Chu Spaces – A New Approach to Describing Uncertainty in Systems

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***Abstract*—This paper proposes the use of a specific type of categories for modeling and fusing information in complex systems in which uncertainty of various types need to be taken into account.**

I. Introduction

Uncertain systems in general, and fusion systems in particular, all share a number of common basic features, namely:

i) facts, rules, observed data are information or knowledge of some type;

ii) there exist interactions among things in i);

iii) we need to combine the above pieces of information in order, e.g., to reach some decision.

Typical examples are *probabilistic systems* and *fuzzy systems*:

- In *probabilistic systems*, the modeling and fusion of information can be carried out by using the *well established calculus* of probabilities, coupled with methodologies such as Bayesian techniques, probability logic, etc.
- In *fuzzy systems*, there are *ad hoc* procedures to be used; only some of them are justified – e.g., by Stone-Weierstrass theorem on approximation of functions.

In *general complex systems*, in which information contains both randomness and fuzziness as uncertainty, there seems to be a need to develop new mathematics for a "quality science" which could produce reliable products!

A moment of thought shall reveal that the elements i), ii), and iii) listed above fall perfectly into the framework of *category theory*. Indeed:

i) pieces of information are objects,

ii) interactions among objects can be modeled as morphisms, and

iii) operations in category theory, such as colimit, can be used as fusion operators.

For related works on using category theory in general (and Chu category in particular) in Data fusion, Information flow in distributed systems, and Concurrency modeling in computer science, see [1]–[8]

In this paper, we will describe the use of Chu category in uncertainty analysis.

II. Background and Motivation

A. What is a category: general definition

A *category* $\mathcal{C}$ consists of:

i) a collection of things called *objects*, denoted as $\mathrm{Ob}(\mathcal{C})$,

ii) a collection of things called *morphisms*, denoted by $\mathcal{M}(\mathcal{C})$; to every morphism f, a pair of objects (a, b) is associated; this association is denoted by $f : a \to b$.

If $f : a \to b$ and $g : b \to c$ are morphisms, then a new morphism $g \circ f : a \to c$ is defined which is called a *composition* of f and g.

If $f : a \to b$, $g : b \to c$, and $h : c \to d$, then $h \circ (g \circ h) = (h \circ g) \circ f$, i.e., composition is *associative*.

Finally, for each object $b \in \mathrm{Ob}(\mathcal{C})$, there is an special *identity morphism* $\mathrm{id}_b : b \to b$ such that for any $f : a \to b$ and for any $g : b \to c$, we have $\mathrm{id}_a \circ f = f$ and $g \circ \mathrm{id}_b = g$.

B. Towards relevant examples of categories

This was a general description of categories. We are interested in categories which are relevant for representing knowledge, especially uncertain knowledge about systems. We would like objects of our category to represent pieces of knowledge, and morphisms to represent relations between different pieces of knowledge. Let us give us some examples of possible relevant categories.

C. First example: category of signatures describing knowledge about systems

A system, by definition, consists of interacting and interrelated subsystems. So, to describe a system, we must describe:

- which elements and subsystems it consists of, and
- what are possible interactions between these objects.

How can we describe this is terms of category theory?

- In order to describe all possible elements and subsystems, we must have a *set* S of all possible states of all possible elements and subsystems of the system.
- A systemic interaction between several elements and/or subsystems usually leads to the emergence of a new combined subsystem. Therefore, an interaction between n different elements or subsystems can be described as a mapping which maps these elements (subsystems) into a new subsystem. In mathematical terms, an interaction is thus a function $S^n \to S$; such functions are usually called (n-ary) *operations* on the set S. In each system, we may have different types of interactions; each interaction is described as an operation. Therefore, to describe all possible types of interaction, we must use a set of all operations corresponding to these types.

As a result, a system is represented as a pair (S, Ω) consisting of a set S and of a collection Ω of operations on this set. In mathematical logic and model theory, such a pair is called a *signature*. Therefore, the authors of this notion called the category of such objects the *category of signatures* [3].

How can we define a *morphism* for such a category? In general, speaking informally, a morphism is a mapping which preserves the main properties of the described objects; e.g.:

- in the category of topological spaces, morphisms are mapping which preserve continuity;
- in the category of groups, morphisms (homomorphisms) are mappings which preserve the group operation (multiplication), etc.

In our case, a "morphism" between two systems (S, Ω) and (S', Ω') would mean that we assign to every element or subsystem $s \in S$ of the first system some element or subsystem $f(s) \in S'$ of the second system, in such a way that each operation $\varphi \in \Omega$ of arbitrary arity n from the first system maps into an appropriate operation $g(\varphi) \in \Omega'$ of the second system, appropriate in the sense that:

- its arity is the same ($= n$) as the arity of φ, and
- for every n elements/subsystems $s_1, \ldots, s_n \in S$, the result of applying the new operation $g(\varphi)$ to the images $f(s_i)$ of the objects s_i is the same as the image of the result $\varphi(s_1, \ldots, s_n)$ of applying the original operation φ to s_i.

Formally, a morphism can be thus defined as follows: a morphism $\sigma : (S, \Omega) \to (S', \Omega')$ is a pair $\sigma = (f, g)$, with $f : S \to S'$, $g : \Omega \to \Omega'$ such that for any $\varphi \in \Omega$, $\varphi : S^n \to S$, the following condition holds:

$$(g(\varphi))(f(s_1), \ldots, f(s_n)) = f(\varphi(s_1, \ldots, s_n))$$

for all $(s_1, \ldots, s_n) \in S^n$.

D. Second example: category of signatures can also describe uncertainty

The same general concept of a category of signatures can be used not only to describe elements and subsystems of a system, it can also describe elements of our knowledge. For example:

- as S, we can take the set of possible words of natural language, and
- as Ω, the collection of all logical operations such as "and", "or", etc.

In particular, if we use *fuzzy sets* to describe words from natural language, then logical operations correspond to operations with fuzzy sets. In this case, we get a category of pairs (S, Ω), where S is a class of fuzzy concepts, and Ω is a collection of operations on fuzzy sets [3].

E. Third example: category representation of probabilistic knowledge

A large part of knowledge consists of if-then rules, i.e., of statements of the type "if a then b" ($a \Rightarrow b$). Typically, we are not 100% sure about the universal validity of each rule; different if-then rules may have different "degree of validity", i.e., different "strength". A natural way to describe the strength of a rule is to estimate the portion of cases in which this rule works, or, in other words, to estimate the conditional probability $P(b|a)$ that b is true under the condition that a is true. So, in general, a probabilistic knowledge can be described as a mapping which assigns, to rules of the type $a \Rightarrow b$, the corresponding conditional probability. In general, the set of all possible conditions a and the set of all possible conclusions b can be the same set; however, in many practical cases, these sets are different: e.g., in medical diagnostics, a is a symptom, and b is a diagnosis.

How can we describe the corresponding object in terms of category theory? Probabilities are usually described in terms of a *probability space* $(\Omega, \mathcal{A}, P)$, where Ω is the set of *elementary events*, $\mathcal{A}$ is the set of *events* (i.e., subsets of Ω), and $P : \mathcal{A} \to [0, 1]$ is a function whose value $P(a)$ is called the *probability* of the event a.

In terms of probability space, the set of all conditional probabilities $P(b|a)$ can be described as a *triple* $(\mathcal{A}, r, \mathcal{B})$, where $\mathcal{A}$ and $\mathcal{B}$ are sets of events, and r is a function $r : \mathcal{A} \times \mathcal{B} \to [0, 1]$ which describes conditional probability: $r(a, b) = P(b|a)$. (For $P(b|a)$ to be properly defined, $\mathcal{A}$ and $\mathcal{B}$ must be subspaces of the same probability space.) Therefore, to describe probabilistic uncertainty, it is natural to consider a category whose objects are triples of this type.

F. Fourth example: category representation of fuzzy knowledge

For fuzzy systems, the very meaning of a *fuzzy concept* (like "small") on a set X is that for every element $x \in X$, and for every fuzzy concept a, we only know the *degree* $a(x)$ to which the object x is a particular case of the fuzzy concept a.

This degree is usually represented by a number from the interval $[0, 1]$. Therefore, fuzzy knowledge can be represented as a triple (X, r, A), where X is a set of objects, A is a set of fuzzy concepts, and $r : X \times A \to [0, 1]$ is a function which describes to what extent the object x satisfies the concept a.

Due to this representation, a fuzzy concept a can be described as a function which maps every object $x \in X$ into a number $r(x, a) \in [0, 1]$. So, the set of all possible fuzzy concepts can be described as the set $[0, 1]^X$ of all possible functions from X to $[0, 1]$. Therefore, if we allow all possible fuzzy concepts, we can take $A = [0, 1]^X$; in this case, $r(x, a) = a(x)$.

G. A natural generalization of the fourth example: Chu spaces

In addition to the interval $[0, 1]$, other sets K have been successfully used to represent different fuzzy degrees, e.g., the set K of all subintervals of the interval $[0, 1]$. In this more general case, a fuzzy knowledge can be represented as a triple (X, r, A), where X is a set of objects, A is a set of fuzzy concepts, and $r : X \times A \to K$. Such a triple is called a *Chu space* [1].

It is therefore natural to consider Chu spaces (with a fixed set K) as *objects* of the new category. How can naturally define *morphisms* of this new category? In other words, how can we naturally define a morphism between two K-Chu spaces $\mathbf{A} = (X, r, A)$ and $\mathbf{B} = (Y, s, B)$? Similarly to the category of signatures, we want to map every object $x \in X$ into the corresponding object $f(x) \in Y$. For category of signatures, we also mapped each operation on the first system into an operation on the second system. At first glance, it may seem that in our case, we need a mapping $h : A \to B$ such that if x satisfies a, then $f(x)$ satisfies $h(a)$ with the same satisfaction degree: $r(x, a) = s(f(x), h(a))$. Such a definition would describe, for all elements $f(x)$, which properties b of the type $h(a)$ they have and to what extent: this extent is equal to $s(f(x), b) = r(x, g(b))$, where $g(b)$ is a (partially defined) inverse function to $h(a)$. However, this definition does not say anything about properties $b \in B$ which cannot be represented as $h(a)$ for some $a \in A$. To cover these properties too, it is natural to assume that the function $g(b)$ is defined for *all* properties $b \in B$. This, we arrive at the following definition:

For $\mathbf{A} = (X, r, A)$ and $\mathbf{B} = (Y, s, B)$, a *morphism* $\varphi : \mathbf{A} \to \mathbf{B}$ is a pair of maps (f, g) where $f : X \to Y$ and $g : B \to A$ such that, $\forall x \in X$, $\forall b \in B$, $r(x, g(b)) = s(f(x), b)$.

Composition of Chu morphisms is naturally defined as follows: For $\psi = (f, g) : \mathbf{A} \to \mathbf{B}$, $\varphi = (u, v) : \mathbf{B} \to \mathbf{C} = (Z, t, C)$, $\varphi \circ \psi : \mathbf{A} \to \mathbf{C}$ is defined as $\varphi \circ \psi = (u \circ f, g \circ v)$.

III. Interactions and Fusion Operators

A. Interaction

Interactions among pieces of information are difficult to model in general:

- For random variables, *independence* is easy to characterize, but the *dependency* possesses various forms.
- In Bayesian networks, *conditional probabilities* can be used to model causal effects (including bi-directional causal effects).
- For *fuzzy knowledge*, dependence is even more difficult to describe.

When pieces of information are of more general nature, it seems that a general concept of morphisms in category theory is useful to model their interactions. In particular, the Chu morphisms are attractive because they exhibit bi-directional interaction S.

- If the set of morphisms from $\mathbf{A}$ to $\mathbf{B}$, denoted as $\mathcal{M}(\mathbf{A}, \mathbf{B})$, is empty, we can interpreted if as $\mathbf{A}$ and $\mathbf{B}$ are *non-interactive.*
- When $\mathcal{M}(\mathbf{A}, \mathbf{B}) \neq \emptyset$ or $\mathcal{M}(\mathbf{B}, \mathbf{A}) \neq \emptyset$, $\mathbf{A}$ and $\mathbf{B}$ are *connected.*
- Modeling a *specific interaction* between $\mathbf{A}$ and $\mathbf{B}$ boils down to a search for a morphism, say, from $\mathbf{A} \to \mathbf{B}$, among $\mathcal{M}(\mathbf{A}, \mathbf{B})$.

B. Fusion

Now, within the framework of Category Theory, there exist various operations on objects. In the context of Chu categories, if turns out that we can combine objects while taking into account their interaction.

For example, if $\varphi = (f, g)$ is a morphism from $\mathbf{A} = (X, r, A)$ to $\mathbf{B} = (Y, s, B)$, then the relation $\alpha(x, b) =$

$r(x, g(b)) = s(f(x), b)$ or $X \times B$ realizes a Chu space (X, α, B); this Chu space is denoted by $(X, \alpha, B) = \mathbf{A} \times_\varphi \mathbf{B}$ and called a *cross product* of $\mathbf{A}$ and $\mathbf{B}$.

IV. Examples of Chu Categories

A. Probabilistic System

Take $K = [0, 1]$. To embed probabilistic systems into a Chu category, we proceed as follows. We start out with a category $\mathcal{P}$, where objects $\mathrm{Ob}(\mathcal{P})$ are probability spaces, and morphisms $\mathcal{M}(\mathcal{P})$ are measure-preserving maps. From $\mathcal{P}$, we consider the dual category $\mathcal{P}^*$, where objects $\mathrm{Ob}(\mathcal{P}^*)$ are measure-preserving maps (i.e., $\mathrm{Ob}(\mathcal{P}^*) = \mathcal{M}(\mathcal{P})$), and morphisms $\mathcal{M}(\mathcal{P}^*)$ are defined as follows:

> If $\alpha : (\Omega, \mathcal{A}, P) \to (\Sigma, \mathcal{B}, Q)$, and $\alpha' : (\Omega', \mathcal{A}, P') \to (\Sigma', \mathcal{B}', Q')$, then a morphism in $\mathcal{P}^*$, $\varphi : \alpha \to \alpha'$ is a measure-preserving map $\varphi : (\Sigma, \mathcal{B}, Q) \to (\Omega', \mathcal{A}', P')$.

We embed $\mathcal{P}^*$ into a Chu category by the following (covariant) functor F:

- For $\alpha \in \mathrm{Ob}(\mathcal{P}^*)$, $\alpha : (\Omega, \mathcal{A}, \mathcal{P}) \to (\Sigma, \mathcal{B}, Q)$, define $F(\alpha)$ to be the Chu space $(\mathcal{A}, \alpha^*, \mathcal{B})$, where $\alpha^*(a, b) = P(a|\alpha^{-1}(b))$.
- For $\varphi : \alpha \to \alpha'$ in $\mathcal{M}(\mathcal{P}^*)$, define
$$F(\varphi) = (\varphi \circ \alpha, (\alpha' \circ \varphi)^{-1}).$$

B. Fuzzy systems

Take again $K = [0, 1]$. Objects of the category $\mathcal{FUZZ}$ will be of the form $(X, r, [0, 1]^Y)$ where X, Y are sets (e.g., input and output spaces of a fuzzy controller, respectively) and r is a mapping $r : X \times [0, 1]^Y \to [0, 1]$.

Let $\mathcal{SET}$ denote the category of ordinary sets in which morphisms are arbitrary maps. Consider the category $\mathcal{SET}^*$, where objects $\mathrm{Ob}(\mathcal{SET}^*)$ are maps between sets, and morphisms $\mathcal{M}(\mathcal{SET}^*)$ are defined as follows:

> For $\alpha : X \to Y$, $\alpha' : X' \to Y'$, define $\varphi : \alpha \to \alpha'$ as a map $\varphi : Y \to X'$.

The category $\mathcal{FUZZ}$ is the image of the category $\mathcal{SET}^*$ under the (covariant) functor G defined below:

- For $\alpha \in \mathrm{Ob}(\mathcal{SET}^*)$, $\alpha : X \to Y$, $G(\alpha) = (X, e_\alpha, [0, 1]^Y)$, where $e_\alpha : X \times [0, 1]^Y \to [0, 1]$ is defined as $e_\alpha(x, b) = b(\alpha(x))$.
- For $\varphi : \alpha \to \alpha'$ in $\mathcal{M}(\mathcal{SET}^*)$,
$$G(\varphi) = (\varphi \circ \alpha, \varphi^* \circ (\alpha')^*),$$
where, e.g., $\varphi^* : [0, 1]^{X'} \to [0, 1]^Y$ is defined as $\varphi^*(a)(y) = a(\varphi(y))$.

V. Conclusions

We propose to use a specific type of categories, known as Chu categories, to model and fuse information. The rationale for this approach is as follows:

i) Chu morphisms are suitable for modeling interactions among pieces of information.

ii) New fusion operators can be derived in the framework of Chu categories.

iii) Chu spaces are very general and hence represent various types of knowledge in a single mathematical setting, suitable for analysis and inference.

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