Range Estimation is NP-Hard for ε^2 Accuracy and Feasible for $\varepsilon^{2-\delta}$

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Abstract

The basic problem of interval computations is: given a function $f(x_1, \ldots, x_n)$ and n intervals $[\underline{x}_i, \overline{x}_i]$, find the (interval) range $\mathbf{y}$ of the given function on the given intervals. It is known that even for quadratic polynomials $f(x_1, \ldots, x_n)$, this problem is NP-hard. In this paper, following the advice of A. Neumaier, we analyze the complexity of *asymptotic* range estimation, when the bound ε on the width of the input intervals tends to 0. We show that for small $c > 0$, if we want to compute the range with an accuracy $c \cdot \varepsilon^2$, then the problem is still NP-hard; on the other hand, for every $\delta > 0$, there exists a feasible algorithm which asymptotically, estimates the range with an accuracy $c \cdot \varepsilon^{2-\delta}$.

1 Formulation of the Problem

The basic problem of interval computations is: given a function $f(x_1, \ldots, x_n)$ and n intervals $[\underline{x}_i, \overline{x}_i]$, find the (interval) range $\mathbf{y}$ of the given function on the given intervals.

Some algorithms for solving this problem require computation time which grows exponentially with the number of variables n and which are, therefore, not feasible for large n. This exponential growth is easy to illustrate. The endpoints of the range interval are the global minimum and the global maximum of a given function $f(x_1, \ldots, x_n)$ on a given box $[\underline{x}_1, \overline{x}_1] \times \ldots \times [\underline{x}_n, \overline{x}_n]$. For a function of a single variable ($n = 1$), the global maximum is attained either at the endpoints, or at a point where its derivative is equal to 0. After computing the value of $f(x_1)$ for both endpoints $\underline{x}_1$ and $\overline{x}_1$ and for all stationary points of the function $f(x_1)$, we can compute the upper endpoint $\overline{y}$ of the range $\mathbf{y}$ as the maximum of these values, and the lower endpoint $\underline{y}$ as the minimum of these same values.

For a function of several variables ($n > 1$), for each of the variables x_i, the maximum is attained either at the left endpoint $\underline{x}_i$ of the corresponding interval, or at its right endpoint $\overline{x}_i$, or at a point where the partial derivative $\dfrac{\partial f}{\partial x_i}$ of the function f with respect to x_i is equal to 0. For each of n variables, we have 3 possibilities, so totally, for all n variables, we have 3^n possible combinations.

Since existing algorithms often require exponential time, it is natural to ask whether there exist feasible algorithms for solving this problem, i.e., algorithms whose running time is limited by a polynomial of the length of the input. In 1981, A. A. Gaganov has proven [2, 3] that for even for polynomial functions $f(x_1, \ldots, x_n)$, this problem is NP-hard. If we follow the belief of most computer scientists that P$\neq$NP, this result implies that no feasible algorithm is possible which would solve all particular cases of the basic problem (for exact definitions of NP-hardness, see, e.g., [4, 9]; in interval computations context, the corresponding definitions are given in [7]). In 1991, S. A. Vavasis has shown [10], in effect, that this problem is NP-hard even for quadratic polynomials with rational coefficients (all these results are reproduced in [7]). In precise terms, the following result is true:

Definition 1. *Let $\alpha > 0$, and let $\mathcal{P}$ be a class of polynomials with rational coefficients. By the $\alpha-$approximate basic problem of interval computations for $\mathcal{P}$, we mean the following problem:*

GIVEN:

- *n rational intervals $\mathbf{x}_i$, and*
- *a polynomial $f(x_1, \ldots, x_n) \in \mathcal{P}$;*

COMPUTE: rational numbers $\tilde{\underline{y}}$ and $\tilde{\overline{y}}$ that are α-close to the range's endpoints, i.e., for which $|\tilde{\underline{y}} - \underline{y}| \le \alpha$ and $|\tilde{\overline{y}} - \overline{y}| \le \alpha$, where:

$$\mathbf{y} = [\underline{y}, \overline{y}] = f(\mathbf{x}_1, \ldots, \mathbf{x}_n) =$$

$$\{y \mid y = f(x_1, \ldots, x_n) \text{ for some } x_1 \in \mathbf{x}_1, \ldots, x_n \in \mathbf{x}_n\}$$

Denotation. *Let $\mathcal{P}_{\text{all}}$ denote the class of all polynomials with rational coefficients, and let $\mathcal{P}_2 \subseteq \mathcal{P}_{\text{all}}$ denote the class of all quadratic polynomials*

$$f(x_1, \ldots, x_n) = a_0 + \sum_{i=1}^{n} a_i \cdot x_i + \sum_{i=1}^{n} \sum_{j=1}^{n} a_{i,j} \cdot x_i \cdot x_j$$

with rational coefficients.

Theorem. (Gaganov 1981) *For $\mathcal{P}_{\text{all}}$, for every $\alpha > 0$, the $\alpha-$approximate basic problem of interval computations is NP-hard.*

Theorem. (Vavasis 1991) *For $\mathcal{P}_2$, for every $\alpha > 0$, the $\alpha-$approximate basic problem of interval computations is NP-hard.*

It is also known that these problems remain NP-hard if we restrict ourselves only to intervals of sufficiently small width [6, 7]. These results mean, crudely speaking, that we cannot get feasible algorithms for computing the interval range of a given polynomial (even quadratic polynomial) with a given accuracy. On the other hand, there exist many efficient algorithms which compute the range of a given computable function with reasonable accuracy (see, e.g., [1, 5, 8]).

So, while it is impossible (unless P=NP) to have a feasible algorithm that always computes the range with an arbitrarily given accuracy, some accuracy can be feasibly achieved. It is therefore desirable to find out for which asymptotic accuracies, the problem is NP-hard and for which it is feasible. This question was formulated by A. Neumaier in an email discussion. In this paper, we analyze the computational complexity of *asymptotic* range estimation, when the bound ε on the width of the input intervals tends to 0, and give the answer to Neumaier's question.

2 Definitions and the Main Result

Definition 2. *Let $(f, \mathbf{x}_1, \ldots, \mathbf{x}_n)$ be a particular case of the basic problem of interval computations; then:*

- *n is called the number of variables;*
- *the degree of a polynomial f is denoted by d and called a degree of the problem, and*
- *the largest width $\max\limits_i |\overline{x}_i - \underline{x}_i|$ of the input intervals is denoted by ε and called the width of the case.*

For a linear function $f(x_1, \ldots, x_n)$, i.e., for a polynomial $f(x_1, \ldots, x_n) = \sum M_k(x_1, \ldots, x_n)$ for which all monomials $M_k(x_1, \ldots, x_n)$ are either of 0-th degree (i.e., constants) or of the 1-st degree (i.e., constants multiplied by variables x_i), the basic problem of interval computations is easy to solve. The difficulty starts when a polynomial $f(x_1, \ldots, x_n)$ has non-linear terms, i.e., when the second derivatives $\dfrac{\partial f}{\partial x_i \partial x_j}$ are not all equal to 0. For each monomial of degree d, the second derivative is a polynomial of degree $d-2$. It is therefore reasonable to estimate the degree of non-linearity of a monomial as a product of the absolute value of the coefficient and the $(d-2)$-th power of the largest input bound. As a result, we arrive at the following definition:

Definition 3. *Let $(f, [\underline{x}_1, \overline{x}_1], \ldots, [\underline{x}_n, \overline{x}_n])$ be a particular case of the basic problem of interval computations; then:*

- *by a largest input bound B, we mean the value*

$$B \stackrel{\text{def}}{=} \max(|\underline{x}_1|, |\overline{x}_1|, \ldots, |\underline{x}_n|, |\overline{x}_n|);$$

- *by a non-linearity degree D of a monomial $c_{i_1\ldots i_n}\cdot x_1^{i_1}\cdot\ldots\cdot x_n^{i_n}$ of degree $d=i_1+\ldots+i_n>1$, we mean a product $D\stackrel{\rm def}{=}|c_{i_1\ldots i_n}|\cdot B^{d-2}$;*

- *by a non-linearity degree D of a polynomial $f=\sum M_k$, we mean the largest of non-linearity degrees D_k of its monomials M_k: $D\stackrel{\rm def}{=}\max(D_k)$.*

Definition 4. *Let $\mathcal{P}$ be a class of polynomials, and let $F(n,d,D,\varepsilon)$ be a given function.*

- *We say that an algorithm $\mathcal{U}$ solves the basic problem of interval computations with accuracy $F(n,d,D,\varepsilon)$ if, for every case of this problem with $f\in\mathcal{P}$, the algorithm $\mathcal{U}$ computes the estimates for the range endpoints with accuracy $F(n,d,D,\varepsilon)$.*

- *We say that an algorithm $\mathcal{U}$ solves the basic problem of interval computations with an asymptotic accuracy $F(n,d,D,\varepsilon)$ if:*
  - *for every n, d, and D, there exists an ε_0 such that for every case of this problem with $f\in\mathcal{P}$ with width $\varepsilon\le\varepsilon_0$, the algorithm $\mathcal{U}$ computes the estimates for the range endpoints with accuracy $F(n,d,D,\varepsilon)$, and*
  - *for any fixed D and d, we can, given n, compute this ε_0 in time which is bounded by a polynomial of n.*

Our first result is that, in principle, it is possible to estimate the range with quadratic accuracy:

Proposition 1. *For $\mathcal{P}_{\rm all}$, there exists a function $C(n,d,D)$ and a feasible algorithm $\mathcal{U}$ which solves the basic problem of interval computations with accuracy $C(n,d,D)\cdot\varepsilon^2$.*

Corollary. *For $\mathcal{P}_2$, there exists a function $C(n,D)$ and a feasible algorithm $\mathcal{U}$ which solves the basic problem of interval computations for second order polynomials with accuracy $C(n,D)\cdot\varepsilon^2$.*

(For reader's convenience, all the proofs are given in the last section.)

In the above estimate, the coefficient C at ε^2 is fixed. It turns out that if we want to be able to arbitrarily decrease the value of this coefficient, then the problem becomes NP-hard even for quadratic polynomials:

Theorem 1. *Let $D>0$ be a rational number. Then, there exists a number $c(D)>0$ such that for every $c\le c(D)$, the problem of asymptotically solving the basic problem of interval computations for all quadratic polynomials and all cases of non-linearity degree $\le D$ with accuracy $c\cdot\varepsilon^2$ is NP-hard.*

If, instead of $\sim\varepsilon^2$, we allow estimates with an asymptotic $\sim\varepsilon^{2-\delta}$ for some $\delta>0$, then the problem becomes feasible for arbitrary polynomials (not necessarily quadratic):

Theorem 2. *For every $\delta > 0$, $D > 0$, d, and $c > 0$, there exists a feasible algorithm which asymptotically solves the basic problem of interval computations for all polynomials of degree $\le d$ and non-linearity degree $\le D$ with accuracy $c \cdot \varepsilon^{2-\delta}$.*

3 Proofs

Proof of Proposition 1. We can get the desired estimate as follows: we take a midpoint $x_i^{(0)}$ of each of the input intervals $\mathbf{x}_i = [\underline{x}_i, \overline{x}_i]$, and represent each input variable as $x_i = x_i^{(0)} + \Delta x_i$, where Δx_i is a new variable which takes values from the symmetric interval $[-\Delta_i, \Delta_i]$, with $\Delta_i = (1/2) \cdot (\overline{x}_i - \underline{x}_i)$.

For an analytical function $F(\theta)$ of one variable, the Taylor expansion formula with Lagrange's remainder takes the form

$$F(\theta) = F(0) + F'(0) \cdot \theta + \frac{1}{2} \cdot F''(\varphi) \cdot \theta^2$$

for some $\varphi \in [0, \theta]$. A similar formula can be written for a function $f(x_1, \ldots, x_n)$ of several variables, if we apply the above expression to the function

$$F(\theta) = f(x_1^{(0)} + \theta \cdot \Delta x_1, \ldots, x_n^{(0)} + \theta \cdot \Delta x_n)$$

and the value $\theta = 1$. For this choice of F and θ, we have $F(0) = f(x_1^{(0)}, \ldots, x_n^{(0)})$; we will denote this value by $y^{(0)}$. Here,

$$F'(\theta) = \sum_{i=1}^{n} \frac{\partial f}{\partial x_i}(x_1^{(0)} + \theta \cdot \Delta x_1, \ldots, x_n^{(0)} + \theta \cdot \Delta x_n) \cdot \Delta x_i.$$

Hence, $F'(0) = \sum\limits_{i=1}^{n} y_i^{(0)} \cdot \Delta x_i$, where we denoted $y_i^{(0)} = \dfrac{\partial f}{\partial x_i}(x_1^{(0)}, \ldots, x_n^{(0)})$ and

$$F''(\varphi) = \sum_{i=1}^{n} \sum_{j=1}^{n} \frac{\partial^2 f}{\partial x_i \partial x_j}(x_1^{(0)} + \varphi \cdot \Delta x_1, \ldots, x_n^{(0)} + \varphi \cdot \Delta x_n) \cdot \Delta x_i \cdot \Delta x_j.$$

Thus, the Taylor expansion formula for $F(\theta)$ leads to the following expression:

$$f(x_1, \ldots, x_n) = y^{(0)} + \sum_{i=1}^{n} y_i^{(0)} \cdot \Delta x_i + \frac{1}{2} \cdot \sum_{i=1}^{n} \sum_{j=1}^{n} \frac{\partial^2 f}{\partial x_i \partial x_j}(\xi_1, \ldots, \xi_n) \cdot \Delta x_i \cdot \Delta x_j,$$

where $\xi_i = x_i^{(0)} + \theta \cdot \Delta x_i \in [x_i^{(0)} - \Delta x_i, x_i^{(0)} + \Delta x_i] = \mathbf{x}_i$. For the linear terms, we know the exact range when $\Delta x_i \in [-\Delta_i, \Delta_i]$: it is the interval

$$\left[y^{(0)} - \sum_{i=1}^{n} \left| y_i^{(0)} \right| \cdot \Delta_i, y^{(0)} + \sum_{i=1}^{n} \left| y_i^{(0)} \right| \cdot \Delta_i \right].$$

Quadratic terms can be estimated as follows. The polynomial f is a sum of monomials $f = \sum M_k$. Thus,

$$\frac{1}{2}\cdot\sum_{i=1}^n\sum_{j=1}^n \frac{\partial^2 f}{\partial x_i \partial x_j}(\xi_1,\ldots,\xi_n)\cdot\Delta x_i\cdot\Delta x_j =$$

$$\frac{1}{2}\cdot\sum_k\sum_{i=1}^n\sum_{j=1}^n \frac{\partial^2 M_k}{\partial x_i \partial x_j}(\xi_1,\ldots,\xi_n)\cdot\Delta x_i\cdot\Delta x_j,$$

hence

$$\left|\frac{1}{2}\cdot\sum_{i=1}^n\sum_{j=1}^n \frac{\partial^2 f}{\partial x_i \partial x_j}(\xi_1,\ldots,\xi_n)\cdot\Delta x_i\cdot\Delta x_j\right| \le$$

$$\sum_k\sum_{i=1}^n\sum_{j=1}^n \frac{1}{2}\cdot\left|\frac{\partial^2 M_k}{\partial x_i \partial x_j}(\xi_1,\ldots,\xi_n)\right|\cdot|\Delta x_i|\cdot|\Delta x_j|.$$

For each monomial $M_k = c_{i_1\ldots i_n}\cdot x_1^{i_1}\cdot\ldots\cdot x_n^{i_n}$ of degree $d_k = i_1+\ldots+i_n$, the absolute value of each second derivative is bounded by $d_k^2\cdot|c_{i_1\ldots i_n}|\cdot B^{d_k-2}$, i.e., by $d_k^2\cdot D_k$. Since $d_k \le d$ and $D_k \le D$, we conclude that this absolute value is bounded by $d^2\cdot D$. Each value Δx_i cannot exceed the half-width of the corresponding interval $\mathbf{x}_i$, hence $|\Delta x_i| \le \varepsilon/2$, and so,

$$\frac{1}{2}\cdot\left|\frac{\partial^2 M_k}{\partial x_i \partial x_j}(\xi_1,\ldots,\xi_n)\right|\cdot|\Delta x_i|\cdot|\Delta x_j| \le \frac{1}{2}\cdot d^2\cdot D\cdot\left(\frac{\varepsilon}{2}\right)^2 = \frac{d^2\cdot D}{8}\cdot\varepsilon^2.$$

For each monomial M_k, there are n^2 such terms, so the sum of the terms coming from each monomial is bounded by

$$\frac{n^2\cdot d^2\cdot D}{8}\cdot\varepsilon^2.$$

How many monomials can there be? For a polynomial of degree d, the number of possible monomials is equal to the number of possible sequences $(i_1,\ldots,i_n)$ for which $i_1+\ldots+i_n \le d$. Each such sequence can be uniquely described if we place i_1 dots, then place a separating asterisk $*$, then i_2 dots, then again a separating asterisk, ..., then i_n dots, the separating asterisk, and then $d-(i_1+\ldots+i_n)$ remaining dots. In total, we have n separators and d dots. So, each monomial corresponds to a placing of n separating asterisks among $n+d$ places. The total amount of such placings is $\binom{n+d}{n}$, hence there are no more than $\binom{n+d}{n}$ monomials. For each of these monomials, the sum of the corresponding second-derivative terms is bounded by $\frac{1}{8}\cdot n^2\cdot d^2\cdot D\cdot\varepsilon^2$, hence the overall sum of these terms is $\le C(n,d,D)\cdot\varepsilon^2$, where we denoted

$$C(n,d,D) \stackrel{\text{def}}{=} \binom{n+d}{n}\cdot\frac{n^2\cdot d^2\cdot D}{8}.$$

Thus, the estimate coming from the linear terms is accuracte with the accuracy $\le C(n,d,D)\cdot\varepsilon^2$. The proposition is proven.

Proof of Theorem 1. This proof is a modification of a proof of Theorem 6.1 from [7].

We will show that a known NP-hard problem, namely, the *propositional satisfiability* problem for 3−CNF formulas, can be reduced to our problem (of asymptotically estimating the range), and therefore, our problem is also NP-hard.

This propositional satisfiability problem consists of the following: Suppose that an integer v is fixed, and a formula F of the type $F_1\&F_2\&\ldots\&F_k$ is given, where each of the expressions F_j has the form $a\vee b$ or $a\vee b\vee c$, and a,b,c are either the variables $z_1,\ldots,z_v$, or their negations $\neg z_1,\ldots,\neg z_v$ (these $a,b,c,\ldots$ are called *literals*). For *example*, we can take a formula

$$(z_1\vee\neg z_2)\&(\neg z_1\vee z_2\vee\neg z_3).$$

If we assign arbitrary Boolean values ("true" or "false") to v variables $z_1,\ldots,z_v$, then, applying the standard logical rules, we get the truth value of F. We say that a formula F is *satisfiable* if there exist truth values $z_1,\ldots,z_v$ for which the truth value of the expression F is "true". The problem is: given F, check whether it is satisfiable.

Let us show the desired reduction, i.e., let us show that if we are able to compute the desired interval $\mathbf{y}$ for quadratic polynomials $f(x_1,\ldots,x_n)$, then we will be able to solve propositional satisfiability problem for 3-CNF formulas.

Indeed, let $F=F_1\&\ldots\&F_k$ be a 3-CNF formula with the (Boolean) variables $z_1,\ldots,z_v$. (In the computer, usually, "true" is represented as 1, and "false" as 0.) In this formula, we have k disjunctions F_j. Each of these conjunctions F_j is a disjunction of two or three literals, and each literal is either a variable or a negation of a variable. By $i(j,1)$, we will denote the ordinal number of the variable corresponding to the first literal of the disjunction F_j; by $i(j,2)$, we will denote the variable corresponding to the second literal of the conjunction F_j, etc. For example, if a disjunction F_3 has the form $z_2\vee\neg z_5$, then $i(3,1)=2$ and $i(3,2)=5$.

The corresponding quadratic problem will have $v+3k+2v\cdot k$ variables $x_{10},\ldots,x_{v0},p_1,\ldots,p_k,q_1,\ldots,q_k,k_1,\ldots,k_k$, and $x_{i,j},y_{i,j}$, $1\le i\le v$, $1\le j\le k$.

The meaning of these variables is as follows: the real-valued variable $x_{i,j}$ will correspond to possible occurrences of the propositional variable z_i in a disjunction F_j, and the real-valued variable $y_{i,j}$ will correspond to possible occurrences of the negation $\neg z_i$ of the propositional variable z_i in a disjunction F_j.

We construct the following auxiliary quadratic polynomial:

$$G(\{x_{i,j}\},\{y_{i,j}\},\{p_j\},\{q_j\},\{k_j\})=$$

$$\sum_{i=1}^{v}\sum_{j=0}^{k}x_{i,j}\cdot(1-x_{i,j})+\sum_{i=1}^{v}\sum_{j=1}^{k}y_{i,j}\cdot(1-y_{i,j})+$$

$$\sum_{i=1}^{v}\sum_{j=1}^{k}(x_{i,j-1}-y_{i,j})^2+\sum_{i=1}^{v}\sum_{j=1}^{k}(y_{i,j}-x_{i,j})^2+\sum_{j=1}^{k}G_j^2,$$

where, for a two-literal formula F_j of the form $a\vee b$, we have

$$G_j=p_j+r_{i(j,1),j}+r_{i(j,2),j}-k_j;$$

for a three-literal formula F_j of the type $a\vee b\vee c$, we have

$$G_j=p_j+q_j+r_{i(j,1),j}+r_{i(j,2),j}+r_{i(j,3),j}-k_j,$$

and $r_{q,j}$ denotes either $x_{q,j}$ or $y_{q,j}$ depending on whether the corresponding literal is z_q or $\neg z_q$.

For example, if a disjunction F_3 has the form $z_2\vee\neg z_5$ (i.e., $i(3,1)=2$ and $i(3,2)=5$), then $r_{i(3,1),3}=r_{2,3}=x_{2,3}$ (because the corresponding literal is z_2), $r_{i(3,2),3}=r_{5,3}=y_{5,3}$ (because the corresponding literal is $\neg z_5$), and the expression G_3 has the form

$$G_3=p_3+x_{2,3}+y_{5,3}-k_3.$$

Let us consider the problem of estimating the range of this auxiliary function G on the intervals $\mathbf{x}_{i,j}=\mathbf{y}_{i,j}=\mathbf{p}_j=\mathbf{q}_j=[0,1]$ and $k_j=[c_j,c_j]$, where c_j is the total number of positive literals in the formula F_j (i.e., literals of the type x_i). We will show that if the original formula F is satisfiable, then the actual lower bound $\underline{G}$ of the range of G is equal to 0, and that if the original formula is not satisfiable, then $\underline{G}\geq 0.09$.

For all values of the variables form the given intervals, all the terms in the sum which defines the polynomial G are non-negative, so $G\geq 0$ and therefore, the lower endpoint $\underline{G}$ of its range is also non-negative.

If the formula F is satisfiable, i.e., it is true for some propositional vector $z_1,\ldots,z_v$, then we take $x_{i,j}=y_{i,j}=z_i$ for all i and j (i.e., $x_{i,j}=y_{i,j}=1$ if $z_i=$"true" and $x_{i,j}=y_{i,j}=0$ if $z_i=$"false"). The values of p_j and q_j are chosen as follows:

- If $F_j=a\vee b$, and both a and b are true for z_i, then we take $p_j=q_j=0$.
- If $F_j=a\vee b$, and only one of the literals a and b is true for a given choice of z_i, then we take $p_j=1$ and $q_j=0$.
- If $F_j=a\vee b\vee c$, and all three literals are true, then $p_j=q_j=0$.
- If $F_j=a\vee b\vee c$, and two out of three literals are true, then $p_j=1$ and $q_j=0$.
- If $F_j=a\vee b\vee c$, and only one of the three literals is true, then $p_j=q_j=1$.

In all five cases, $G_j = 0$ for all j. For these values, $x_{i,j} \cdot (1 - x_{i,j}) = y_{i,j} \cdot (1 - y_{i,j}) = x_{i,j-1} - y_{i,j} = y_{i,j} - x_{i,j} = 0$; therefore, $G = 0$. Hence, $\underline{G} = \min G \leq 0$. Since we know that $\underline{G} \geq 0$, we conclude that $\underline{G} = 0$.

Let us prove that if the formula F is not satisfiable, then $\underline{G} \geq 0.09$. We will prove this statement by reduction to a contradiction: we will assume that $\underline{G} < 0.09$, and conclude that F is satisfiable.

From the fact that $\underline{G} = \min G < 0.09$, it follows that there exist values of the variable within given intervals for which $G = \min G < 0.09$. Since G is the sum of non-negative terms, from this inequality, it follows that each term is < 0.09.

In particular, it follows that $x_{i,j} \cdot (1 - x_{i,j}) < 0.09$ and $y_{i,j} \cdot (1 - y_{i,j}) < 0.09$. The function $x \cdot (1 - x)$ is increasing for $x < 0.5$ and decreasing afterwards. So, from $x_i(1 - x_i) < 0.09$ and from the fact that $0.1 \cdot (1 - 0.1) = 0.9 \cdot (1 - 0.9) = 0.09$, it follows that $x_{i,j} < 0.1$ or $x_{i,j} > 0.9$ for all i and j, and similarly, that either $y_{i,j} < 0.1$ or $y_{i,j} > 0.9$.

Let us take $z_i =$"true" if $x_{i,0} > 0.9$, and $z_i =$"false" if $x_{i,0} < 0.1$, and let us show that these propositional values make the formula F true (i.e., they make all the expressions F_j true).

Indeed, from the condition that $(x_{i,j-1} - y_{i,j})^2 < 0.09$, we conclude that $|x_{i,j-1} - y_{i,j}| < 0.3$. Both values $x_{i,j-1}$ and $y_{i,j}$ either belong to $[0, 0.1)$ or to $(0.9, 1]$. If they belong to two different subintervals, then the difference between them is at least $0.9 - 0.1 = 0.8$; so, from the fact that the difference between the two values is smaller than 0.3, we can conclude that for every i and j, the values $x_{i,j-1}$ and $y_{i,j}$ belong to the same subinterval. Similarly, from the restriction $(y_{i,j} - x_{i,j})^2$, we conclude that $x_{i,j}$ and $y_{i,j}$ belong to the same subinterval. Thus, for every i, if $x_{i,0} < 0.1$, then $y_{i,1} < 0.1$, $x_{i,1} < 0.1$, $\ldots$, and $x_{i,j} < 0.1$ and $y_{i,j} < 0.1$ for all j. Similarly, if $x_{i,j} > 0.9$, then $x_{i,j} > 0.9$ and $y_{i,j} > 0.9$ for all j.

Now:

- If $F_j = a \vee b$, then from $G_j^2 < 0.09$, it follows that $p_j + r_{i(j,1),j} + r_{i(j,2),j} - k_j > -0.3$, hence, if we denote $f_j[z_i] = x_{i,j}$ and $f_j[\neg z_i] = 1 - x_{i,j}$, we have $f_j[a] + f_j[b] > 1.7 - p_j$. Since $p_j \leq 1$, we conclude that $f_j[a] + f_j[b] > 0.7$. Therefore, the values $f_j[a]$ and $f_j[b]$ cannot be both < 0.1. Therefore, one of these two values is > 0.9. The corresponding literal is equal to "true", and hence, F_j is true.

- If $F_j = a \vee b \vee c$, then from $G_j = (f_j[a] + f_j[b] + f_j[c] + p_j + q_j - 3)^2 < 0.09$, it follows that $f_j[a] + f_j[b] + f_j[c] + p_j + q_j - 3 > -0.3$, and $f_j[a] + f_j[b] + f_j[c] > 2.7 - p_j - q_j$. Since $p_j \leq 1$, we conclude that $f_j[a] + f_j[b] + f_j[c] > 0.7$. Therefore, the values $f_j[a]$, $f_j[b]$, and $f_j[c]$ cannot be all < 0.1. Therefore, one of these three values is > 0.9. The corresponding literal is equal to "true", and hence, F_j is true.

So, F is satisfiable. Hence:

- if the original formula F is satisfiable, then the actual lower bound $\underline{G}$ of the range of G is equal to 0; and
- if the original formula is not satisfiable, then $\underline{G} \geq 0.09$.

Let us now estimate the non-linearity degree D_0 for the above case of the interval computation problem. Let us first show that the absolute values of all the coefficients of the polynomial G also cannot exceed 3.

Indeed, if we open all the parentheses, we will see that G is a quadratic polynomial with no constant terms. Its only linear terms are $x_{i,j}$ and $y_{i,j}$ (with coefficient 1), and all other terms are either products of two different variables, or squares.

- Each *product* of different variables comes from some square. For every two different variables, there is at most one squared term with these two terms, so, the coefficient at this product is either 0 (if there is no such term at all) or ± 2 (if there is exactly one such term).
- The *square* of each variable p_j, q_j, and k_j occurs only once; so, these squares come with coefficient 1.
- Each square $x_{i,j}^2$ comes from no more than 4 terms: a negative term $x_{i,j} \cdot (1 - x_{i,j})$, two positive terms $(y_{i,j} - x_{i,j})^2$, $(x_{i,j} - y_{i,j+1})^2$, and, possibly, a positive term G_j^2. Thus, the coefficient at $x_{i,j}^2$ is equal to -1, 0, 1, or 2. In all these cases, the absolute value of this coefficient does not exceed 3.
- Similarly, the absolute value of the coefficient at $y_{i,j}^2$ cannot exceed 3.

Thus, the absolute values of all the coefficients of the corresponding monomials are ≤ 3. All these monomials are quadratic, so, for each of them, the non-linearity degree coincides with the absolute value of the corresponding coefficient and is, therefore, ≤ 3. Thus, the non-linearity degree of the polynomial G – which was defined as the largest non-linearity degree of all its monomials – is also ≤ 3. In other words, for the polynomial G, we have $D_0 \leq 3$.

Let us show that Theorem 1 holds for $c(D) = 0.12/D$. Indeed, let us assume that for some $c \leq c(D)$, we can solve the basic problem of interval computations with an asymptotic accuracy $\leq c \cdot \varepsilon^2 \leq (0.12/D) \cdot \varepsilon^2$. Let us show how we can use this asymptotically accurate algorithm to find the lower endpoint $\underline{G}$ and thus, to check the satisfiability of the original formula F. Indeed, by definition of asymptotic accuracy, for any given n, we cam feasibly compute ε_0 such that for widths $\varepsilon \leq \varepsilon_0$, the algorithm computes the endpoints with the desired accuracy. Let us pick one such value ε and form a new polynomial

$$f(t_1, t_2, \ldots) = (D/3) \cdot \varepsilon^2 \cdot G(t_1/\varepsilon, t_2/\varepsilon, \ldots).$$

We will apply the hypothetic asymptotically accurate algorithm $\mathcal{U}$ to this polynomial f and to intervals which are ε times the original intervals for G (i.e.,

$[0, \varepsilon]$ instead of $[0, 1]$, $[2\varepsilon, 2\varepsilon]$ instead of $[2, 2]$, etc.). For this new problem, all the widths are bounded by ε, and the absolute values of all the coefficients are (for sufficiently small ε) bounded by $(D/3) \cdot 3 = D$ – hence the non-linearity degree is also bounded by D. Thus, the algorithm $\mathcal{U}$ will produce an estimate $\widetilde{f}$ for the lower endpoint $\underline{f}$ of the range $[\underline{f}, \overline{f}]$ with the accuracy $\leq (0.12)/D \cdot \varepsilon^2$. The lower endpoint $\underline{f}$ is equal to $(D/3) \cdot \varepsilon^2 \cdot \underline{G}$. Thus, from the inequality $|\widetilde{f} - \underline{f}| \leq (0.12)/D \cdot \varepsilon^2$, we can conclude that $|\widetilde{G} - \underline{G}| \leq 0.04$, where we denoted $\widetilde{G} = (D/3) \cdot \varepsilon^{-2} \cdot \widetilde{f}$.

If $\widetilde{G} > 0.04$, this means that $\underline{G} > 0$ and hence, the original formula F is not satisfiable. If $\widetilde{G} < 0.04$, this means that $\underline{G} < 0.04 + 0.04 = 0.08 < 0.09$ and hence, the original propositional formula F is satisfiable. The reduction is completed and thus, the theorem is proven.

Proof of Theorem 2. According to the definition of asymptotic accuracy, we must prove the existence of a width ε_0 such that for all intervals with widths $\varepsilon \leq \varepsilon_0$, the algorithm computes the estimates for the endpoints with the given accuracy.

The estimate from the proof of Proposition 1 gives an estimate with an accuracy $\leq C \cdot \varepsilon^2$ for some C. We want to show that this same estimate provides us with asymptotic accuracy $c \cdot \varepsilon^{2-\delta}$. To show this, we must prove that there exists a value ε_0 such that for all $\varepsilon \leq \varepsilon_0$, the accuracy provided by Proposition 1 is indeed sufficient, i.e., that $C \cdot \varepsilon^2 \leq c \cdot \varepsilon^{2-\delta}$.

It is easy to see that such a value ε_0 does exist for every δ: namely, it is sufficient to pick ε_0 for which $\varepsilon_0^\delta \leq c/C$, i.e., to pick an arbitrary value $\varepsilon_0 \leq (c/C)^{1/\delta}$.

For thus chosen ε_0, for every $\varepsilon \leq \varepsilon_0$, we have $C \cdot \varepsilon^2 \leq c \cdot \varepsilon^{2-\delta}$ and therefore, the simple estimate from the proof of Proposition 1 is the desired asymptotically accurate one. The theorem is proven.

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