Fundamental Properties of Pair-Wise Interactions Naturally Lead to Quarks and Quark Confinement: A Theorem

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Abstract

In traditional mechanics, most interactions are pair-wise; if we omit one of the particles from our description, then the original pair-wise interaction can sometimes only be represented as interaction between triples, etc. It turns out that, vice versa, every possible interaction between N particles can be represented as pair-wise interaction if we represent each of the original N particles as a triple of new ones (and two new ones are not enough for this representation). The resulting three "particles" actually represent a single directly observable particles and in this sense, cannot be separated. So, this representation gives a fundamental reason for the three-quark model of basic barions, and explains quark confinement. The representation is based on a deep mathematical result (namely, a minor modification of Kolmogorov's solution to Hilbert's 13th problem) which has already been used in foundations of physics – to explain why fundamental physical equations are of second order, and why all these fundamental equations naturally lead to non-smooth solutions like singularity.

Formulation of the problem. In traditional (Newtonian) particle mechanics, most interactions are pair-wise (see, e.g., [6]): the total force $\vec{F}^{(a)}$ acting on a particle a is equal to the sum of all the forces $\vec{F}^{(ab)}$ with which all other particles

b acts on this a:

$$\vec{F}^{(a)} = \sum_{b \neq a} \vec{F}^{(ab)}(\vec{x}^{\,(a)}, \vec{x}^{\,(b)}); \tag{1}$$

here, $\vec{x}^{\,(a)}$ denotes all the parameters which characterize the state of a particle a (its coordinates, its charge, etc.). For example, the Newtonian gravitational interaction has the form

$$\vec{F}^{(a)} = \sum_{b \neq a} \frac{G \cdot m_a \cdot m_b \cdot (\vec{r}^{\,(a)} - \vec{r}^{\,(b)})}{\|\vec{r}^{\,(a)} - \vec{r}^{\,(b)}\|^3};$$

here, $\vec{x}^{\,(a)} = (\vec{r}^{\,(a)}, m_a)$, and

$$\vec{F}^{(ab)} = \frac{G \cdot m_a \cdot m_b \cdot (\vec{r}^{\,(a)} - \vec{r}^{\,(b)})}{\|\vec{r}^{\,(a)} - \vec{r}^{\,(b)}\|^3}.$$

Relativistic non-quantum particle interactions can be described by a similar formula, with only two changes: first, we must use a relativistic expression for the force, and second, in determining the force which acts on a particle a at a given moment of time t, we must use the values $\vec{x}^{\,(b)}$ at the appropriately retarded moments of time $t' < t$ (see, e.g., [2]).

From the general mathematical viewpoint, we can expect more general interactions which are not necessarily pair-wise; namely, we can expect a general dependence of the type

$$\vec{F}^{(a)} = \vec{F}^{(a)}(\vec{x}^{\,(a)}, \vec{x}^{\,(b)}, \vec{x}^{\,(c)}, \ldots). \tag{2}$$

which is not necessarily representable as a sum of type (1).

General formulas of type (2) are not only mathematically possible, they are physically quite possible even when interactions are actually pair-wise: namely, if we omit one of the particles c from our description, then particles b and b' act on a both directly (in the form (1)) and indirectly (via the unaccounted particle c), and in this indirect interaction, terms corresponding to $\vec{x}^{\,(b)}$ and $\vec{x}^{\,(b')}$ may not be separable.

A similar idea can be described in an even clearer way on the example of chemical interactions: Suppose that we have pair-wise interactions in which two substances a and b form a compound ab, and this compound interacts with a third compound c to form a new substance abc:

$$a + b \to ab; \quad ab + c \to abc. \tag{3}$$

If the intermediate compound ab is short-lived, then it is possible to omit it, and treat the reaction as a single (non-pair-wise) reaction between *three* chemical substances a, b, and c:

$$a + b + c \to abc. \tag{4}$$

In this case, a non-pair-wise interactions (4) can be reformulated in a pair-wise form (3) if we add an additional substance ab. In a more general case, we may have to add several additional substances.

Similarly, some non-pair-wise particle interactions (2) can be reformulated in a pair-wise form (1) if we add one or more additional particles. A natural question is: *Can an arbitrary particle interaction be reformulated as pair-wise interaction by adding additional particles?*

In this paper, we will show that the answer to this question is "yes"; this answer will also lead to a fundamental explanation of why a nucleon consists of exactly three (not two and not five) hard-to-separate particles (quarks).

Definition. *Let m and N be positive integers, and let B be a positive real number. The integer n will be called the number of particles, and the number B will be called a bound.*

- *By a state space, we mean a set $S = [-B, B]^m$ of all m-tuples $s = (s_1, \ldots, s_m)$ for which $|s_i| \le B$ for all i.*
- *By a particle interaction between N particles, we mean a sequence of N continuous functions $f^{(a)} : S^N \to R^m$ $(1 \le a \le N)$, i.e., functions which transforms every N-tuple of states $(s^{(1)}, \ldots, s^{(N)})$ into a vector $v \in R^m$: $v^{(a)} = f^{(a)}(s^{(1)}, \ldots, s^{(N)})$.*
- *We say that a particle interaction $(f^{(1)}, \ldots, f^{(N)})$ is pair-wise if every function $f^{(a)}$ can be represented as a sum:*

$$f^{(a)}(s^{(1)}, \ldots, s^{(N)}) = \sum_{b=1}^{N} f^{(ab)}(s^{(a)}, s^{(b)}),\ 1 \le a \le N,$$

  for some functions $f^{(ab)} : S^2 \to R^m$.
- *We say that a particle interaction $(f^{(1)}, \ldots, f^{(N)})$ can be represented in pair-wise form by adding E extra particles, if there exists a particle interaction $(g^{(1)}, \ldots, g^{(N)}, \ldots, g^{(N+E)})$ in which:*

- *for $1 \leq a \leq N$, the functions $g^{(a)}$ may depend on the states of all $N+E$ particles:*

$$g^{(a)}(s^{(1)}, \ldots, s^{(N)}, \ldots, s^{(N+E)}); \tag{5}$$

- *for $N+1 \leq a \leq N+E$, the functions $g^{(a)}$ depend only on the states of the first N particles:*

$$g^{(a)}(s^{(1)}, \ldots, s^{(N)}, \ldots, s^{(N+E)}) = g^{(a)}(s^{(1)}, \ldots, s^{(N)}), \tag{6}$$

  and describe the states $s^{(a)} = v^{(a)}$ of the new particles;

- *for every sequence of N states $(s^{(1)}, \ldots, s^{(N)})$ from S^N and for every a from 1 to N, if we substitute the values $s^{(b)} = g^{(b)}(s^{(1)}, \ldots, s^{(N)})$, $N+1 \leq b \leq N+E$, into the expression (5), we get the value $f^{(a)}(s^{(1)}, \ldots, s^{(N)})$.*

Theorem. *An arbitrary particle interaction can be represented in pair-wise form by adding a finite number of extra particles.*

Proof. This proof uses a result proven by A. Kolmogorov [5] as a solution to the conjecture of Hilbert, formulated as the thirteenth problem [4]: one of 22 problems that Hilbert has proposed in 1900 as a challenge to the XX century mathematics.

We will present this result in a form given by D. Sprecher in [8] (see also [7]): *For an arbitrary integer $n > 0$, and for an arbitrary real number B, there exist $2n+1$ continuous increasing functions $\varphi_q : [-B, B] \to R$, $1 \leq q \leq 2n+1$, and n real numbers $\lambda_1, \ldots, \lambda_n$ such that if we denote*

$$y_q = \sum_{p=1}^{n} \lambda_p \cdot \varphi_q(x_p), \tag{7}$$

then we can represent each continuous function $f : [-B, B]^n \to R$ in the form

$$f(x_1, \ldots, x_n) = \sum_{q=1}^{2n+1} g(y_q) \tag{8}$$

for some continuous function $g : [-B, B] \to R$.

In particular, if we have n different functions $f_1(x_1, \ldots, x_n)$, …, $f_n(x_1, \ldots, x_n)$, then we can represent each of them in this form:

$$f_p(x_1, \ldots, x_n) = \sum_{q=1}^{2n+1} g_p(y_q), \quad 1 \leq p \leq n \tag{9}$$

for some continuous functions $g_1, \ldots, g_n : [-B, B] \to R$.

This result was previously used in *computational physics* (see, e.g., [3]) where it helped to speed up computations, and in *foundations of physics* (see. e.g., [9]), where it was used to explain why fundamental physical equations are of second order, and why all these fundamental equations naturally lead to non-smooth solutions like singularity.

In our physical problem, we have N vector-valued functions of N vector variables. Each of these vector variables has m components, so, we can view each of m components of each of N functions as a function of $n = N \cdot m$ scalar variables:

$$f_i^{(a)}(s^{(1)}, \ldots, s^{(N)}) = f_i^{(a)}(s_1^{(1)}, \ldots, s_m^{(1)}, \ldots, s_1^{(N)}, \ldots, s_m^{(N)}).$$

Thus, we have n functions $f_i^{(a)}$ of n scalar variables $s_i^{(a)} \in [-B, B]$; so, according to Sprecher's theorem, we can find $2n+1$ functions $\varphi_q(x)$ and $g_p(x)$ for which the formulas (7) and (9) hold. Hence, we have $2n+1$ extra variables $y_1, \ldots, y_{2n+1}$, for which each original function f_p is represented as a sum of functions of one variable depending on these new variables (formula (9)), and each of the new variables y_q is represented as a sum of functions of one variable depending only on old variables x_p (formula (7)).

This is *almost* what we want in our definition of "represented in pair-wise form by adding E extra particles"; to get *exactly* what we want, we add $m-1$ fictitious new variables y_q which do not affect anything, and divide the resulting

$$(2n+1) + (m-1) = 2n + m = 2N \cdot m + m = m(2N+1)$$

new variables into $E = 2N + 1$ groups of m variables in each. Then, if we interpret variables from each group as representing a state of one of the new particles, we get the desired representation (5), (6). The theorem is proven.

Physical interpretation of the result. Our result explains why in classical Newtonian mechanics, we only consider pair-wise particle interactions: indeed, as the this theorem shows, by adding extra ("fictitious") particles, we can describe an arbitrary particle interaction as a pair-wise one.

This result does more than simply explain this general possibility; it tells us how many of these additional particles we need to add to describe an arbitrary particle interaction. Indeed, if we start with N particles, then we need $2N+1$ additional ("fictitious") particles, and we can therefore describe the original physical system by pair-wise interactions between the resulting $3N+1$ particles.

If we add one "real" particle to the original physical system (i.e., if we go from N from $N+1$), then, to preserve the pair-wise description of the particle interaction, we need to go from $3N+1$ to $3(N+1)+1 = 3N+4 = (3N+1)+3$ particles in the pair-wise description, i.e., we need to add 3 particles to that description.

In other words, to represent particle interaction as pair-wise, we must represent each original particle as a triple of new ones. The resulting three "particles" actually represent a single directly observable particle and in this sense, cannot be separated. So, this representation gives a fundamental reason for the *three-quark* model of basic barions, and explains *quark confinement*.

We have explained why 3 new particles is enough; the only remaining question is: are 3 new particles really necessary? Can we do the same with *two* new particles instead of each original one? A (partial) negative answer to this question comes from a theorem proven by Doss [1]: that (at least for some n), in Sprecher's result, we cannot use fewer than $2n+1$ new variables y_q. Thus, two new particles are not enough, and we get an explanation of why exactly 3 quarks are needed.

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