A Comment on the Shape of the Solution Set for Systems of Interval Linear Equations with Dependent Coefficients

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Abstract

This article is a short supplement to our previously published paper, in which we proved that each semialgebraic set can be represented as a projection of a solution set of some system of interval linear equations with dependent coefficients. The new result says that interval occurring can be chosen as narrow as wanted. The new result is proved by a simple linear transformation.

In our paper [1], we considered *systems of interval linear equations with dependent coefficients*, i.e., systems of the type

$$\sum_{j=1}^{n} a_{ij} x_j = b_i, \tag{1}$$

where

$$a_{ij} = a_{ij}^{(0)} + \sum_{\alpha=1}^{p} a_{ij\alpha} f_\alpha, \tag{2}$$

$$b_i = b_i^{(0)} + \sum_{\alpha=1}^{p} b_{i\alpha} f_\alpha, \tag{3}$$

$a_{ij}^{(0)}$, $a_{ij\alpha}$, $b_i^{(0)}$, and $b_{i\alpha}$ are given real numbers, ($1 \le i \le m$, $1 \le j \le n$, $1 \le \alpha \le p$), and coefficients f_α can take arbitrary values from the given intervals $\mathbf{f}_\alpha$. These systems are common in practice, when due to measurement uncertainty, we do not know the exact values of the coefficients of linear equations. By a *solution set* of such a system, we mean the set of all solution corresponding to different values $f_\alpha \in \mathbf{f}_\alpha$.

In [1], we described the shape of the solution set. Namely, we showed that each solution set is *semialgebraic*, i.e., it can be represented as a finite union of subsets, each of which is defined by a finite system of polynomial equations $P_r(x_1, \ldots, x_q) = 0$ and inequalities of the types $P_s(x_1, \ldots, x_q) > 0$ and $P_t(x_1, \ldots, x_q) \ge 0$ (for some polynomials P_i). We also showed that for every subset $I = \{i_1, \ldots, i_q\} \subset \{1, \ldots, n\}$, the corresponding *projection* of a solution set, i.e., the set of all vectors $(x_{i_1}, \ldots, x_{i_q}) \in I\!R^q$ that can be extended to a solution $(x_1, \ldots, x_n)$ of a system, is also semialgebraic, and that, vice versa, every semialgebraic set can be represented as a projection of the solution set of some system of interval linear equations with dependent coefficients.

In this representation, however, we allowed intervals $\mathbf{f}_\alpha$ to be arbitrarily wide. In terms of measurements, wide intervals correspond to low measurement accuracy. It is natural to ask the following question: if we only consider narrow intervals, which correspond to high measurement accuracy, will we still get all possible semialgebraic shapes or a narrower class of shapes? In this article, we show that even for narrow intervals, all possible shapes are possible.

Let us recall (see, e.g., [2]) that for a given $\delta > 0$, an interval $\mathbf{x} = [\tilde{x} - \Delta, \tilde{x} + \Delta]$ is called *absolutely δ-narrow* if $\Delta \le \delta$, and is *relatively δ-narrow* if $\Delta \le \delta \cdot |\tilde{x}|$.

Theorem. *For every $\delta > 0$, every semialgebraic set can be represented as a projection of the solution set of some system of interval linear equations with dependent coefficients, whose intervals are both absolutely and relatively δ-narrow.*

Comment. A reader should be aware that both in our paper [1] and in this paper, we assume that we are able to perform operations with arbitrary small or arbitrarily large rational numbers exactly. It is *theoretically* possible to implement such operations on the existing computers (and these operations have indeed been implemented), but in most *practical* computations, we are limited to a certain number of binary digits. Within such limitations, the use of narrow intervals – which correspond to high *measurement* accuracy – does not necessarily mean that the resulting *computational* accuracy will be high.

For example, in the algorithm presented in the following proof, when δ is small, the constant k_α will be much larger than f_α^-; so, to compute l_α, we will have to subtract two numbers, one of which is much larger than the other one. When we are limited to a certain number of binary digits, then such subtraction leads to a rounding error, and this error will add to the inaccuracy of the computational results.

Proof. We have already proven, in [1], that an arbitrary semialgebraic set S can be represented as a projection of the solution set of some system of interval linear equations with dependent coefficients. In other words, we proved that for every semialgebraic set, there exists a system of interval linear equations with dependent coefficients for which the projection of its solution set coincides with S. In that system of interval equations, the intervals $\mathbf{f}_\alpha = [f_\alpha^-, f_\alpha^+]$ can be arbitrarily wide.

To prove the new result, we will prove that for every system of interval linear equations with dependent coefficients and possibly wide intervals $\mathbf{f}_\alpha = [f_\alpha^-, f_\alpha^+]$, there exists a new system of interval linear equations with dependent coefficients which has the exact same solution set as the original system, but in which all corresponding intervals $\mathbf{g}_\alpha = [g_\alpha^-, g_\alpha^+]$ are both absolutely and relatively δ-narrow. Since the new system has the same solution set as the old system, the projection of the solution set of the new system will coincide with the projection of the solution set of the old system, i.e., with the original semialgebraic set S (thus, we will be able to complete the proof of the theorem).

Specifically, we will prove an even slightly stronger statement: that for each interval system, we can design a new interval system in which all intervals $\mathbf{g}_\alpha$ are equal to each other and equal to the interval $[1-\delta, 1+\delta]$ – an interval which is both absolutely and relatively δ-narrow.

To design this new system, we will first show that we can transform this fixed narrow interval into an arbitrary wide interval – in particular, into any of the intervals $\mathbf{f}_\alpha$ which are used in the old interval system. Indeed, we can apply a linear transformation $g_\alpha \to f_\alpha = k_\alpha \cdot g_\alpha + l_\alpha$ with $k_\alpha = (f_\alpha^+ - f_\alpha^-)/(2\delta)$ and $l_\alpha = f_\alpha^- - k_\alpha \cdot (1-\delta)$.

Substituting this linear expression for f_α in terms of g_α into the equations (2) and (3), we get a new system of interval linear equations with dependent coefficients which has the same solution set as the old system, but for which all intervals are narrow. The theorem is thus proven.

Acknowledgments. This work was supported in part by NASA under cooperative agreement NCC5-209, by NSF grant No. DUE-9750858, by the United

Space Alliance grant No. NAS 9-20000 (PWO C0C67713A6), by the Future Aerospace Science and Technology Program (FAST) Center for Structural Integrity of Aerospace Systems, effort sponsored by the Air Force Office of Scientific Research, Air Force Materiel Command, USAF, under grant number F49620-95-1-0518, by the National Security Agency under Grants No. MDA904-98-1-0561 and MDA904-98-1-0564. Part of this work was conducted while one of the authors (V.K.) was visiting the Department of Mechanical and Automation Engineering at the Chinese University of Hong Kong under the support of grant RGC4138/97E.

The authors are thankful to the anonymous referees for useful comments.

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