ARITHMETIC OF COMPLEX SETS: NICKEL'S CLASSICAL PAPER REVISITED FROM A GEOMETRIC VIEWPOINT

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Abstract. *Due to measurement uncertainty, after measuring a value of a physical quantity (or quantities), we do not get its exact value, we only get a* set *of possible values of this quantity (quantities). In case of 1-D quantities, we get an* interval *of possible values. It is known that the family of all real intervals is closed under point-wise arithmetic operations* $(+, -, \cdot)$ *(i.e., this family forms an* arithmetic*). This closeness is efficiently used to estimate the set of possible values for* $y = f(x_1, \ldots, x_n)$ *from the known sets of possible values for* x_i.

In some practical problems, physical quantities are complex-valued; it is therefore desirable to find a similar closed family (arithmetic) of complex *sets. We follow K. Nickel's 1980 paper to show that, in contrast to 1-D interval case, there is no finite-dimensional arithmetic.*

We prove this result by reformulating it as a geometric problem of finding a finite-dimensional family of planar sets which is closed under Minkowski addition, rotation, and dilation.

Data processing: a practical problem which leads to arithmetic of complex sets. In many real-life situations, we are interested in the value of some physical quantity y which is difficult (or even impossible) to measure directly. To estimate y, we measure directly measurable quantities $x_1, \ldots, x_n$ which have a known relationship with y, and then reconstruct y from the results $\widetilde{x}_1, \ldots, \widetilde{x}_n$ of these measurements by using this known relation: $\widetilde{y} = f(\widetilde{x}_1, \ldots, \widetilde{x}_n)$, where f is a known algorithm.

Measurements are never 100% accurate; as a result, the actual value x_i of each measured quantity may differ from the measured value $\widetilde{x}_i$. If we know the upper bound Δ_i for the measurement error $|\Delta x_i| = |\widetilde{x}_i - x_i|$, then after we get the measurement result $\widetilde{x}_i$, we can conclude that the actual value x_i of the measured quantity belongs to the *interval* $\mathbf{x}_i = [\widetilde{x}_i - \Delta_i, \widetilde{x}_i + \Delta_i]$. A natural question is: when $x_i \in \mathbf{x}_i$, what is the resulting interval $\mathbf{y} = f(\mathbf{x}_1, \ldots, \mathbf{x}_n) = \{f(x_1, \ldots, x_n) \,|\, x_i \in \mathbf{x}_i\}$ of possible values of y?

Computing the exact bounds for the range interval is, in general, computationally difficult (see e.g., [Kreinovich et al. 1997]). However, there are efficient methods of computing an *enclosure* $\mathbf{Y} \supseteq \mathbf{y}$ for this range; these methods are called methods of *interval computations* (see, e.g., [Hammer et al. 1993], [Hansen 1992], [Kearfott 1996], [Kearfott et al. 1996], [Moore 1979]). For example, we can use "naive interval computations": describe the algorithm f as a sequence of elementary arithmetic operations $(+, -, \cdot, /)$, and on each step, replace each operation $\odot$ with numbers by the corresponding operation with intervals:

$$\mathbf{x} \odot \mathbf{y} = \{x \odot y \,|\, x \in \mathbf{x}, y \in \mathbf{y}\}. \tag{1}$$

For intervals, we have explicit formulas for these arithmetic operations: e.g., $[\underline{a}, \overline{a}] + [\underline{b}, \overline{b}] = [\underline{a} + \underline{b}, \overline{a} + \overline{b}]$, etc.

For example, to estimate the range of the function $f(x_1) = x_1 \cdot (1 - x_1)$, we describe the algorithm f as a sequence of two arithmetic operations:

- computing the intermediate value $r_1 := 1 - x_1$, and
- computing the product $f := x_1 \cdot r_1$.

So, to estimate the range $f([0, 1])$, we compute $\mathbf{r}_1 := 1 - [0, 1] =$

$[0,1]$, and then get the final enclosure $\mathbf{Y} := \mathbf{x}_1 \cdot \mathbf{r}_1 = [0,1] \cdot [0,1] = [0,1]$ (this is, of course, a superset of the actual range [0,0.25]).

Similar range estimation problems appear when the physical quantities are described by *complex numbers.* It is therefore desirable to find a similar technique for complex numbers. The methodology of naive interval computations is based on the fact that the set of all intervals (including degenerate intervals – real numbers) is closed under point-wise arithmetic operations (1) (except, of course, division by an interval $\mathbf{y}$ containing 0). In other words, arithmetic operations are well defined on the family of all intervals, so we can talk about the *arithmetic of intervals.* Hence, it is desirable to look for families of subsets of complex numbers which are also closed under arithmetic operations, i.e., to look for an *arithmetic of complex sets.*

We want these subsets to be representable in a computer, where we can only store finitely many parameters and therefore, we want these sets to form a finite-dimensional (finite-parametric) family.

Also, we want to take into consideration that real numbers are an important practical case of complex numbers; therefore, real-line intervals (corresponding to imprecisely known real numbers) should be a particular case of this more general family of complex sets.

Reasonable families of complex sets do not form a complex arithmetic: the empirical fact and the resulting question. There are several natural complex analogues of real-line intervals:

- *boxes*, i.e., rectangular parallel to real axis;
- *ellipses* (including real-line intervals as degenerate ellipses), etc.

None of these families is closed under point-wise arithmetic operations (1). Moreover, they are not even closed under a limited set of arithmetic operations which includes addition and multiplication by complex *numbers.* A natural question is: *Is there a finite-dimensional family of complex sets which is closed under these operations?* To answer this question, let us reformulate it in geometric terms.

Reformulating the question in geometric terms. In geometric terms, a complex plane is simply a plane, so we are looking for families of planar sets. The sum (1) of two planar sets is simply their Minkowski sum.

In geometric terms, if we multiply a complex number t by another complex number $z = \rho \cdot \exp(\mathrm{i}\varphi)$, this mean that we first rotate t by an angle φ around the origin $O = (0, 0)$ of the coordinate system, and then dilate the rotated point ρ times. Thus, the pointwise product $z \cdot T$ of a complex number z and a set T means that we first rotate the set T, and then dilate the result of this rotation.

Hence, we arrive at the following definition:

Definition. *Let R^2 be a plane. By an arithmetic of complex sets, we mean a family $\mathcal{F}$ of planar sets which satisfies the following three properties:*

- *$\mathcal{F}$ contains all sub-intervals of the x-axis $R \times \{0\}$;*
- *$\mathcal{F}$ is closed under Minkowski addition, and*
- *$\mathcal{F}$ is closed under rotations and dilations around $O = (0, 0)$.*

A *finite-dimensional* family can be defined in a standard topological way: if we restrict ourselves to bounded and closed (hence, compact) sets, we can use Hausdorff distance between sets to define a topology; once the family is a topological space, we can use standard topological definitions to define its dimension.

The question is: *does there exist a finite-dimensional arithmetic of complex sets?*

Nickel's answer, and why it is not final. In his paper [Nickel 1980], K. Nickel proves that "finite-dimensional" arithmetics of complex sets do not exist. However, in his formulation, he only considers sets with piece-wise smooth boundaries, and he uses a non-standard (and non-topological) definition of dimension.

To be more precise, he calls a family "at least m-dimensional" if this family contains at least one set with m "corner" (non-smooth) points, and he proves that every arithmetic of complex sets is "infinite-dimensional" in this sense by proving that it contains a m-cornered set B_m for each m. From the topological viewpoint, all these sets B_m form a family of dimension 0, and therefore, Nickel's proof does not answer our question.

Final answer. We will show that a minor modification of Nickel's construction does lead to the final answer:

Proposition. *There exists no finite-dimensional arithmetic of complex sets.*

Proof. We will show that every arithmetic of complex sets $\mathcal{F}$ contains, for every n, an n-dimensional subfamily. Indeed, by definition of an arithmetic of complex sets, the family $\mathcal{F}$ contains a horizontal (real-line) interval $I_0 = [0,1] \times \{0\}$, and also the results $I_1, \ldots, I_n$ of its rotation by angles $\varphi_0, 2\varphi_0, \ldots, n \cdot \varphi_0 = \pi/2$, where $\varphi_0 = \pi/(2n)$. Since $\mathcal{F}$ is closed under dilations, for every $n+1$ positive real numbers $\rho_0, \ldots, \rho_n$, this family contains the dilated sets $J_i = \rho_i \cdot I_i$, $0 \le i \le n$. Since $\mathcal{F}$ is closed under Minkowski addition, the family $\mathcal{F}$ also contains their Minkowski sum $J_0 + \ldots + J_n$. One can easily see that this Minkowski sum is a polygon, and if we count its sides starting from the horizontal side, we get sides of lengths $\rho_0, \ldots, \rho_n$ which make angles of $0, \varphi_0, 2\varphi_0, \ldots, n \cdot \varphi_0 = \pi/2$ with the horizontal axes. Thus, different values of $n+1$ parameters ρ_i lead to different sets from $\mathcal{F}$. Hence, the family $\mathcal{F}$ contains a $(n+1)$-dimensional subfamily. The proposition is proven.

Open problem. This result prompts the following open problem: what if, in our Definition, we do not require that a family $\mathcal{F}$ contain real-line intervals? What finite-dimensional families we will then have? For one, we will have a 1-D family of all circles with a center in $O = (0,0)$, a 3-D family of all circles. We will also have several other families of rotation-invariant sets (e.g., circles + circles with a narrow circular gap + circles with a concentric circular holes in them, etc.) Is there any finite-dimensional rotation- and dilation-invariant family of compact sets which is closed under Minkowski addition and whose sets are not rotation-invariant?

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