Choosing a Physical Model: Why Symmetries?

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One of the main applications of Kolmogorov complexity ideas to data processing is via the Minimum Description Length principle (see, e.g., [1, 11, 13] and references therein). According to this principle, if several different models (or theories) are consistent with the same observations, then we should choose a model with the shortest description, i.e., crudely speaking, a model with the smallest value of Kolmogorov complexity. This principle is in perfect agreement with the Occam principle (it actually formalizes Occam's principle), and it has been successfully applied to various problems.

In particular, it has been successfully used in physics, where Occam's principle originated and where it has been successfully used. In modern fundamental physics, however, symmetry groups play such an important role that often, physicists choose not the simplest model, but the model which corresponds to the simplest symmetry group (see, e.g., [2–9], [12, 14]).

At first glance, this restriction to *symmetry*-defined models seem to prevent us from considering possible simple *non-group* models, and thus, make this symmetry-group version of Occam principle worse than the unrestricted one. However, the success of this direction in theoretical physics seems to indicate that this restriction does not bring any disadvantage at all. Our analysis shows that this restriction is indeed non-essential.

To formalize the physicists' idea, we fix a universal programming language ("computer") f, and define a *symmetry* as a program from this language which transforms strings (e.g., binary strings) into strings and which is a bijection (1-1 and onto). By a *complexity* of a symmetry s, we mean the length $\mathrm{len}(s)$ of this program. We say that a symmetry s *defines a string* x *uniquely* if $s(x) = x$ and $s(y) \neq y$ for all $y \neq x$. Now, for every string x, we can define its *group-symmetric complexity* $C_{\mathrm{sym}}(x)$ as the smallest complexity of a symmetry which defines x uniquely. It turns out that this new complexity is asymptotically equivalent to the usual Kolmogorov complexity $C(x) = C_f(x)$:

Proposition. $|C(x) - C_{\mathrm{sym}}(x)| = O(1)$.

Comments.

- Since the Kolmogorov complexity itself is defined modulo the term $O(1)$, we conclude that $C_{\rm sym}$ is asymptotically equivalent to the usual Kolmogorov complexity. Therefore, choosing the model with the simplest symmetry (i.e., a model x for which $C_{\rm sym}(x)$ is the smallest) is indeed (asymptotically) equivalent to choosing the simplest model.
- This result was first announced in [15].
- An alternative (more physics-oriented) Kolmogorov complexity-based justification of the use of symmetries in physics was given in [10].

Proof. To prove that $|C(x)-C_{\rm sym}(x)|=O(1)$, we will first show that $\forall x(C(x)\le C_{\rm sym}(x)+C_1)$ for some constant C_1 and then, that $\forall x(C_{\rm sym}(x)\le C(x)+C_2)$ for some constant C_2.

Let us prove the first inequality. By definition, $C_{\rm sym}(x)$ is the length of a program s for which $s(x)=x$ and $s(y)\ne y$ for all $y\ne x$. To compute x, we can use the following new program p: Generate, sequentially, all possible binary strings Λ, 0, 1, 00, 01, etc., and check, for each string y, whether $s(y)=y$; as soon as we get $s(y)=y$ for some string y, we stop. This new program p clearly computes the desired string x. This program p has finitely many instructions added to a call to s; thus, its length $\mathrm{len}(p)$ is equal to the length of s plus the (constant) length C_1 of these extra instructions. Hence, $\mathrm{len}(p)=\mathrm{len}(s)+C_1=C_{\rm sym}(x)+C_1$. Since the program p generates x, the Kolmogorov complexity $C(x)$ of x – i.e., the shortest length of the program which generates x – cannot exceed $\mathrm{len}(p)=C_{\rm sym}(x)+C_1$. Thus, indeed, $C(x)\le C_{\rm sym}(x)+C_1$. The first inequality is proven.

Let us now prove the second inequality. By definition, $C(x)$ is the length of a program p (to be more precise, the shortest program) which generates x: $C(x)=\mathrm{len}(p)$. Based on this program p, we can construct the desired symmetry $s(y)$ as follows: For any given input y, the program s first calls p to compute the word x. Then, the program s returns the following string $s(y)$:

- If $y=x$, then we return $s(x)=x$.
- If $y=x^*$, where by x^*, we denote a string which differs from x only in the last bit, then we return the empty string: $s(x^*)=\Lambda$.
- If y is an empty string ($y=\Lambda$), then we return $s(\Lambda)=x^*$.
- For all other input strings y, we return $s(y)=y^*$.

One can easily check that this program is indeed a bijection, i.e., a symmetry. For this symmetry, we have $s(x)=x$ and $s(y)\ne y$ for all strings $y\ne x$; thus, this symmetry defined the string x uniquely. The program s has finitely many instructions added to a call to p; hence, its length $\mathrm{len}(s)$ is equal to the length of p plus the (constant) length C_2 of these extra instructions. Thus, $\mathrm{len}(s)=\mathrm{len}(p)+C_2=C(x)+C_2$. Since the symmetry s defines x uniquely, the group-symmetric complexity $C_{\rm sym}(x)$ of x – i.e., the shortest length of the symmetry which defines x uniquely – cannot exceed $\mathrm{len}(s)=C(x)+C_2$. Thus, indeed, $C_{\rm sym}(x)\le C(x)+C_2$. The second inequality is also proven, and so is the proposition.

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