Complex Fuzzy Sets: Towards New Foundations

Hung T. Nguyen[1], Abraham Kandel[2], and Vladik Kreinovich[3]

[1]Department of Mathematical Sciences
New Mexico State University
Las Cruces, NM 88003, USA
email hunguyen@nmsu.edu

[2]Department of Computer Science and Engineering
University of South Florida
4202 East Fowler Avenue
Tampa, FL 33620-5399, USA

[3]Department of Computer Science
University of Texas at El Paso
El Paso, TX 79968, USA
email vladik@cs.utep.edu

Abstract

Uncertainty of complex-valued physical quantities $z = x + iy$ can be described by complex fuzzy sets. Traditionally, such sets have been described by membership functions $\mu(x, y)$ which map the universe of discourse (complex plane) into the interval $[0, 1]$. The problem with this description is that it is difficult to directly translate into words from natural language. To make this translation easier, several authors have proposed to use, instead of a single membership function for describing the complex number, several membership functions which describe different real-valued characteristics of this numbers, such as its real part, its imaginary part, its absolute value, etc. The quality of this new description strongly depends on the choice of these real-valued functions, so it is important to choose them optimally. In this paper, we formulate the problem of optimal choice of these functions and show that, for all reasonable optimality criteria, the level sets of optimal functions are straight lines and circles. This theoretical result is in good accordance with our numerical experiments, according to which such functions indeed lead to a good description of complex fuzzy sets.

1 Many Practical Problems Lead to Complex Fuzzy Sets

Many physical quantities are complex-valued: wave function in quantum mechanics, complex amplitude and impedance in electrical engineering, etc.

In all these problems, expert uncertainty means that we do not know the exact value of the corresponding complex number; instead, we have a fuzzy knowledge about this number.

2 Membership Function Description and Why It Is Not Sufficient

A natural way to represent fuzzy knowledge about a complex number $z = x + iy$ is to use a membership function $\mu(x, y)$ which maps a complex plane C into the interval $[0, 1]$. This approach was proposed and developed in [1].

From the purely mathematical viewpoint, this approach is very natural. However, there is one problem with this approach: A membership function is not something which is natural for a human to understand and to use. It was invented as a way of representing human fuzzy knowledge in a language which is understandable for a computer. From this viewpoint, after we get the desired membership function, we must perform one more step: we must translate it into the natural language.

This translation is relatively easy for real variables, because we have accumulated a lot of intuition, and therefore, we have a lot of words describing different membership functions, such as "small", "close to 0", etc. Unfortunately, complex numbers are much less intuitive, and there are very few terms of natural language which can be naturally used to described the knowledge about complex numbers.

3 A New Approach to Describing Complex Fuzzy Sets

Since the original membership function $\mu : C \to [0,1]$ is difficult to interpret directly, the authors of [2] proposed a new approach. In this approach, to describe a fuzzy knowledge about a complex number, we use, instead of *single* membership function which describes the number $z = x + \mathrm{i}y$, *several* (two or more) membership functions which describe real-valued quantities which are *functions* of this complex number, such as its real part $\mathrm{Re}(z)$, its imaginary part $\mathrm{Im}(z)$, its absolute value $|z|$, its phase ϕ, etc.

4 Which Functions Should We Use?

As shown in [2], the efficiency of the new description in solving practical problems with complex numbers strongly depends on the appropriate choice of the real-valued functions which are used to describe the corresponding fuzzy set: a good choice can drastically improve the quality of the result. It is therefore important to find out which functions are the best here. This is the problem that we will be solving in the present paper.

5 Preliminary Step: Reformulation in Terms of Sets

The membership function $\mu_f(t)$ corresponding to a function $f : C \to R$ can be described by the extension principle:

$$\mu_f(t) = \max_{z:\ f(z)=t} \mu(t).$$

Thus, to be able to compute all the values $\mu_f(t)$, we do not need to compute know the exact functions $f(t)$; it is sufficient to be able to describe their level sets $\{z \mid f(z) = t\}$. So, instead of choosing functions, we can choose a family of sets.

For the above functions, these level sets are straight lines and circles.

Of course, the more parameters we allow in the description of a family, the more elements this family contains, and therefore, the better the representation. So, the question can be reformulated as follows: for a given number of parameters (i.e., for a given dimension of approximating family), which is the best family? In this paper, we formalize and solve this problem.

6 Formalizing the Problem

All proposed families of sets have analytical (or piece-wise analytical) boundaries, so it is natural to restrict ourselves to such families. By definition, when we say that a piece of a boundary is analytical, we mean that it can be described by an equation $F(x,y) = 0$ for some analytical function

$$F(x,y) = a + b \cdot x + c \cdot y + d \cdot x^2 + e \cdot x \cdot y + f \cdot y^2 + \ldots$$

So, in order to describe a family, we must describe the corresponding class of analytical functions $F(x,y)$.

Since we are interested in finite-dimensional families of sets, it is natural to consider finite-dimensional families of functions, i.e., families of the type

$$\{C_1 \cdot F_1(x,y) + \ldots + C_d \cdot F_d(x,y)\},$$

where $F_i(z)$ are given analytical functions, and $C_1, \ldots, C_d$ are arbitrary (real) constants. So, the question is: which of such families is the best?

When we say "the best", we mean that on the set of all such families, there must be a relation $\geq$ describing which family is better or equal in quality. This relation must be transitive (if A is better than B, and B is better than C, then A is better than C). This relation is not necessarily asymmetric, because we can have two approximating families of the same quality. However, we would like to require that this relation be *final* in the sense that it should define a unique *best* family $A_{\rm opt}$ (i.e., the unique family for which $\forall B\,(A_{\rm opt}\ge B)$. Indeed:

- If none of the families is the best, then this criterion is of no use, so there should be *at least one* optimal family.
- If *several* different families are equally best, then we can use this ambiguity to optimize something else: e.g., if we have two families with the same approximating quality, then we choose the one which is easier to compute. As a result, the original criterion was not final: we get a new criterion ($A\ge_{\rm new} B$ if either A gives a better approximation, or if $A\sim_{\rm old} B$ and A is easier to compute), for which the class of optimal families is narrower. We can repeat this procedure until we get a final criterion for which there is only one optimal family.

It is reasonable to require that the relation $A\ge B$ should not change if we add or multiply all elements of A and B by a complex number; in geometric terms, the relation $A\ge B$ should be shift-, rotation- and scale-invariant.

Now, we are ready for the formal definitions.

7 Definitions and the Main Result

Definition 1. *Let $d>0$ be an integer. By a d-dimensional family, we mean a family A of all functions of the type $\{C_1\cdot F_1(x,y)+\ldots+C_d\cdot F_d(x,y)\}$, where $F_i(z)$ are given analytical functions, and $C_1,\ldots,C_d$ are arbitrary (real) constants. We say that a set is defined by this family A if its border consists of pieces described by equations $F(x,y)=0$, with $F\in A$.*

Definition 2. *By an optimality criterion, we mean a transitive relation $\ge$ on the set of all d-dimensional families. We say that a criterion is final if there exists one and only one optimal family, i.e., a family $A_{\rm opt}$ for which $\forall B\,(A_{\rm opt}\ge B)$. We say that a criterion $\ge$ is shift- (corr., rotation- and scale-invariant) if for every two families A and B, $A\ge B$ implies $TA\ge TB$, where TA is a shift (rotation, scaling) of the family A.*

Theorem. ($d\le 4$) *For every final optimality criterion $\ge$ which is shift-, rotation-, and scale-invariant, the border of every set defined by the optimal family $A_{\rm opt}$ consists of straight line intervals and circular arcs.*

Comment. This result is in good accordance with experiments described in [2], according to which such sets indeed provide a good description of complex fuzzy sets.

8 Proof of the Theorem

This proof is similar to the ones from [3].

1. Let us first show that the optimal family $A_{\rm opt}$ is itself shift-, rotation-, and scale-invariant.

Indeed, let T be an arbitrary shift, rotation, or scaling. Since $A_{\rm opt}$ is optimal, for every other family B, we have $A_{\rm opt}\ge T^{-1}B$ (where T^{-1} means the inverse transformation). Since the optimality criterion $\ge$ is invariant, we conclude that $TA_{\rm opt}\ge T(T^{-1}B)=B$. Since this is true for every family B, the family $TA_{\rm opt}$ is also optimal. But since our criterion is final, there is only one optimal family and therefore, $TA_{\rm opt}=A_{\rm opt}$. In other words, the optimal family is indeed invariant.

2. Let us now show that all functions from $A_{\rm opt}$ are polynomials.

Indeed, every function $F\in A_{\rm opt}$ is analytical, i.e., can be represented as a Taylor series (sum of monomials). Let us combine together monomials $c\cdot x^a\cdot y^b$ of the same degree $a+b$; then we get

$$F(z)=F_0(z)+F_1(z)+\ldots+F_k(z)+\ldots,$$

where $F_k(z)$ is the sum of all monomials of degree k. Let us show, by induction over k, that for every k, the function $F_k(z)$ also belongs to $A_{\rm opt}$.

Let us first prove that $F_0(z)\in A_{\rm opt}$. Since the family $A_{\rm opt}$ is scale-invariant, we conclude that for every $\lambda>0$, the function $F(\lambda\cdot z)$ also belongs to $A_{\rm opt}$. For each term $F_k(z)$, we have $F_k(\lambda z)=\lambda^k\cdot F_k(z)$, so

$$F(\lambda z)=F_0(z)+\lambda\cdot F_1(z)+\ldots\in A_{\rm opt}.$$

When $\lambda\to 0$, we get $F(\lambda\cdot z)\to F_0(z)$. The family $A_{\rm opt}$ is finite-dimensional hence closed; so, the limit $F_0(z)$ also belongs to $A_{\rm opt}$. The induction base is proven.

Let us now suppose that we have already proven that for all $k < s$, $F_k(z) \in A_{\rm opt}$. Let us prove that $F_s(z) \in A_{\rm opt}$. For that, let us take

$$G(z) = F(z) - F_1(z) - \ldots - F_{s-1}(z).$$

We already know that $F_1, \ldots, F_{s-1} \in A_{\rm opt}$; so, since $A_{\rm opt}$ is a linear space, we conclude that

$$G(z) = F_s(z) + F_{s+1}(z) + \ldots \in A_{\rm opt}.$$

The family $A_{\rm opt}$ is scale-invariant, so, for every $\lambda > 0$, the function

$$G(\lambda \cdot z) = \lambda^s \cdot F_s(z) + \lambda^{s+1} \cdot F_{s+1}(z) + \ldots$$

also belongs to $A_{\rm opt}$. Since $A_{\rm opt}$ is a linear space, the function

$$H_\lambda(z) = \lambda^{-s} \cdot G(\lambda z) = F_s(z) + \lambda \cdot F_{s+1}(z) + \lambda^2 \cdot F_{s+2}(z) + \ldots$$

also belongs to $A_{\rm opt}$.

When $\lambda \to 0$, we get $H_\lambda(z) \to F_s(z)$. The family $A_{\rm opt}$ is finite-dimensional hence closed; so, the limit $F_s(z)$ also belongs to $A_{\rm opt}$. The induction is proven.

Now, monomials of different degree are linearly independent; therefore, if we have infinitely many non-zero terms $F_k(z)$, we would have infinitely many linearly independent functions in a finite-dimensional family $A_{\rm opt}$ – a contradiction. Thus, only finitely many monomials $F_k(z)$ are different from 0, and so, $F(z)$ is a sum of finitely many monomials, i.e., a polynomial.

3. Let us prove that if a function $F(x,y)$ belongs to $A_{\rm opt}$, then its partial derivatives $F_{,x}(x,y)$ and $F_{,y}(x,y)$ also belong to $A_{\rm opt}$.

Indeed, since the family $A_{\rm opt}$ is shift-invariant, for every $h > 0$, we get $F(x+h,y) \in A_{\rm opt}$. Since this family is a linear space, we conclude that a linear combination $h^{-1}(F(x+h,y) - F(x,y))$ of two functions from $A_{\rm opt}$ also belongs to $A_{\rm opt}$. Since the family $A_{\rm opt}$ is finite-dimensional, it is closed and therefore, the limit $F_{,x}(x,y)$ of such linear combinations also belongs to $A_{\rm opt}$. (For $F_{,y}$, the proof is similar).

4. Due to Parts 2 and 3 of this proof, if any polynomial from $A_{\rm opt}$ has a non-zero part F_k of degree $k > 0$, then it also has a non-zero part ($(F_k)_{,x}$ or $(F_k)_{,y}$) of degree $k-1$. Similarly, it has non-zero parts of degrees $k-2, \ldots, 1, 0$.

So, in all cases, $A_{\rm opt}$ contains a non-zero constant and a non-zero linear function $F_1(x,y) = b \cdot x + c \cdot y$. We can now use the fact that the family $A_{\rm opt}$ is rotation-invariant; let T be a rotation which transforms (b,c) into the x-axis, then we conclude that

$$F_1(Tz) = b'x \in A_{\rm opt},$$

and hence $x \in A_{\rm opt}$. Similarly, $y \in A_{\rm opt}$. So, the family $A_{\rm opt}$ contains at least 3 linearly independent functions: a non-zero constant, x, and y.

If $d = 3$, then the 3-D family $A_{\rm opt}$ cannot contain anything else, and all the pieces of borders $F(x,y) = 0$ of all the sets defined by this family are straight lines.

If $d = 4$, then we cannot have any cubic or higher order terms in $A_{\rm opt}$, because then, due to Part 3, we would have both this cubic part *and* a (linearly independent) quadratic part, and the total dimension of $A_{\rm opt}$ would be at least $3 + 2 = 5$. So, all functions from $A_{\rm opt}$ are quadratic. Since $\dim(A_{\rm opt}) = 4$, and the dimension of 0- and 1-D parts is 3, the dimension of possible parts of second degree is 1. Since $A_{\rm opt}$ is rotation-invariant, the quadratic part $d \cdot x^2 + e \cdot x \cdot y + f \cdot y^2$ must be also rotation-invariant (else, we would have two linearly independent quadratic terms in $A_{\rm opt}$: the original expression and its rotated version). Thus, this quadratic part must be proportional to $x^2 + y^2$.

Hence, every function $F \in A_{\rm opt}$ has the form

$$F(x,y) = a + b \cdot x + c \cdot y + d \cdot (x^2 + y^2),$$

and therefore, all the pieces of borders $F(x,y) = 0$ of all the sets defined by this family are either straight lines or circular arcs. The proposition is proven.

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