2)$$

Theorem 2. *Let $\mathcal{A}$ be a set, and let L be the set of all lotteries over $\mathcal{A}$.*

- *(consistency) For every family of convexity-preserving functions $u : L \to V$ from L to linearly ordered affine spaces V, the relations* (1) *and* (2) *define a preference relation.*
- *(existence) For every preference relation $\langle \prec, \sim \rangle$, there exists a family of linearly ordered vector utility function which describes this preference.*

Example. Let us consider a simple case in which the quality of each alternative a is described by the value of a single quantity $q(a)$ (e.g., profit), and the partialness of the preference relation is caused by the fact that we do not know the exact values of this quantity; instead, for each alternative a, we know the *interval* $[q^-(a), q^+(a)]$ of possible values of this quantity. In such a case, it is natural to define preference as follows:

- $a \preceq a'$ if and only if $q^-(a) \le q^-(a')$ and $q^+(a) \le q^+(a')$;
- $a \sim a'$ if and only if $q^-(a) = q^-(a')$ and $q^+(a) = q^+(a')$.

If, e.g., $[q^-(a), q^+(a)] = [1, 2]$, and $[q^-(a'), q^+(a')] = [0, 3]$, then neither of the two alternatives is preferable to the other one. It is easy to check that the function $a \to (q^-(a), q^+(a))$ from A to the vector space R^2 (with component-wise order) forms a vector utility function for this preference. One can also check that, as a *family* of linearly ordered utility functions, we can take functions of the type $u(a) = \alpha \cdot q^+(a) + (1-\alpha) \cdot q^-(a)$ which correspond to different values $\alpha \in [0, 1]$.

5 How to describe degrees of belief ("subjective probabilities") for partially ordered preferences?

In traditional (scalar) utility theory, it is possible to describe our degree of belief $ps(E)$ in each statement E, e.g., as follows: We pick two alternatives a_0 and a_1 with utilities 0 and 1, and as the degree of belief in E, we take the utility of a conditional alternative "if E then a_1 else a_0" (or $(E|a_1|a_0)$, for short). This utility is also called *subjective probability* because if E is a truly random event which occurs with probability p, then this definition leads to $ps(E) = p$: Indeed, according to the convexity-preserving property of a utility function, we have

$$ps(E) = u(E|a_1|a_0) = p \cdot u(a_1) + (1 - p) \cdot u(a_0) =$$

$$p \cdot 1 + (1 - p) \cdot 0 = p.$$

How can a similar description look like for *partially* ordered preferences? Before we formulate our result, let us first explain our reasoning that led to this result.

The linear-ordered case definition of subjective probability $ps(E)$ can be rewritten as follows: for every two lotteries $\ell, \ell' \in L$, we have

$$u(E|\ell|\ell') = ps(E) \cdot u(\ell) + (1 - ps(E)) \cdot u(\ell'),$$

or, equivalently,

$$u(E|\ell|\ell') = ps(E) \cdot (u(\ell) - u(\ell')) + u(\ell').$$

In other words, we can interpret $ps(E)$ as a *linear operator* which transforms the utility difference $u(\ell) - u(\ell')$ into an expression

$$u(E|\ell|\ell') - u(\ell') = ps(E) \cdot (u(\ell) - u(\ell')).$$

It is, therefore, reasonable to expect that for *partially* ordered preferences, when we have multi-dimensional (vector) utilities with values in a vector space V, $ps(E)$ would also be a linear operator, but this time from V to V (and not from R to R). We will now show that this expectation is indeed true.

Definition 10. *Let $\mathcal{A}$ be a set, let L be the set of all lotteries over $\mathcal{A}$, and let E be a formula (called event). By a conditional lottery, we mean an expression of the type $\sum p_i \cdot \ell_i + \sum q_k \cdot (E|\ell'_k|\ell''_k)$, where $\sum p_i + \sum q_k = 1$, and ℓ_i, ℓ'_k, and ℓ''_k are lotteries. We will denote the set of all conditional lotteries by $L(E)$.*

The meaning of a conditional lottery is straightforward: with probability p_i, we run a lottery ℓ_i, and with probability q_k, we run a conditional event "if E then ℓ'_k else ℓ''_k".

Definition 11. *Let $\mathcal{A}$ be a set, and let $L(E)$ be the set of all conditional lotteries over $\mathcal{A}$. By a preference relation, we mean a pair $\langle \prec, \sim \rangle$, where $\prec$ is a (strict) order on $L(E)$, $\sim$ is an equivalence relation on $L(E)$, which satisfies conditions 1)–6) from Definition 2 plus the following additional conditions:*

C1) *if $\ell \sim \ell'$, then $(E|\ell|\ell'') \sim (E|\ell'|\ell'')$;*

C2) *if $\ell' \sim \ell'''$, then $(E|\ell|\ell') \sim (E|\ell|\ell'')$;*

C3) $(E|\ell|\ell) \sim \ell$;

C4) $(E|p \cdot \ell + (1-p) \cdot \ell'|\ell'') \sim$
$p \cdot (E|\ell|\ell'') + (1-p) \cdot (E|\ell'|\ell'')$;

C5) $(E|\ell|p \cdot \ell' + (1-p) \cdot \ell'') \sim$
$p \cdot (E|\ell|\ell') + (1-p) \cdot (E|\ell|\ell'')$;

C6) $(E|p \cdot \ell + (1-p) \cdot \ell''|p \cdot \ell' + (1-p) \cdot \ell'') \sim$
$p \cdot (E|\ell|\ell') + (1-p) \cdot \ell''$;

C7) *if $\ell \preceq \ell'$, then $\ell \preceq (E|\ell|\ell') \preceq \ell'$.*

The meaning of all these conditions is straightforward; e.g., C7) means that $(E|\ell|\ell')$ is better (or of the same quality) than ℓ because in the conditional alternative, both possibilities ℓ and ℓ' are at least as good as A.

In accordance with our Theorem 1, the utility of such events can be described by a vector utility function.

Definition 12. *Let V be an ordered vector space.*

- *A linear operator $T : V \to V$ is called non-negative (denoted $T \geq \mathbf{0}$) if $x > 0$ implies $Vx \geq 0$.*
- *A linear operator T is called a probability operator if both T and $\mathbf{1} - T$ are non-negative (where $\mathbf{1}$ is a unit transformation $v \to v$).*

Theorem 3.

- *Let $u : L \to V$ be a vector utility function and let $T : V \to V$ be a strict probability operator. Then, a function $u^* : L(E) \to V$ defined as*

$$u^*\left(\sum_i p_i \cdot \ell_i + \sum_k q_k \cdot (E|\ell'_k|\ell''_k)\right) =$$

$$\sum_i p_i \cdot u(\ell_i) + \sum_k q_k \cdot u^*(E|\ell'_k|\ell''_k),$$

  with $u^*(E|\ell|\ell') = Tu(\ell) + (\mathbf{1} - T)u(\ell')$, is a vector utility function which describes a preference relation on $L(E)$.

- *Let $\langle \prec, \sim \rangle$ be a preference relation on $L(E)$, and let $u : L(E) \to V$ be a vector utility function which describes this preference. Then, there exists a probability operator $T : A(u(L)) \to V$ for which $u(E|\ell|\ell') = Tu(\ell) + (\mathbf{1} - T)u(\ell')$ for all ℓ and ℓ'.*

Thus, we get a generalization of subjective probabilities, from scalar values $p \in [0, 1]$ (which, in our description, correspond to scalar matrices) to general linear *probability operators.*

> It is worth mentioning that a similar generalization from a scalar to a matrix (linear operator) happens in quantum mechanics.

If we have *several* different events $E_1, E_2, \ldots$, then for each event E_i, we can thus describe its "subjective probability by the corresponding probabilistic operator $T(E_i)$.

The main objective of describing subjective probabilities is to use them to describe expert knowledge. Thus, if we have n events $E_1, \ldots, E_n$ which we want to evaluate, then we must extract, from the experts, their degrees of belief in all these events. This information must then be placed in a knowledge base. We would like the resulting knowledge-based system to be able to answer queries about the domain of knowledge. These queries may take a form of Boolean combinations of the original events, e.g., "$E_1 \vee E_2$?", "$E_1 \& E_2$?". In normal (1-D) probability theory, if we only know the probabilities $p(E_i)$ of the events E_i, then, in general, we cannot uniquely determine the probabilities of their Boolean combinations; however, there are two important cases when we *can* do that:

- if we know that the events E_1 and E_2 are *incompatible*, then we can conclude that $p(E_1 \vee E_2) = p(E_1) + p(E_2)$;
- if we know that the events E_1 and E_2 are *independent*, then we can conclude that $p(E_1 \& E_2) = p(E_1) \cdot p(E_2)$.

Let us show that similar formulas hold for multi-D case as well.

Let us first formalize the notion of incompatible events. In general, for every two events E_1 and E_2, the following two composite conditions should coincide:

- we can choose one of the conditions E_1 and E_2 with probability 1/2, and then check the chosen condition;
- alternatively, we can choose one of the conditions $E_1 \vee E_2$ and $E_1 \& E_2$ with probability 1/2, and then check the chosen condition.

In classical (1-D) probability, the probability of the first composite condition is $(p(E_1) + p(E_2))/2$, and the probability of the second composite condition is $(p(E_1 \vee E_2) + p(E_1 \& E_2))/2$, and it is known that these values coincide. It is therefore natural to require that these two composite conditions coincide for partial preferences as well.

To formalize this condition, we must have events and their Boolean combinations, i.e., we must have a *Boolean algebra* of events, with events 0 (always false) and 1 (always true).

Definition 13. *Let $\mathcal{B}$ be a Boolean algebra, and let conditional lotteries be defined for all events $E \in \mathcal{B}$. We say that these lotteries are consistent if for every two lotteries $\ell, \ell' \in L$, the following three conditions hold:*

- *$E(1|\ell|\ell') \sim \ell$;*
- *$E(0|\ell|\ell') \sim \ell'$;*
- *for every two events E_1 and E_2, we have*

$$\frac{1}{2} \cdot (E_1|\ell|\ell') + \frac{1}{2} \cdot (E_2|\ell|\ell') \sim$$

$$\frac{1}{2} \cdot (E_1 \vee E_2|\ell|\ell') + \frac{1}{2} \cdot (E_2 \& E_2|\ell|\ell').$$

Theorem 4. *Let $\mathcal{B}$ be a Boolean algebra of events, and let a consistent preference relation be defined on the set $\mathcal{B}$ of all $\mathcal{B}$-conditional lotteries. Then:*

- *$T(0) = \mathbf{0}$;*
- *$T(1) = \mathbf{1}$; and*

- *for every two events E_1 and E_2, we have*

$$T(E_1) + T(E_2) = T(E_1 \vee E_2) + T(E_1 \& E_2).$$

In particular, if the events E_1 and E_2 are incompatible (i.e., if $E_1 \& E_2 = 0$), then

$$T(E_1 \vee E_2) = T(E_1) + T(E_2).$$

In other words, for incompatible events, we get a formula similar to the formula for 1-D probabilities; the operators $T(E)$ corresponding to different events are, thus, *additive* (in the same sense in which probability of different events is additive). Additive functions on a Boolean algebra are usually called *measures*; hence, we can say that operators $T(E)$ form an *operator-valued additive measure* on the Boolean algebra $\mathcal{B}$ of all events.

Let us now formalize what *independent* means. We want to evaluate our degree of belief in $E_1 \& E_2$, i.e., we want to evaluate the utility of conditional alternatives of the type "if $E_1 \& E_2$ then ℓ else ℓ'" for different lotteries ℓ and ℓ'. This if-then statement with a composite condition can be equivalently rewritten as a composition of two if-then statements with simple conditions, e.g., as "if E_1 then (if E_2 then ℓ else ℓ') else ℓ'".

> Indeed, in both conditional alternatives (with a single composite condition and with two simple conditions) ℓ will be chosen if $E_1 \& E_2$ is true, otherwise, ℓ' will be chosen.

Intuitively, when E_1 and E_2 are independent, then we should be able to compute our degree of belief in the composite statement in two steps:

- first, we find the degree of belief in the internal if-then conditional alternative "if E_2 then ℓ else ℓ'", i.e., to be more precise, we find a lottery ℓ'' which is equivalent to this conditional alternative;
- then, we substitute ℓ'' into the composite statement instead of the conditional alternative, and find the degree of belief of the resulting statement "if E_1 then ℓ'' else ℓ'".

We expect the resulting degree of belief to coincide with the degree of belief in the original conditional alternative with a composite condition. In other words, we define independence as follows:

Definition 14. *We say that the set of lotteries is complete if for every event E, and for every two lotteries ℓ and ℓ', there exists a lottery ℓ'' for which $\ell'' \sim (E_2|\ell|\ell')$.*

In the remaining part of this section, we will consider events for which the set of lotteries is complete.

Definition 15. *We say that the events E_1 and E_2 are independent if for every three lotteries ℓ, ℓ', and ℓ'', the following two conditions hold:*

- *If $\ell'' \sim (E_2|\ell|\ell')$, then $(E_1|\ell''|\ell') \sim (E_1 \& E_2|\ell|\ell')$;*
- *If $\ell'' \sim (E_1|\ell|\ell')$, then $(E_2|\ell''|\ell') \sim (E_1 \& E_2|\ell|\ell')$.*

Theorem 5. *Events E_1 and E_2 are independent if and only if $T(E_1 \& E_2) = T(E_1)T(E_2) = T(E_2)T(E_1)$.*

For 1-D probabilities, as a particular case of this theorem, we get the classical independence formula $p(E_1 \& E_2) = p(E_1) \cdot p(E_2)$. The formula of Theorem 5 is a natural generalization of this classical formula, from numbers to linear operators. This generalization takes into consideration that unlike two real numbers, two linear operators do not necessarily always commute, so in addition to requiring that $T(E_1 \& E_2) = T(E_1)T(E_2)$, we must also require that $T(E_1 \& E_2) = T(E_2)T(E_1)$.

6 A surprising corollary: if preferences are partially ordered, then we can control them

As a somewhat unexpected corollary of our results, we can show that if a person's preferences are only partially ordered, then we can *control* the person's choices: Namely, if a person has originally selected a lottery ℓ, and the person does not feel that some other lottery ℓ' is worse or better than ℓ, (i.e., $\ell \not\preceq \ell'$ and $\ell' \not\preceq \ell$), then we can, in principle, convince the person to change his preference to ℓ'. After the new choice "settles in", we can convince the person to change his preferences once again, to some third lottery which is not worse than ℓ', etc. In turns out that if we are allowed sufficiently many steps of this type, then, under a natural continuity assumption, we can change the person's choice from any lottery to any other. Let us describe this result formally.

Definition 16.

- *We say that a sequence of lotteries $\ell_n = \langle p_n, A\rangle$, with the same finite set A, converges to a lottery $\ell = \langle p, A\rangle$ if for every $a \in A$, we have $p_n(a) \to p(a)$.*
- *We say that a preference relation is continuous if from $\ell_n \preceq \ell'$ and $\ell_n \to \ell$, we can conclude that $\ell \preceq \ell'$.*
- *We say that lotteries ℓ and ℓ' are indifferent (and denote it $\ell \parallel \ell'$) if $\ell \not\preceq \ell'$ and $\ell' \not\preceq \ell$.*

- *We say that we can change ℓ to ℓ' if there exists a sequence $\ell_1 = \ell$, ℓ_2, ..., $\ell_m = \ell'$ for which, for every i, $\ell_i \parallel \ell_{i+1}$.*

- *We say that we can completely control preferences if the set of all pairs (ℓ, ℓ') for which we can change ℓ to ℓ' is everywhere dense in the set of all pairs of lotteries, i.e., if every pair of lotteries (ℓ, ℓ') can be represented as a limit of sequences (ℓ_n, ℓ'_n) for which $\ell_n \to \ell$, $\ell'_n \to \ell'$, and for every n, we can change ℓ_n to ℓ'_n.*

Theorem 6. *If the preference relation is continuous and there exist two indifferent lotteries $\ell^{(1)} \parallel \ell^{(2)}$, then we can completely control preferences.*

Comments.

- The continuity condition exclude *lexicographic* orders in which, e.g., $(1 - 1/n, 1) \prec (1, 0)$ but $(1, 1) = \lim (1 - 1/n, 1) \not\preceq (1, 0)$.

- From the *practical* viewpoint, it is OK that we cannot change from ℓ to *exactly* ℓ', but we can change to a lottery which is as close to ℓ' as we want. However, from the *theoretical* viewpoint, it is worth mentioning that there are cases when we cannot get *exactly* ℓ': e.g., if $A = [0, 1] \times [0, 1]$ is the set of all pairs of real numbers (x_1, x_2) with $x_i \in [0, 1]$ and component-wise order, then the alternative $a = (1, 1)$ is better than any other alternative (and thus, than any lottery) and hence, we cannot change any other lottery ℓ to exactly a.

- As we will, the proof requires arbitrary long chains of lotteries. One can show that we cannot *a priori* limit the length N of the chain of changing lotteries.

- Theorem 6 has a natural interpretation in special relativity theory: Namely, if we take as A the set of all points in space-time, and by $a \prec a'$, the notion of causality, then $a \parallel a'$ means, in physical terms, that the interval between a and a' is *spatial*. In these terms, Theorem 6 says that if we could travel instantaneously from any point a to any other spatially separated point a', i.e., if we had a "*space* machine", then we would be able to travel to any event in space-time, i.e., we would be able to design a *time* machine as well.

7 Proofs

Proof of Theorem 1. The proof of *consistency* is straightforward: we can simply check all the conditions from Definition 2. Let us now prove *existence*. For this proof, we will expand the set of lotteries to an affine space. Namely, a lottery is defined as an expression of the type $\ell = p_1 \cdot a_1 + \ldots + p_n \cdot a_n$ with $p_i \geq 0$ and $\sum p_i = 1$. The set of all lotteries is not an affine space, because we cannot have negative coefficients p_i. So, we will consider a set S of all possible expressions of the same type with $\sum p_i = 1$ in which the coefficients p_i can be both positive and negative. This set is already an affine space, and the set L of all lotteries is its subset.

(Readers who are more familiar with ordered *vector* spaces can re-do this proof in terms of vector spaces only. For this, we need to pick an arbitrary element $s_0 \in L \subseteq S$ as 0, and then define a structure of the vector space as follows: $\lambda \cdot s$ as $\lambda \cdot s - (\lambda - 1) \cdot s_0$ and $s + s'$ as $s + s' - s_0$. All further arguments can be changed accordingly.)

Let us show that the preference relation can be naturally extended from $L \subseteq S$ to the entire set S. Let s and s' are elements of S. We will show that a question of whether $s \prec s'$ is true can be naturally reduced to the question of comparing two auxiliary lotteries. Without losing generality, we can assume that s and s' contain the same elements of the set $\mathcal{A}$ (otherwise, we can add the remaining elements a_i with 0 probabilities $p(a_i) = 0$), i.e., that $s = \sum p_i \cdot a_i$ and $s' = \sum p'_i \cdot a_i$, where $\sum p_i = \sum p'_i = 1$. Then, if we want the extended relation to be affine-invariant, we must have $s \prec s'$ iff $0 \prec s' - s$, i.e., $0 \prec \sum d_i \cdot a_i$, where $d_i = p'_i - p_i$. Here, the sum of all differences d_i is 0: $\sum (p'_i - p_i) = \sum p_i - \sum p'_i = 0$. Thus, some difference are non-negative: $d_i = d_i^+ \geq 0$, some are negative: $d_i = d_i^- < 0$. We can move all negative terms to the left-hand side, and get an equivalent formula $\sum |d_j^-| \cdot a_j \prec \sum d_k^+ \cdot a_k$. Since the sum $\sum d_i = \sum d_j^- + \sum d_k^+$ of all the differences d_i was 0, we can conclude that the sum of weights on both sides of the preference is the same: $\sum |d_j^-| = \sum d_k^+$. Thus, if we divide both sides of the desired formula by this sum $D = \sum d_k^+$, we get the equivalent expression $\ell^- \prec \ell^+$, where $\ell^- = \sum p_j^- \cdot a_j$, $\ell^+ = \sum p_k^+ \cdot a_k$, $p_j^- = |d_j^-|/D$, and $p_k^+ = d_k^+/D$. Here, $p_j^- \geq 0$, $p_k^+ \geq 0$, $\sum p_j^- = 1$, and $\sum p_k^+ = 1$; hence, both ℓ^- and ℓ^+ are lotteries. So, intuitively, we expect $s \prec s'$ to be true in S if and only if $\ell^- \prec \ell^+$ is true in L.

It is therefore natural to *define* $s \prec s'$ as $\ell^- \prec \ell^+$. Similarly, we can define $s \sim s'$ as $\ell^- \sim \ell^+$. One can check that thus defined relations $\prec$ and $\sim$ on the set S are indeed extensions of the relations $\prec$ and $\sim$ on L, and that they have the following property:

- The set S with the relation $\prec$ is an ordered affine space.

- The relation $\sim$ is an equivalence relation, which is consistent with the affine structure (i.e., if $a_i \sim a'_i$, then $\sum p_i \cdot a_i \sim \sum p_i \cdot a'_i$). Thus, the *factor-set*

$V = S/\sim$ is also an affine space.

- The relation $\prec$ is consistent with the equivalence relation $\sim$ (i.e., if $s \prec t$, $s \sim s'$, and $t \sim t'$, then $s' \prec t'$), and therefore, this relation induces an ordering $<$ on the factor-space $V = S/\sim$.

It is straightforward to check that the affine space V with the relation $<$ is an ordered affine space. Now, we can define, for every element $\ell \in L$, the value $u(\ell)$ as an equivalence class

$$u(\ell) = (\ell/\sim) \in (L/\sim) \subseteq (S/\sim) = V$$

to which the lottery ℓ belongs. Then, by definition of a factor space, $\ell \prec \ell'$ if and only if $u(\ell) < u(\ell')$, and $\ell \sim \ell'$ if and only if $u(\ell) = u(\ell')$. Hence, the utility function u indeed describes the original preference relation. The existence is proven.

Finally, to prove uniqueness, we must take into consideration that the our construction of the utility function u uniquely followed from the desire to express the original preference relation in an affine-ordered way. Thus, this utility function is indeed defined uniquely modulo an arbitrary isomorphism of ordered affine spaces. The theorem is proven.

Proof of Theorem 2. *Consistency* is straightforward. To prove *existence*, we will start with the utility function $u : L \to V$ constructed in the proof of Theorem 1. We will show that:

- As desired linearly ordered utility functions, we can take a similar map $u^* : L \to V^*$, where V^* is exactly the same affine space (and for every lottery ℓ, $u^*(\ell)$ is exactly the same element as $u(\ell)$), but instead of the original *partial* order $<$ on V, we now have a *linear* order $<^*$ which extends $<$.

- As a family U of utility functions, we take the functions which correspond to all possible linear extensions $<^*$.

How can we extend a partial order on a vector space to a linear order? Defining a partial order is equivalent to defining a *cone* of all positive elements $C = \{a \mid a > 0\}$. As such a cone, we can take any set $C \subseteq V$ which does not contain 0, and which is closed under addition and under multiplication by a positive real number. A cone C defines a *linear* order if and only for every $v \in V$, either $v \in C$, or $-v \in C$. If the cone *does not* define a linear order, then there exists an element $v \in V$ for which neither $v \in C$ nor $-v \in C$.

- If we want to add v to the set of positive elements, then we must consider a new cone which contains both the old cone and the vector v. The smallest such cone is a so-called *conic hull* of the union $C \cup \{v\}$, i.e., the set $Cone(C \cup \{v\})$ of all possible expressions of the type $\lambda_1 \cdot v_1 + \ldots + \lambda_n \cdot v_n$, where $\lambda_i \geq 0$ and $v_i \in C \cup \{v\}$. It is easy to check that the resulting set is indeed a cone which does not contain 0 and therefore, that it defines the desired extension of the original order, and extension in which $v > 0$.

- Similarly, by taking $Cone(C \cup \{-v\})$, we can define a new extension in which $v < 0$. If none of these extensions defines a linear order, we can extend each of them further, etc.

Since the union of any monotonically increasing family of a cones is still a cone, we can use Zorn's lemma and show that each of the two extensions $Cone(C \cup \{v\})$ and $Cone(C \cup \{-v\})$ can be extended to a linear order.

Thus, for every $v \in V$, the only possibility to have $v > 0$ in *all* such extensions is to have $v > 0$ in the original partial order. So, since for every two lotteries $\ell, \ell' \in L$, $\ell \prec \ell'$ is equivalent to $u(\ell) < u(\ell')$, it is also equivalent to the condition that $u^*(\ell) < u^*(\ell')$ for all extensions u^* (i.e., for all linearly ordered utility functions from our family U). Hence, the condition (1) is satisfied (and so it condition (2)). So, the family U indeed describes the given preference relation. The theorem is proven.

Proof of Theorem 3. The proof of the first part of Theorem 3, that the preference described by the given expression indeed satisfies all the conditions from Definition 11, is straightforward.

Let us prove its second part, that the preference relation on the set of all conditional lotteries can be indeed described by the given expression. Indeed, Theorem 1 is applicable to a general case, in particular, it is applicable to the case when as alternatives, we allow not only the original alternatives, but also conditional statements $(E|\ell|\ell')$. Thus, from Theorem 1, we can conclude that the preference relation $\langle \prec, \sim \rangle$ on the set $L(E)$ can be described by a utility function $u : L \to V$ with values in some ordered affine space. We want to show that there exists a probabilistic operator T for which $u(E|\ell|\ell') = Tu(\ell) + (\mathbf{1} - T)u(\ell')$ for all lotteries $\ell, \ell' \in L$.

Indeed, due to the conditions C1) and C2), the value of $u(E|\ell|\ell')$ can only depend on $u(\ell)$ and $u(\ell')$, i.e., $u(E|\ell|\ell') = f(u(\ell), u(\ell'))$ for some function f. We will show that this expression can be reduced to a function of one variable, and then we will show that this function of one variable indeed corresponds to a probabilistic operator.

Let us first reduce the function f to a function of one variable. Due to conditions C4)–C5), the mapping f is convexity-preserving in each of its variables, and therefore, it can be extended (similarly to what we did in the proof of Theorem 1) to a convexity-preserving mapping of the corresponding affine hulls $f : A(u(L)) \times A(u(L)) \to V$. Specifically, since every element from the affine hull $A(u(L))$ has the form $\sum p_i \cdot v_i$ for some real numbers p_i and for some $v_i = u(\ell_i) \in V$, this extension can be defined as follows: $f(\sum p_i \cdot v_i, \sum p'_j \cdot v'_j) = \sum p_i \cdot p'_j \cdot f(v_i, v'_j)$. After this extension, the condition C6) holds not only for $p \in [0,1]$, but for arbitrary real numbers p; in other words, in terms of the function $f(v, v')$, we get the following formula: $f(p\cdot v + (1-p)\cdot v'', p\cdot v' + (1-p)\cdot v'') = p \cdot f(v, v') + (1-p)\cdot v''$. If we move $(1-p)\cdot v''$ into the other side and divide both sides by p, we get the following equation:

$$f(v, v') = p^{-1} \cdot f(p\cdot v + (1-p)\cdot v'', p\cdot v' + (1-p)\cdot v'') - (p^{-1} - 1)\cdot v''.$$

In particular, if we fix some element $v_0 \in V$, and take $p = 1/2$, and $v'' = 2v_0 - v'$, we conclude that

$$f(v, v') = 2f(v_0 + (1/2)\cdot(v - v'), v_0) - 2v_0 + v'.$$

Thus, to describe the function $f(v, v')$ for all v and v', it is sufficient to describe a function $g(v) = f(v, v_0)$ of a single variable v. So,

$$f(v, v') = 2g(v_0 + (1/2)\cdot(v - v')) - 2v_0 + v'. \quad (3)$$

If we take v_0 as 0, and interpret $\sum \lambda_i \cdot v_i$ as $\sum \lambda_i \cdot v_i - (\sum \lambda_i - 1)\cdot v_0$, then we get the corresponding vector space. In this vector space, the condition C3) leads to $g(0) = 0$, and hence, the condition that f is an affine transformation leads to *linearity* of the function g, i.e., to the fact that $g(v) = Tv$ for some linear operator T. In the vector space, in which $v_0 = 0$, the formula (3) takes the form $f(v, v') = 2g((1/2)\cdot(v - v')) + v'$. Substituting $g(v) = Tv$ into this formula, we get $f(v, v') = Tv + (\mathbf{1} - T)v'$. By definition of the function f, this means that for some linear operator T, the formula $u(E|\ell|\ell') = Tu(\ell) + (\mathbf{1} - T)u(\ell')$ holds for every two lotteries $\ell, \ell' \in L$. The fact that T is a probabilistic operator follows from the condition C7). The theorem is proven.

Proof of Theorem 4 directly follows from Theorem 3 and from the conditions contained in Definition 13.

Proof of Theorem 5 directly follows from Theorem 3 and from the conditions contained in Definition 15.

Proof of Theorem 6. To prove this theorem, let us start with what we know. According to the condition of the theorem, there exist lotteries $\ell^{(1)}$ and $\ell^{(2)}$ for which $\ell^{(1)} \parallel \ell^{(2)}$.

Let us now describe the general properties of the indifference relation $\parallel$ and of the corresponding *changeability* relation $\ell \, C \, \ell'$.

- First, since the indifference relation is symmetric ($\ell \parallel \ell'$ iff $\ell' \parallel \ell$), the changeability relation is also symmetric.
- Second, by definition of changeability, this relation is *transitive*, i.e., if $\ell \, C \, \ell'$ and $\ell' \, C \, \ell''$, then $\ell \, C \, \ell''$.

  Due to this transitivity, it is sufficient to show that every lottery $\ell \in L$ can be approximated by lotteries which can be changed from $\ell^{(1)}$ (i.e., that ℓ can be represented as a limit of lotteries which can be changed from $\ell^{(1)}$). Indeed, if we prove this result, then, to change from ℓ to ℓ', we will be able to:

  * first change from ℓ to $\ell^{(1)}$, and
  * then change from $\ell^{(1)}$ to ℓ'.

- Finally, for the proof, we will need the following technical properties: if $\ell \parallel \ell'$ and $p \in (0,1)$, then for every lottery ℓ'', we have

  $$p\cdot\ell + (1-p)\cdot\ell'' \parallel p\cdot\ell' + (1-p)\cdot\ell''.$$

  Indeed, e.g., from

  $$p\cdot\ell + (1-p)\cdot\ell'' \preceq p\cdot\ell' + (1-p)\cdot\ell'',$$

  we would be able to conclude, by using properties 4) and 6) from Definition 2, that $\ell \preceq \ell'$.

Similarly, if we know that $\ell\, C\, \ell'$, and we apply the above property of indifference relation to all the elements in a chain which proves changeability, we can conclude that

$$[p\cdot \ell + (1-p)\cdot \ell'']\, C\, [p\cdot \ell' + (1-p)\cdot \ell''].$$

Let us prove the "limit changeability" from ℓ to $\ell^{(1)}$. If $\ell^{(1)} \parallel \ell$ or $\ell^{(1)} \sim \ell$, then we can change from $\ell^{(1)}$ to ℓ in one step. Thus, it is sufficient to consider the remaining cases, when $\ell^{(1)} \prec \ell$ or $\ell \prec \ell^{(1)}$. Without losing generality, let us consider the first case (the second is proven in a similar way).

In this first case (when $\ell^{(1)} \prec \ell$), the proof will depend on the relation between ℓ and $\ell^{(2)}$.

- If $\ell \parallel \ell^{(2)}$ or $\ell \sim \ell^{(2)}$, then we can change from $\ell^{(1)}$ to ℓ in two steps: via $\ell^{(2)} \parallel \ell^{(1)}$.
- The relation $\ell \prec \ell^{(2)}$ is impossible, because it would lead to $\ell^{(1)} \prec \ell \prec \ell^{(2)}$ and $\ell^{(1)} \prec \ell^{(2)}$, which contradicts to our choice of $\ell^{(i)}$.

Thus, to complete the proof, it is sufficient to consider the remaining case, when $\ell^{(2)} \prec \ell$ (and $\ell^{(1)} \prec \ell$).

In this case of $\ell^{(1)} \prec \ell$ and $\ell^{(2)} \prec \ell$, we will show that the lottery ℓ can be represented as a limit of the lotteries ℓ_n for which $\ell_n\, C\, \ell^{(1)}$. Moreover, we will show that as ℓ_n, we can take the lotteries $\ell(p) = p\cdot \ell + (1-p)\cdot \ell^{(1)}$ for appropriate values $p_n \to 1$, for which $\ell(p_n) \to \ell$. (Since $\ell^{(1)} \prec \ell$, the function $p \to \ell(p)$ is strictly increasing.)

Let us first prove a weaker form of the desired statement: that there *exists* a value $p_0 \in (0,1)$ for which $\ell(p_0)\, C\, \ell^{(1)}$. We will prove that:

- as this p_0, we can take the value for which $\ell^{(2)} \not\preceq \ell(p_0)$, and that
- the existence of such p_0 can be proven by reduction to a contradiction.

Let us start with the reduction to a contradiction. If we had $\ell^{(2)} \preceq \ell(p)$ for all $p \in (0,1)$, then, in the limit, we would have $\ell^{(2)} \preceq \lim \ell(p) = \ell(0) = \ell^{(1)}$, which constradicts to our choice of $\ell^{(1)} \parallel \ell^{(2)}$. Thus, such a p_0 indeed exists.

For this p_0, we thus cannot have $\ell^{(2)} \preceq \ell(p_0)$; similarly, we cannot have $\ell(p_0) \preceq \ell^{(2)}$, because then we would have $\ell^{(1)} \prec \ell(p_0) \preceq \ell^{(2)}$ and $\ell^{(1)} \prec \ell^{(2)}$. Thus, $\ell(p_0) \parallel \ell^{(2)}$. So, we can change from $\ell^{(1)}$ to $\ell(p_0)$ in two steps: $\ell^{(1)} \parallel \ell(p_0) \parallel \ell^{(2)}$, i.e., we have $\ell(p_0)\, C\, \ell^{(1)}$.

From this proven changeability, and from the third property of the changeability relation, we can now conclude that

$$[p_0\cdot \ell + (1-p_0)\cdot \ell(p_0)]\, C\, [p_0\cdot \ell + (1-p_0)\cdot \ell^{(1)}].$$

The right-hand side of this relation is exactly $\ell(p_0)$, and the left-hand side, if we substitute the expression $\ell(p_0)$, becomes

$$p_0\cdot \ell + (1-p_0)\cdot p_0\cdot \ell + (1-p_0)^2\cdot \ell^{(1)} =$$

$$[1-(1-p_0)^2]\cdot \ell + (1-p_0)^2\cdot \ell^{(1)} = \ell(1-(1-p_0)^2).$$

Thus, from $\ell(p_0)\, C\, \ell^{(1)}$, we conclude that

$$\ell(1-(1-p_0)^2)\, C\, \ell^{(1)},$$

and therefore, due to the transitivity of the changeability relation, we can conclude that

$$\ell(1-(1-p_0)^2)\, C\, \ell^{(1)}.$$

Similarly, by applying the similar argument to this new changeability, we can conclude that

$$\ell(1-(1-p_0)^3)\, C\, \ell^{(1)},\quad \ell(1-(1-p_0)^4)\, C\, \ell^{(1)},$$

and, in general, $\ell(1-(1-p_0)^n)\, C\, \ell^{(1)}$ for an arbitrary integer n. As $n \to \infty$, we get $1-(1-p_0)^n \to 1$, and hence, $\ell(1-(1-p_0)^n) \to \ell(1) = \ell$. Thus, an arbitrarily chosen lottery ℓ is indeed the limit of lotteries which are changeable with $\ell^{(1)}$.

Thus, due to transitivity, using $\ell^{(1)}$ as an intermediate step, we get a limit changeability between arbitrary two lotteries ℓ and ℓ'. The theorem is proven.

Proof of a comment after Theorem 6. Let us give an example of a preference relation in which for every N there exist two lotteries which require, for changing between them, a chain of at least N intermediate lotteries. Indeed, we can take $A = [0,1]\times R$, with equality as $\sim$, and $(x_1,x_2) \prec (x_1',x_2')$ iff $x_2' > x_1' + |x_1 - x_1'|$. Then, since both x_1 and x_1' belong to the interval $[0,1]$, we have $|x_1 - x_1'| \le 1$. Hence, if $x_2' > x_2 + 1$, we have $(x_1,x_2) \prec (x_1',x_2')$. Thus, if the alternatives $a = (x_1,x_2)$ and $a' = (x_1',x_2')$ are indifferent, then:

- we cannot have $x_2' > x_2 + 1$, because then we would have $a \prec a'$;
- we also cannot have $x_2 > x_2' + 1$, because then we would have $a' \prec a$.

Thus, we must have $|x_2 - x_2'| \le 1$. Hence, if we want to change from an event $a = (x_1, x_2)$ to an event $a' = (a_1', a_2')$ with $|x_2 - x_2'| > N$, we need at least N intermediate steps. The statement is proven.

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