Intervals is All We Need: An Argument

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***Abstract*— In many practical applications of fuzzy methodology, it is desirable to go beyond the interval $[0, 1]$ and to consider more general fuzzy values: e.g., intervals, or more general sets of values. In this paper, we show that under some reasonable assumptions, there is no need to go beyond intervals.**

I. Introduction

The ultimate goal of using fuzzy logic is to make decisions. It is therefore desirable to be able, given an alternative A, to decide whether A is preferable, or its negation $\neg A$ is preferable, or whether A and $\neg A$ are equivalent to each other.

This possibility is there if we use *real numbers* to describe uncertainty: indeed, in this case, both the degree of belief a in A and $\neg a = 1 - a$ in $\neg A$ are real numbers, and for every two real numbers, either the first is larger, or the second is larger, or they are equal to each other.

In some applications, it is desirable to use not only real numbers but also *subintervals* of the interval [0.1] to describe uncertainty; see, e.g., [1, 2, 3, 4, 5] and references therein. For *intervals*, the fact that we can always compare A and $\neg A$ is a little bit less trivial but still true: Indeed, if we use intervals as degrees of belief, with the relation $[a^-, a^+] \le [b^-, b^+]$ if and only if $a^- \ge b^-$ and $a^+ \ge b^+$ and $\neg[a^-, a^+] = [1 - a^+, 1 - a^-]$, then $[a^-, a^+] \ge \neg[a^-, a^+]$ if and only if $a^- \ge 1 - a^+$ and $a^+ \ge 1 - a^-$. Both inequalities are equivalent to $a^- + a^+ \ge 1$. Similarly, $[a^-, a^+] \le \neg[a^-, a^+]$ if and only if $a^- + a^+ \le 1$. So, the result of comparing A and $\neg A$ depends on the comparison between $a^- + a^+$ and 1:

- if $a^- + a^+ > 1$, we have $A > \neg A$;
- if $a^- + a^+ < 1$, we have $A < \neg A$;
- if $a^- + a^+ = 1$, we have $A \sim \neg A$;

Thus, for subintervals of the interval $[0, 1]$, we can always decide whether A is better than $\neg A$, or $\neg A$ is better than A, or A and $\neg A$ are of equal quality.

As we have mentioned, it is sometimes desirable to go from a linearly ordered set $[0, 1]$ to a more general partially ordered set V of truth values. Of course, the set of intervals is, by itself, not linearly ordered. However, we may go one step further, and consider intervals over intervals, etc. A natural question is: if this further step necessary? Will the property of comparison between A and $\neg A$ hold for intervals over intervals? Or, even more generally, for what sets V will the similar property hold for "intervals" in V? In this paper, we show that if we want to preserve the comparison property, then there is no need to go beyond intervals: Intervals is all we need.

II. Definitions and the Main Result

We want to consider "intervals" over arbitrary sets V. How can we define such intervals I? The set I describes possible values of degree of truth. Therefore, a natural requirement on I is that if two values $a < b$ are possible, i.e., if for some statement S, it is possible that the actual degree of belief in S is a, and it is also possible that the actual degree of belief in S is larger (and equal to b), then all intermediate values c (i.e., values for which $a < c < b$) are possible as well. Thus, we arrive at the following definitions:

Definition 1. *By a set of values, we mean an ordered set V with an operation $\neg : V \to V$ which satisfies the following two properties:*

- *$\neg a \ge \neg b$ if and only if $a \le b$, and*
- *$\neg(\neg a) = a$ for all a.*

Definition 2. *Let V be a set of values. By an interval $I \subseteq V$ we mean a subset of V which has the following property: for every a, b, and c, if $a < c < b$, $a \in I$, and $b \in I$, then $c \in I$.*

Standard intervals are a particular case of this definition:

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Definition 3. *By a proper interval, we mean a set* $[a,b] = \{v \mid a \le v \le b\}$ *for some* $a > b$.

Proposition 1. *If* $a \le b$*, then every proper interval is an interval (in the sense of Definition 2).*

For *linearly ordered* sets of values, this definition leads exactly to intervals; to be more precise, it leads to intervals with endpoints in limit points which are not necessarily in V (e.g., for rational numbers we may get the set of all rational numbers from 0 to $\sqrt{2}$):

Definition 4. *Let* V *be a linearly ordered set. For each point* $v \in V$*, we define*

$$Q^-(v) = \{w \in V \mid w \le v\}.$$

By a limit point, we mean a set Q *which contains, with each point* v*, all smaller points. For convenience, we will associate each point* v *with a limit point* $Q^-(v)$*. If* Q *and* Q' *are two limit points, we say that* $Q \le Q'$ *if* $Q \subseteq Q'$*. The set of all limit points will be denoted by* V^*.

It is easy to check that for elements from V, $Q^-(v) \le Q^-(w)$ if and only if $v \le w$, i.e., that the order on the set V^* of all limit points extends the original order on V. It is also known that for real numbers, every limit point is either equal to $(-\infty, a)$, or to $(-\infty, a]$, or to $(-\infty, \infty)$.

Proposition 2. *If the set of values* V *is linearly ordered, then:*

- *every interval* I *(in the sense of Definition 2) has the form* $\{v \mid a \le v \le b\}$ *for some limit points* $a, b \in V^*$*, and*
- *for every two limit points* $a, b \in V^*$*, the set* $\{v \mid a \le v \le b\}$ *is an interval (in the sense of Definition 2).*

How do we compare two intervals? Since both sets I and J represent possible sets of degrees in belief in some statement A and B, the meaning of $I \ge J$ is as follows:

- whatever possible value $a \in I$ we take, it is possible that it is larger than (or equal to) the degree of belief in B, i.e., $a \ge b$ for some value $b \in J$;
- similarly, whatever possible value $b \in J$ we take, it is possible that it is smaller than (or equal to) the degree of belief in B, i.e., $a \ge b$ for some value $a \in I$.

So, we arrive at the following definition:

Definition 5. *Let* I *and* J *be intervals. We say that* $I \ge J$ *if and only if* $\forall a_{a \in I} \exists b_{b \in J} \, (a \ge b)$ *and* $\forall b_{b \in J} \exists a_{a \in I} \, (a \ge b)$.

For proper intervals, this relation coincides with the standard one:

Proposition 3. *For proper intervals* $A = [a^-, a^+]$ *and* $B = [b^-, b^+]$*,* $A \le B$ *if and only if* $a^- \le b^-$ *and* $a^+ \le b^+$.

Negation for intervals is defined element-wise:

Definition 6. *For every interval* I*, its negation* $\neg I$ *is defined as* $\neg I = \{\neg v \mid v \in I\}$.

For proper intervals, this operation coincides with the standard one:

Proposition 4. *For a proper interval* $A = [a^-, a^+]$*,* $\neg A = [\neg a^+, \neg a^-]$.

Now, we are ready to formulate the main result of this paper:

Definition 7. *We say that an element* $a \in V$ *is a midpoint if* $a = \neg a$*. We say that a set* V *is almost linearly ordered if for every two elements* $a, b \in V$ *which are not midpoints,* $a \le b$ *or* $b \le a$.

In other words, an almost linearly ordered set is a linearly ordered set in which, instead of a single element a with $a = \neg a$, we may have several elements with this same property; these elements may be unrelated to each other or to other elements from V, but all other elements are linearly ordered.

To give an example of an almost linearly ordered set, we can take an interval $[0, 1]$ and replace the point 0.5 by many points $a = (0.5, x)$ for all of which $a = \neg a$, so that $\alpha < (0.5, x) < \beta$ for all $\alpha < 0.5 < \beta$, and $(0.5, x) \parallel (0.5, y)$ for $x \ne y$ (where $a \parallel b$ means $a \not\le b$ and $b \not\le a$).

Theorem 1. *For every set of values* V*, the following two conditions are equivalent to each other:*

- *The set* V *is almost linearly ordered.*
- *For every interval* A*, either* $A \ge \neg A$*, or* $\neg A \ge A$.

In other words, if we want to be always able to compare A with $\neg A$, then the underlying set V should be almost linearly ordered.

III. First Auxiliary Result: Pairs OK, Triples Not

In particular, we should not have intervals over intervals (i.e., quadruples). It turns out that we cannot even consider *triples*:

Definition 8. *Let* V *be a set of values. By the corresponding set of triples, we mean the set* V_3 *of all triples* (v_1, v_2, v_3) *with* $v_1 \le v_2 \le v_3$ *with a component-wise order* $(v_1, v_2, v_3) \le (v_1', v_2', v_3')$ *if and only if* $v_1 \le v_1'$*,* $v_2 \le v_2'$*, and* $v_3 \le v_3'$*, and the component-wise negation* $\neg(v_1, v_2, v_3) = (\neg v_3, \neg v_2, \neg v_1)$.

Proposition 5. *Let* V *be a linearly ordered set of values with at least four elements. Then, there exist a triple* $A \in V_3$ *for which neither* $A \le \neg A$ *nor* $\neg A \le A$.

IV. Second Auxiliary Result: In Our Proof, We Need All Intervals, Not Only Proper Ones

The following counterexample shows that to conclude that the set is almost linearly ordered, we must require the above property for *all* intervals A, not just for all *proper* intervals:

Proposition 6. *There exists a set of values V which is not almost linearly ordered, and for which for every proper interval A, either $A \le \neg A$, or $\neg A \le A$.*

V. Third Auxiliary Result: We Want Logic To Be a Lattice

There is another reason for V to be linearly ordered: normally, we consider *lattices*, i.e., ordered sets with operations $\vee$ and $\wedge$ (corresponding to logical "or" and "and"), with the properties that $a \vee a = a$, $a \vee b \ge a$, $a \wedge b \le a$, etc. These lattice operations are easy to generalize to intervals: if we do not know the exact degree of belief a in a statement S, we only know the set A of possible values of degree, and we also know the set B of possible values of degree of belief b in some other statement T, then, naturally, we want to define the set of possible degrees of belief in $S \vee T$ as the set $A \vee B$ of all possible values $a \vee b$ with $a \in A$ and $b \in B$. It is natural to expect that these intervals should also form a lattice. It turns out that this is only true for linearly ordered sets V:

Definition 9.

- *By a logic, we mean a set of values V with two lattice operations $\vee$ and $\wedge$.*
- *Let V be a logic. For every two intervals $A, B \subseteq V$, we define $A \vee B = \{a \vee b \,|\, a \in A, b \in B\}$ and $A \wedge B = \{a \wedge b \,|\, a \in A, b \in B\}$.*

Proposition 7. *Let V be a logic. If the set of intervals over V forms a lattice, then V is linearly ordered.*

It turns out that V is linearly ordered even if we simply require that the logical combination of two intervals be an interval:

Definition 10. *A logic is called continuous if for every $a < b$, there exists a $c \in V$ for which $a < c < b$.*

Proposition 8. *Let V be a continuous logic for which, for every two intervals A and B, the sets $A \vee B$ and $A \wedge B$ are also intervals. Then, V is linearly ordered.*

VI. Proofs

Proof of Theorem 1. If the set V is linearly ordered, then every interval A in the sense of Definition 2 is an interval in a usual sense. For such intervals, $[a^-, a^+] \le [b^-, b^+]$ in the sense of Definition 4 if and only if $a^- \le b^-$ and $a^+ \le b^+$, and $\neg[a^-, a^+] = [\neg a^+, \neg a^-]$. Therefore, $A = [a^-, a^+] \le \neg A = [\neg a^+, \neg a^-]$ if and only if $a^- \le \neg a^+$ and $a^+ \le \neg a^-$. Due to Definition 1, these two inequalities are equivalent. Therefore, if $a^- \le \neg a^+$, we have $a^+ \le \neg a^-$ and $A \le \neg A$. Similarly, if $a^- \ge \neg a^+$, then $A \ge \neg A$. Since V is linearly ordered, we either have $a^- \le \neg a^+$ (and $A \le \neg A$), or $a^- \ge \neg a^+$ (and $A \ge \neg A$).

Similarly, one can prove that the same is true for *almost* linearly ordered sets V. The first part of the theorem is proven.

To complete the proof of the theorem, let us now assume that for every interval A over a set V, we have either $A \le \neg A$ or $\neg A \le A$. From this assumption, we want to conclude that the original set V is almost linearly ordered, i.e., if $a \parallel b$, then either $a = \neg a$, or $b = \neg b$.

Let us first consider the one-point (degenerate) interval $A = \{a\}$. For this interval, the above property of V means that either $a \le \neg a$ or $a \ge \neg a$. Similarly, we can conclude that either $b \le \neg b$ or $\neg b \le b$. So, to prove that either $a = \neg a$ or $b = \neg b$, we must therefore exclude the following four possibilities:

- $a < \neg a$ and $b > \neg b$;
- $a < \neg a$ and $b < \neg b$;
- $a > \neg a$ and $b > \neg b$;
- $a > \neg a$ and $b < \neg b$.

Let us show that in the first case ($a < \neg a$ and $b > \neg b$), we get a contradiction. Indeed, let us now consider the set $A = \{a, b\}$ (it is easy to see that since $a \parallel b$, this set is also an interval). For this set, either $A \le \neg A$ or $\neg A \le A$.

- If $A = \{a, b\} \le \neg A = \{\neg b, \neg a\}$, this means, in particular, that for $b \in A$, there exists an element $c \in \neg A$ for which $b \le c$. Since $b > \neg b$, we cannot have $c = \neg b$ and therefore, $c = \neg a$ and $b \le \neg a$.
- Similarly, if $A = \{a, b\} \ge \neg A = \{\neg b, \neg a\}$, this means, in particular, that for $a \in A$, there exists an element $c \in \neg A$ for which $a \ge c$. Since $a < \neg a$, we cannot have $c = \neg a$ and therefore, $c = \neg b$ and $a \ge \neg b$.

So, either $b \le \neg a$ or $a \le \neg b$.

To be able to exclude these possibilities, let us show that if $a \parallel b$, then $a \parallel \neg b$. We will prove this by reduction to a contradiction. If $a \not\parallel \neg b$, this means that either $a \le \neg b$ or $\neg b \le a$.

- In the first case $a \le \neg b$, a proper interval $A = [a, \neg b]$ is an interval and therefore, either $A \le \neg A$ or $\neg A \le A$.

– If $A = [a, \neg b] \leq \neg A = [b, \neg a]$, then, due to Proposition 4, we have $a \leq b$, which contradicts to the assumption that $a \parallel b$.

– If $A = [a, \neg b] \geq \neg A = [b, \neg a]$, then, due to Proposition 4, we have $a \geq b$, which also contradicts to the assumption that $a \parallel b$.

So, this case is impossible.

- The second case $\neg b \leq a$ is similarly impossible.

In both cases, we get a contradiction and therefore, $a \parallel \neg b$.

So, if $a \parallel b$, we cannot have $a < \neg a$ and $b > \neg b$. If $a < \neg a$ and $b < \neg b$, then we can take $b' = \neg b$. As we have just proven, $a \parallel b$ implies $a \parallel b'$, and for the new elements a and b', we have $a < \neg a$ and $b' > \neg b'$. We have already shown that this is impossible. Similarly, we can show that the cases ($a > \neg a$ and $b > \neg b$) and ($a > \neg a$ and $b < \neg b$) are also impossible. The theorem is proven.

Proof of Proposition 5. Let us first show that in a linearly ordered set V, there is at most one midpoint element m, i.e., at most one element m with $m = \neg m$. Indeed, if $n \neq m$, then either $n < m$ and then $\neg n > \neg m = m > n$, or $n > m$ and then $\neg n < \neg m = m < n$. In both cases, $n \neq \neg n$.

Since a set V can only have one midpoint, and the set V has at least four elements, it must have an element n which is not a midpoint. If $\neg n < n$, then we can take $n' = \neg n$ and get $n' < \neg n'$. Therefore, without losing generality, we can assume that $n < \neg n$.

Since the set V has at least four elements, it must have at least one element p in addition to the (two or) three elements n, $\neg n$, and maybe m. Since $p \neq m$, this p cannot be a midpoint and therefore, $p \neq \neg p$. Similarly to the above case, without losing generality, we can assume that $p < \neg p$.

Now, we have two different points n and p for which $n < \neg n$ and $p < \neg p$. Since $n \neq p$ and the set V is linearly ordered, we have $n < p$ or $n > p$. If $n > p$, then we can switch n and p and therefore, without losing generality, we can assume that $n < p$. So, we have $n < p < \neg p < \neg n$. Let us now take $A = (n, \neg p, \neg p)$. Then, $\neg A = (p, p, \neg n)$. Here, $n < p$ and $\neg p > p$, so we cannot have neither $A \leq \neg A$ nor $\neg A \leq A$. The proposition is proven.

Proof of Proposition 6. Let us take, as V, the set of all pairs (a, s), where $a \in [0,1]$ and $s \in \{-,+\}$, with the order $(a_1, s_1) \leq (a_2, s_2)$ if and only if $s_1 = s_2$ and $a_1 \leq a_2$. Then, every proper interval A is an interval within one of the linearly ordered components $[0,1] \times \{-\}$ or $[0,1] \times \{+\}$, and therefore, $A \leq \neg A$ or $\neg A \leq A$. On the other hand, we have $(a_1, -) \parallel (a_2, +)$ for any a_1 and a_2 and therefore, the set V is not almost linearly ordered. The proposition is proven.

Proof of Propositions 7 and 8. We will prove these proposition by reduction to a contradiction. Let V be not linearly ordered. This means that there exist $a, b \in V$ for which $a \parallel b$. Then, $A = \{a, b\}$ is an interval. For this interval,

$$A \vee A = \{a \vee a = a, a \vee b, b \vee b = b\}.$$

By definition of a lattice, $a \vee b \geq a$ and $a \vee b \geq b$. Since $a \parallel b$, we cannot have $a \vee b = b$, because else, $b \geq a$. Thus, $a \vee b > a$ and $a \vee b > b$.

In the proof of Proposition 7, we now conclude that $a \vee b \neq a$, $a \vee b \neq b$, and therefore, $A \vee A \neq A$, which contradicts to our assumption that the set of all intervals is a lattice. This contradiction proves that $a \parallel b$ is impossible and hence, the set V is linearly ordered. The proposition is proven.

In the proof of Proposition 8, we conclude that due to continuity of the logic V, there exists an intermediate value c such that $a \vee b > c > b$. Since $c \notin A \vee A$, this contradicts to our assumption that the set $A \vee A$ is an interval. This contradiction proves that $a \parallel b$ is impossible and hence, the set V is linearly ordered. The proposition is proven.

Acknowledgments. This work was supported in part by NASA under cooperative agreement NCC5-209, and by the Hong Kong UGC Direct Grant 2050148.

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