Extending T-Norms Beyond [0,1]: Relevant Results of Semigroup Theory

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Abstract

Originally, fuzzy logic was proposed to describe human reasoning. Lately, it turned out that fuzzy logic is also a convenient approximation tool, and that moreover, sometimes a better approximation can be obtained if we use real values outside the interval $[0,1]$; it is therefore necessary to describe possible extension of t-norms and t-conorms to such new values. It is reasonable to require that this extension be associative, i.e., that the set of truth value with the corresponding operation form a semigroup. Semigroups have been extensively studied in mathematics. In this short paper, we describe several results from semigroup theory which we believe to be relevant for the proposed extension of t-norms and t-conorms.

Introduction. In classical (two-valued) logic, there are only two truth values: "true" (which, in the computer, is usually denoted by 1) and "false" (which is usually denoted by 0). To represent the uncertainty of human reasoning, L. Zadeh proposed, in his *fuzzy logic*, to use additional truth values, including truth values which are intermediate between "true" and "false", i.e., intermediate between 1 and 0 [25]. Due to this motivation, the most natural choice of the set of truth values is the interval $[0,1]$. This set is indeed used in most applications of fuzzy logic.

It turned out that fuzzy logic leads to a good description of general systems. This fact, in itself, should not be surprising, because fuzzy logic reflects human reasoning which is known to be good in controlling different systems. Indeed, it was shown that for some reasonable criteria, fuzzy logic indeed leads to an optimal approximation (see, e.g., [9]). When we consider fuzzy logic as a method of representing human uncertainty, then it is usually natural to restrict ourselves to values from the interval $[0, 1]$ (usually, but not always: see, e.g., [10]). However, when we consider fuzzy logic as simply a convenient approximation tool (see, e.g., [8]), then this restriction is no longer justified. Moreover, it has been shown [24] that if we allow negative values and values greater than 1, we sometimes drastically decrease the number of rules which are necessary to describe the same input-output behavior with the same accuracy.

For such values, traditional fuzzy operations, e.g., "and" and "or" operations (t-norms and t-conorms) are not applicable. We therefore need to analyze the possibility of extending traditional fuzzy definitions from the interval [0,1] to more general domains.

It is reasonable to require that this extension be associative, i.e., in mathematical terms, that the set of truth value S with the corresponding operation form a *semigroup*.

Semigroups have been extensively studied in mathematics (see, e.g., [1, 11]), especially *topological semigroups*, i.e., semigroups in which the underlying space is endowed with a topology, and the semigroup operation is continuous in this topology. Some semigroups are too large (infinite-dimensional), such as the semigroup of all operators in an (infinite-dimensional) Hilbert space, etc. These semigroups are useful, e.g., in physics, but for our purpose, the most interesting case is when the topological space S is not too large. In mathematical terms, we will consider the situation when this set S is a *compact* space.

In this paper, we describe several results from compact topological semigroup theory which we believe to be relevant for the proposed extension of t-norms and t-conorms. Our main goal is to give the knowledge-representation interpretation of these results; our related secondary goals are: to attract the attention of researchers in fuzzy theory to these results, and hopefully, to promote new research which will bring these new results closer to practical applications.

Main Result – Classification Theorem for Topological Semigroups, and Its Relation to Evolutionary Computation and to Chu Spaces. In [11], it is shown that an arbitrary compact topological semigroup can be represented as a combination of semigroups with a certain property which are called *simple* semigroups, and a complete classification is given for these simple semigroups (Theorem 1.3.10, p. 40 of [11]). Namely, it turns out that for every simple semigroup S, there exist compact spaces X, Y, a compact group H, and a continuous function $\varphi : Y \times X \to H$ such that S is isomorphic to the set $X \times H \times Y$ of all triples (x, h, y), where $x \in X$, $h \in H$, $y \in Y$, and the semigroup operation $*$ has the form

$$(x_1, h_1, y_1) * (x_2, h_2, y_2) = (x_1, h_1 \cdot \varphi(y_1, x_2) \cdot h_2, y_2). \tag{1}$$

Vice versa, whenever we have two compact spaces X, Y, a compact group H, and a function $\varphi : Y \times X \to H$, the formula (1) defines a simple semigroup.

From this result, we can make three conclusions. Let us first consider the case when we are interested only in *commutative* t-norms, for which $(x_1, h_1, y_1) * (x_2, h_2, y_2) = (x_2, h_2, y_2) * (x_1, h_1, y_1)$ for all x_1, x_2, h_2, h_2, y_1, and y_2. Substituting the formula (1) into this commutativity formula, we can conclude that $x_1 = x_2$ for all $x_1, x_2 \in X$, and similarly, $y_1 = y_2$ for all $y_1, y_2 \in Y$. In other words, we can conclude that both the space X and the space Y consist of a single point. Therefore, $S = X \times H \times Y$ is isomorphic to the group H. So, the only commutative simple semigroup is a topological group, and a general commutative semigroup is a combination of topological groups. For example, $[0,1]$ with the standard multiplication operation is a combination of a set $(0,1]$ which can be extended to a group, and a set $\{0\}$ which is equivalent to a 1-element group.

We have just shown that the commutative operations correspond to the degenerate case when X and Y are one-point sets, and S is isomorphic to H. We can also consider the "opposite" degenerate case in which, vice versa, the group H consists of a single element, and so S is isomorphic to the set $X \times Y$ of all pairs (x, y). For such pairs, (1) reduces to a formula $(x_1, y_1) * (x_2, y_2) = (x_1, y_2)$. In other words, when we combine two elements $s_1 = (x_1, y_1) \in S$ and $s_2 = (x_2, y_2) \in S$, then $s = s_1 * s_2$ is obtained by taking the "head" (x_1) from the element s_1 and adding the "tail" from the element s_2. This operation is exactly the *recombination* operation which is used in genetic algorithms (and, more generally, in evolutionary computations) to simulate the interaction between parents' genes in producing a child's DNA (see, e.g., [2, 3, 4]). Thus, genetic algorithms, which normally are described as a part of soft computing whose origins and ideas are independent of fuzzy logic, appear naturally when we extend fuzzy logic to a more general semigroup operation. Moreover, the most general semigroup operation (1) can be viewed as a combination of fuzzy-logic type operation which corresponds to $S = H$, and a genetic algorithm-type combination which corresponds to $S = X \times Y$.

Our final remark is that according to the above theorem, simple semigroups are in 1-1 correspondence with functions $Y \times X \to H$. Such functions form the basis of a new approach to foundations of concurrency and foundations of computer science in general which is promoted by V. R. Pratt from Stanford under the name of *Chu spaces* (see, e.g, [5, 6, 7, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22, 23]). Thus, a general extension of t-norms naturally leads us to Chu spaces.

Auxiliary Results and Their Relationship With the Existence and Borderline Character of Classical Truth Values. According to [1], Theorem 1.8, and [11], Theorem 1.4.2, if a compact topological semigroup S is *not* a group (and if it satisfies a natural condition that every element $s \in S$ can be represented as $s_1 * s_2$ for some $s_1, s_2 \in S$), then S must have at least two *idempotents* (i.e., values s for which $s * s = s$). Since $s * s = s$ for both "classical" values 0 and 1 ("true" and "false"), this means that we must have *at least* *two* "quasi-classical" values. This result explains the fundamental role (and the origin) of 2-valued logic as the basis of all other logics.

According to [11], Theorem 2.4.9, if S is not a group, a unit element 1 cannot have a Euclidean neighborhood U for which it is inside it; in other words, a unit element must be on the *border* of the semigroup (as in the interval $[0,1]$).

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