Discrete (Granular) Logics: A New (Natural) Notion of Continuity, With a Complete Description of All Continuous Granular Logics

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Abstract. *In most knowledge-based systems, the experts' uncertainty is described by a real number from the interval* $[0,1]$ *(this number is called subjective probability, degree of certainty, etc.). However, experts usually use a small finite set of words to describe their degree of unecratinty; thus, to adequately describe the expert's optinion, it is desirable to use a finite (granular) logic. If all we know about the expert's opinion on two statements* A *and* B *is this expert's degrees of certainty* $d(A)$ *and* $d(B)$ *in these two statements, and the user asks a query "A&B?", then we need to estimate the degree* $d(A\&B)$ *based on the given values* $d(A)$ *and* $d(B)$. *In this paper, we formalize the natural demand that gradual changes in* $d(A)$ *and* $d(B)$ *must lead to gradual changes in our estimate for* $d(A\&B)$ *(we called it continuity). We show that the only continuous &-operation is* $\min(a,b)$. *Likewise, the only continuous* $\vee$*-operation is* $\max(a,b)$, *the only continuous "not"−operation corresponds to* $f(a)=1-a$, *etc.*

1 Discrete (Granular) Logics Are Necessary for Adequately Describing Expert's Uncertainty

A large part of our knowledge comes from experts. Experts are often uncertain about the statements they make; it is therefore desirable to represent this degree of uncertainty in a knowledge-based system. There exist many formalisms for describing this uncertainty (see, e.g., [14]). In most of these formalisms, the expert's uncertainty in each statement is represented by a number (subjective probability, degree of certainty, etc.) which can take any value from the interval $[0,1]$.

One way to elicit this number from an expert is to ask her to estimate her degree of confidence in a given statement on a scale, say, from 0 to 10; then, if we get 7 on a scale of 0 to 10, we can say that the expert's degree of certainty in a given statement is 7/10=0.7. An expert can probably meaningfully distinguish between degree of certainty 0.7 and 0.8 (or, at least, 0.6 and 0.8), but it is highly improbable that an expert would be able to meaningfully distinguish between, say degrees of certainty 0.7 and 0.701. Indeed, it is known in psychology that, in general, humans are most comfortable with 5 to 9 items to choose from ("7 plus minus 2" law, see, e.g., [7, 8]); thus (see, e.g., [4]), an expert can probably use no more than 9 different values to describe her degree of certainty.

This conclusion is in good agreement with the fact that experts usually describe their degrees of certainty by *words*, and there are very few words which are actually used (like "probably", "certainly", etc.). Thus, to adequately describe the expert's subjective degree of confidence, we must restrict ourselves to a finite set of possible degrees, i.e., to a *finite* (discrete) logic.

2 There Is Also a Computational Advantage in Using Discrete Logics

Description of uncertainty in terms of a finite logic is *granular* – granules correspond to different words – while the description in terms of real numbers is not granular.

There is an additional *computational* advantage of using finite (granular) logics instead of arbitrary real numbers: when we use a finite logic, we need fewer bits to store the information about uncertainty, and hopefully, smaller computation time to process it.

3 We Must Define & and $\vee$-Operations

Representing the truth values (= degrees of certainty) inside a computer is not all: we must be able to process these values. For example, suppose that we know the truth values $d(A)$ and $d(B)$ of two statements A and B, and the user asks a query "$A\&B$?". Since we are not sure whether A and B are true, we are also not sure whether $A\&B$ is true or not. Therefore, the only possible answer that we can give to this query is to describe a (reasonable) degree of belief $d(A\&B)$ in $A\&B$. If the only information that we have about A and B consists of their truth values, then we must somehow produce this reasonable estimate $d(A\&B)$ based on the known values $d(A)$ and $d(B)$. In other words, we must have a function (moreover, an algorithm) that would transform $d(A)$ and $d(B)$ into $d(A\&B)$. If we denote this function by $f_\&$, then we can describe the resulting "reasonable" estimate for $d(A\&B)$ as $f_\&(d(A), d(B))$.

In case both $d(A)$ and $d(B)$ coincide with "true" or "false", this function must coincide with the usual $\&-$operation that is defined on a classical set of truth values $\{0, 1\}$. Therefore, this function $f_\&$ is called an *&-operation.*

Likewise, there must exist a function $f_\vee$ that corresponds to $\vee$ and is therefore called an $\vee$-operation, and a function $f_\neg$ (an $\neg$-operation) that generalizes "not" to the bigger set of truth values.

A set with logical operations on it ("and", "or", and "not") is usually called a *logic.* A logic that is a finite set is called a *finite logic.* Our finite set of truth values has all these operations, and is therefore a finite logic.

Therefore, an ideal representation of degrees of uncertainty must form a finite logic.

4 How to Choose &- and $\vee$-Operations For Finite Logics: Empirical Solution

Since our main objective is to represent experts' beliefs in the most adequate manner, it is reasonable to choose &- and $\vee$-operations so as to provide the best description of the human reasoning with uncertainty. To do this, we must first ask the experts to estimate their degrees of belief in different statements and their logical combinations. Then, we choose a function $f_\&$ as follows: For every pair of degrees of belief a and b, we find all the statements in our record for which the degree of belief was a ($d(A) = a$), and all the statement B for which $d(B) = b$. For different A and B, we look for the truth values that the experts assigned to the statements $A\&B$. For different A and B, these truth values may be different; we find the "average" one (e.g., the one that is most frequent) and use it as $f_\&(a, b)$.

In a similar way, we can experimentally determine $f_\vee(a, b)$.

This is (in essence) the method that was originally used to choose &- and $\vee$-operations in one the first successful expert systems MYCIN (see, e.g., [2]). More recently, a similar method was efficiently used to produce &- and $\vee$-operations on finite logics in a MILORD system [1, 13].

5 How to Choose &- and $\vee$-Operations For Finite Logics: The Need For Theoretical Foundations

If we can afford to perform the above-described procedure, fine, this procedure is the ideal solution to the choice problem. However, already the authors of MYCIN noticed that it is a very expensive and time-consuming procedure [2]. So, what to do if we cannot afford it, but still have to choose &- and $\vee$-operations?

In this case, we need to develop theoretical methods to choose these operations. The authors of MILORD formulated reasonable conditions that &- and $\vee$-operations must satisfy [1, 13]. However, these are several different operations that satisfy all these conditions. Hence, the problem of choice remains.

6 Formulation of the Problem

At present, this choice problem is solved in the following manner. In the majority of actual expert systems the set of possible truth values is infinite (see, e.g., [2, 14]; MILORD is one of the few exceptions). Usually, the numbers from the interval [0,1] are used to represent degrees of belief. The reason for choosing this interval is very simple: inside the computer, "true" is usually represented as 1, and "false" as 0. So, it is reasonable to represent all intermediate degrees of belief by real numbers that are intermediate between 0 and 1.

If we assume that all numbers from [0,1] are possible, then we need to define &- and $\vee$-operations as functions from $[0,1] \times [0,1]$ to [0,1]. There exist several reasonable approaches that enable us to make a choice of such a function (see, e.g., [9]).

These approaches provide us with reasonable &- and $\vee$-operations, but they essentially depend on the assumption that *all* numbers from the interval [0,1] can be "truth values". Strictly speaking, this assumption is not true. Therefore, it is reasonable to formulate the following problem: if we are unable to elicit these operations from the experts, can we still choose them using only the actual truth values?

7 How We Are Going to Solve This Problem: The Notion of Continuity

In order to solve this problem, we will assume that both &- and $\vee$-operations $f_{\&}(a, b)$ and $f_{\vee}(a, b)$ are "continuous" in the following sense. If we gradually (= without skipping any intermediate values) increase our degrees of belief $a = d(A)$ and $b = d(B)$, then the resulting degrees of belief $d(A\&B) = f_{\&}(a, b)$ and $d(A \vee B) = f_{\vee}(a, b)$ must also change gradually.

It turns out that this reasonable demand is satisfied by only one pair of operations: min and max. This result is in good accordance with the known experiments [5, 12, 15], according to which in many situations, min and max describe human reasoning better than other possible &- and $\vee$-operations.

Definition 1. *By a finite logic, we understand a (partially) ordered finite set L that contains two elements T and F such that $F \le a \le T$ for every $a \in L$. The elements of L will be called truth values, or degrees of belief.*

Motivation. We consider finitely many truth values, that represent different degrees of belief. Sometimes, we are certain that belief expressed by a degree a is stronger than the belief that is expressed by a degree b. For example, $a =$"for certain" is stronger than $b =$"maybe'. We will denote this by $a > b$. So, on our set of truth values, there is a ordering relation.

In particular, if we denote the degree of belief that expresses our absolute certainty in A, by T (T from "true"), and the degree of belief that expresses the absolute belief in $\neg A$ by F (from "false"), then $F \le a \le T$ for an arbitrary degree of belief a.

It is possible that for some words that describe uncertainty, there is no clear understanding which of them corresponds to greater belief (e.g., it is difficult to compare "probable" and "possible"). Therefore, we do not require that this ordering is a total (linear) order, it can be only partial.

Definition 2. *Let L be a finite logic. By an &-operation on L we mean a function $f_\& : L \times L \to L$ with the following properties:*

- $f_\&(a, b) \le a$;
- $f_\&(a, b) = f_\&(b, a)$;
- $f_\&(a, F) = F$;
- *if* $a \le a'$ *and* $b \le b'$, *then* $f_\&(a, b) \le f_\&(a', b')$.

Motivations.

- The first property is motivated by the following: if we believe in A *and* B, then we must believe in both statements A and B; therefore, our belief in $A\&B$ is either of the same strength or less strong than our belief in A.
- The second property is motivated by the fact that "$A\&B$" and "$B\&A$" are equivalent statements, so it is reasonable to demand that our estimated degree of belief in $A\&B$ ($= f_\&(d(A), d(B))$) is the same as the estimated degree of belief in $B\&A$ ($= f_\&(d(B), d(A))$).
- The third property expresses the following: if B is false, then "A and B" is false for all A.
- The fourth property means that if the degree of belief in A and B increases (i.e., if we found additional reasons to believe in A or B), then the resulting degree of belief in $A\&B$ must either increase, or stay the same.

Comment. This definition is similar to the usual definition of a t-norm (see, e.g., [6, 11]) and to the definition of an &-operation on a finite logic from [1, 13]. The reader may notice, however, that we do not require some additional properties that are usually required for a t-norm, like associativity ($f_\&(a, f_\&(b, c)) = f_\&(f_\&(a, b), c)$)). The reason is that in our case, as we will see from our results, associativity automatically follows from the other properties.

Definition 3. *Let L be a finite logic. By an $\vee$-operation on L we mean a function $f_\vee : L \times L \to L$ with the following properties:*

- $f_\vee(a, b) \ge a$;
- $f_\vee(a, b) = f_\vee(b, a)$;
- $f_\vee(a, T) = T$;
- *if* $a \le a'$, *and* $b \le b'$, *then* $f_\vee(a, b) \le f_\vee(a', b')$.

Motivations for these demands are similar to the ones given for an &-operation.

Definition 4. *We say that an element $a' \in L$ immediately follows a (and denote it by $a \ll b$, or $b \gg a$) if $a < a'$, and there exists no c such that $a < c < a'$. We say that a function $f : L \to L$ is discontinuous if there exist elements a, a', c such that $a \ll a'$, and either $f(a) < c < f(a')$, or $f(a') < c < f(a)$.*

Motivation. If such values a, a', c exist, this means that when we gradually increase our degree of belief from a to a' (gradually in the sense that we do not skip any intermediate values), then the resulting value of f "jumps" from $f(a)$ to $f(a')$, skipping an intermediate value c. So, in this sense, the function f is discontinuous.

We can use a similar definition for a function of two variables:

Definition 5. *A function $f : L \times L \to L$ is called discontinuous is there exist the values a, a', b, b', c for which the following three conditions are true:*

- $a \ll a'$, $a' \ll a$, or $a = a'$;
- $b \ll b'$, $b' \ll b$, or $b = b'$;
- $f(a, b) < c < f(a', b')$, or $f(a', b') < c < f(a, b)$.

Comment. The first condition means that a gradually changes into a' (i.e., either a' immediately follows a, or a immediately follows a', or a' equals a). The second condition means that b gradually changes into b'. The third condition means that there is a "gap" between $f(a, b)$ and $f(a', b')$.

Definition 6. *A function is called continuous if it is not discontinuous.*

Comments.

- If a function f is continuous in the intuitive sense of this word, then it cannot have discontinuities in the sense of Definitions 4 and 5, and therefore it will be continuous in the sense of Definition 6. We do not claim, however, that an arbitrary function that satisfies Definition 6 is intuitively continuous, because there may be other types of discontinuity. We will prove that this weak continuity is sufficient to select &- and $\vee$-operations.
- It is worth mentioning that usually in mathematics, continuity is understood as continuity with respect to some topology. For finite sets, however, this notion is not applicable: on a finite set, we either have a discrete topology (in which case all functions are continuous), or a topology that is reduced to an ordering relation, in which case monotonic functions and only they are continuous (see, e.g., [3]). This monotonicity is not enough for us: we have already included monotonicity in our definitions of &- and $\vee$-operations, and we want to formalize the evident fact that some monotonic operations are "continuous" (in intuitive sense), and some are not. Hence, we had to use new definitions of continuity.

Now, we are ready to formulate the main results of this paper:

8 Main Results

Proposition 1. *If f is a continuous &-operation on a finite logic L, then L is linearly ordered, and $f(a, b) = \min(a, b)$.*

Comments.

- For a linearly ordered set, $\min(a, b)$ is defined as the smallest of a and b.
- For reader's convenience, all the proofs are placed in the last section of the paper.
- This result was first mentioned in our survey [10].

Proposition 2. *If f is a continuous $\vee$-operation on a finite logic L, then L is linearly ordered, and $f(a, b) = \max(a, b)$.*

9 Example

Let us give an example of an &-operation that is different from min, and show that it is really discontinuous. As a finite logic, let us take the set of 11 numbers $\{0, 0.1, 0.2, \ldots, 0.9, 1.0\}$ with natural order. We thus defined L as a subset of the interval [0,1]. From the statistical viewpoint, it is natural to consider the following &-operation which describes "and" for independent events: $f(a, b) = a \cdot b$. This operation, unlike min, cannot be directly applied to the chosen values, because, e.g., $0.6 \cdot 0.6 = 0.36$ and 0.36 does not belong to the set of 11 chosen values. This difficulty is, however, easy to overcome: we can take as $f(a, b)$ the number from L that is the closest to $a \cdot b$ (and if there are two closest numbers, like 0.2 and 0.3 for $0.25 = 0.5 \cdot 0.5$, choose the biggest of these two). For this operation, we will have $f(0.6, 0.6) = 0.4$, $f(0.3, 0.5) = 0.2$, etc.

Let us now consider the case when we have two statements A and B, and our degree of belief in each of them is equal to 0.9. Then, our degree of belief in $A\&B$ is equal to $f(0.9, 0.9) = 0.8$. In the chosen set L, 1.0 immediately follows 0.9, which means that an increase in the degree of belief from 0.9 to 1.0 can be called gradual. So, we can consider the possibility that our degrees of belief in both A and B gradually increase from 0.9 to 1.0. After this increase, the degree of belief in $A\&B$ becomes equal to $f(1.0, 1.0) = 1.0$. So, we gradually increased our degrees of belief in A and B, but the resulting degree of belief in $A\&B$ "jumped" from 0.8 to 1.0, skipping the value 0.9. Hence, this function f is discontinuous.

In Definition 5, we can thus take $a = b = 0.9$, $a' = b' = 1.0$, and $c = 0.9$.

10 Continuous Operations Describing Other Logical Connectives

Let us now describe continuous operations with degrees of belief that correspond to other logical connectives.

Definition 7. *By a $\neg$-operation on L we mean a function $f : L \to L$ such that $f(T) = F$ and $f(F) = T$.*

Motivation. This condition simply means that if A is absolutely true, then $\neg A$ is absolutely false, and vice versa.

Proposition 3. *If $L = \{F = a_0 < a_1 < a_2 < \ldots < a_n = T\}$ is a linearly ordered finite logic, and f is a continuous $\neg$-operation on L, then $f(a_i) = a_{n-i}$.*

Comment. We can represent this result in a manner that is closer to the traditional representation of uncertainty, if we describe each degree of belief a_i by a real number i/n. Then, for each truth value a, $f_\neg(a) = 1 - a$. This is exactly the operation natural for probabilities.

Let us now describe the implication operations.

Definition 8. *Let L be a finite logic. By an $\to$-operation on L we mean a function $f_\to : L \times L \to L$ with the following properties:*

- $f_\to(F, a) = T$;
- $f_\to(T, a) = a$;
- $f_\to(a, T) = T$;
- $f_\to(a, a) = 1$;
- *if* $a \le a'$, *then* $f_\to(a, b) \ge f_\to(a', b)$.

Motivations. The intended meaning of the function $f_\to(a, b)$ is as follows: if we know the degrees of belief $a = d(A)$ and $b = d(B)$ in some statements A and B, then $f_\to(a, b)$ is a reasonable degree of belief in the statement $A \to B$ ("A implies B"). With this interpretation in mind:

- The first of the above properties states that anything follows from a false statement.
- The second property states that to believe that A follows from an an absolutely true statement is the same as to believe that A is true, and therefore, the corresponding degrees of belief must coincide.
- The third condition means that a true statement follows from everything.
- The fourth condition means that for any statement A, A follows from A (and therefore, the degree of belief in $A \to A$ must be equal to T).
- The last condition is related to the third one: Namely, the third one says that if A is false, then $A \to B$ is always true. Therefore, if for some reason our degree of belief in a statement A decreases (from a' to a), then our belief that A can be false will correspondingly increase. Therefore, our degree of belief that $A \to B$ is true, will also increase. Hence, it is reasonable to demand that $f_\to(a', b) \le f_\to(a, b)$.

Proposition 4. *If $L = \{F = a_0 < a_1 < \ldots < a_n = T\}$ is a linearly ordered finite logic, and f is a continuous $\to$-operation on L, then $f(a_i, a_j) = a_{\min(n, n+j-i)}$.*

Comment. If we describe a_i by a real number i/n, then this $\to$-operation turns into $f(a, b) = \min(1, 1 + b - a)$.

11 Proofs

11.1 Proof of Proposition 1

1°. Let us first prove that every element $a \in L$ can be connected to T by a finite chain $T = a_0 \gg a_1 \gg \ldots \gg a_k = a$ $(k \geq 0)$.

Indeed, if $a = T$, then we already have a chain, with $k = 0$.

If $a \neq T$, then according to our definition of a finite logic, we have $a < T$. If $a \ll T$, then we have a chain $a_0 = T$, $a_1 = a$. If $a \not\ll T$, then, according to the definition of $\ll$, it means that there exists a c such that $T > c > a$. If $T \gg c$, and $c \gg a$, then we have a desired chain. Else, we can insert additional elements in between them, etc.

On each step of this procedure, we either have a chain, or we can insert more elements into a sequence $T = a_0 > a_1 > \ldots > a_n = a$. Since there are only finitely many elements in the set L, and all a_i are different, this insertion cannot go on forever. Therefore, sooner or later, it will stop, and we will get the desired chain.

2°. Let us now prove that $f(a,a) = a$ for every $a \in L$.

Indeed, suppose that $a \in L$ is given. According to 1°, there exists a chain $T = a_0 \gg a_1 \gg \ldots \gg a_k = a$ that connects T and a.

If $k = 0$, then $a = T$, and $f(T,T) = T$ follows from the properties of an &-operation.

So, we can assume that $k > 0$. We will prove that $f(a,a) = a$ by reduction to a contradiction. Indeed, suppose that $f(a,a) \neq a$. Hence, $f(a_0,a_0) = a_0$, and $f(a_k,a_k) \neq a_k$. Let us denote by p the smallest integer for which $f(a_p,a_p) \neq a_p$. From this definition of p it follows, in particular, that $f(a_{p-1},a_{p-1}) = a_{p-1}$.

Since f is an &-operation, we can conclude that $f(a_p,a_p) \leq a_p$. Since $f(a_p,a_p) \neq a_p$ (by the choice of p), we conclude that $f(a_p,a_p) < a_p$.

Therefore, we have $a_p \ll a_{p-1}$, and $f(a_p,a_p) < a_p < a_{p-1} = f(a_{p-1},a_{p-1})$, i.e., f is discontinuous (here, $a = b = a_p, a' = b' = a_{p-1}$, and $c = a_p$). However, we assumed that f is continuous.

This contradiction proves that $f(a,a)$ cannot be different from a, so $f(a,a) = a$ for all a.

3°. Let us prove that L is linearly ordered, i.e., for every two elements $a, b \in L$, either $a = b$, or $a < b$, or $b < a$.

Indeed, let us take $a, b \in L$. Following 1°, we will form chains $T = a_0 \gg a_1 \gg \ldots \gg a_k = a$, and $T = b_0 \gg b_1 \gg \ldots \gg b_l = b$. Let us denote by p the biggest integer for which a_p and b_p are both defined and equal to each other ($a_p = b_p$).

If $p = k = l$, then $a = a_k = a_p = b_p = b_l = b$, i.e., $a = b$.

If $p = k \neq l$, then $a = a_k = b_p \gg b_{p+1} \gg \ldots \gg b_l = b$, therefore $a > b_{p+1} > \ldots > b_l = b$, and $a > b$.

Likewise, if $p = l \neq k$, then $b > a$.

Let us prove that the remaining case when $p < k$ and $p < l$, is impossible. Indeed, in this case, both a_{p+1} and b_{p+1} are defined and different from each other. Since f is an &-operation, we can conclude that $f(a_{p+1},b_{p+1}) \leq a_{p+1}$ and $f(a_{p+1},b_{p+1}) = f(b_{p+1},a_{p+1}) \leq b_{p+1}$.

The first inequality means that we have two possibilities: $f(a_{p+1},b_{p+1}) = a_{p+1}$, and $f(a_{p+1},b_{p+1}) < a_{p+1}$. We will show that in both cases, we have a contradiction.

Suppose first that $f(a_{p+1},b_{p+1}) = a_{p+1}$. We already know that $f(a_{p+1},b_{p+1}) \leq b_{p+1}$, so $a_{p+1} \leq b_{p+1}$. We chose p in such a way that $a_{p+1} \neq b_{p+1}$ (and $a_p = b_p$), therefore $a_{p+1} < b_{p+1}$. So, $a_{p+1} < b_{p+1} < b_p = a_p$. The existence of the intermediate value b_{p+1} contradicts the assumption that $a_{p+1} \ll a_p$. So, in this case, we have a contradiction.

Let us now consider the case when $f(a_{p+1},b_{p+1}) < a_{p+1}$. Since $a_p = b_p$ (because of our choice of p), and $f(a,a) = a$ for all a (this we have proved), we have $f(a_{p+1},b_{p+1}) < a_{p+1} < a_p = f(a_p,a_p) = f(a_p,b_p)$. Therefore, in this case, $a_{p+1} \ll a_p$, $b_{p+1} \ll a_p$, and $f(a_{p+1},b_{p+1}) < a_{p+1} < f(a_p,b_p)$. Hence, we have a proof that f is discontinuous (with $a = a_{p+1}, b = b_{p+1}, a' = a_p, b' = b_p$, and $a_{p+1} = c$). This contradicts to our assumption that f is continuous.

Summarizing: in both cases the assumption that $p < k$ and $p < l$ led us to a contradiction. So, either $p = k$, or $p = l$, in which cases, as we have already proved, either $a = b$, or $a < b$, or $b < a$. We have thus proved that L is linearly ordered.

4°. It now remains to prove that $f(a,b)=\min(a,b)$ for all a,b.

Since L is finite and linearly ordered, we can order all its elements into a sequence $F=a_0<a_1<\ldots<a_{n-1}<a_n=T$. So, each element of L has the form a_i, and $a_i<a_j$ if and only if $i<j$.

In these terms, it is necessary to prove that $f(a_i,a_j)=a_{\min(i,j)}$. If $i=j$, this follows from 2°. Let us now consider the case, when $i<j$, and prove that in this case, $f(a_i,a_j)=a_i$.

Let us fix j. For every i, the value of $f(a_i,a_j)\in L$ is equal to a_k for some k. Let us denote this k by $\phi(i)$. So, in these denotations, $f(a_i,a_j)=a_{\phi(i)}$. The desired equality can be then expressed as $\phi(i)=i$ for all $i\le j$.

We already know the value of this function $\phi(i)$ for $i=0$ and $i=j$: Indeed, since f is an &-operation, we have $f(T,a_j)=T$, i.e., in our notations, $f(a_0,a_j)=a_0$, hence $\phi(0)=0$. From 2°, it follows that $f(a_j,a_j)=a_j$, so $\phi(j)=j$.

Since f is an &-operation, it is monotonically non-decreasing, hence ϕ is also non-decreasing: $0=\phi(0)\le\phi(1)\le\phi(2)\le\ldots\le\phi(j)=j$.

Since $a_i\ll a_{i+1}$, and f is continuous, there cannot be a gap between $F(a_i)$ and $F(a_{i+1})$. Therefore, for each i, we must either have $\phi(i+1)=\phi(i)$, or $\phi(i+1)=\phi(i)+1$. Since

$$j=j-0=\phi(j)-\phi(0)=(\phi(j)-\phi(j-1))+\ldots+(\phi(2)-\phi(1))+(\phi(1)-\phi(0)),$$

the number j is the sum of j differences, each of which is ≤ 1. If one of these differences was smaller than 1, then the entire sum would be smaller than j. Since this sum is equal to j, none of these differences can be smaller than 1. Therefore, $\phi(i+1)-\phi(i)=1$ for all i. This equality is equivalent to $\phi(i+1)=\phi(i)+1$.

So, we have $\phi(0)=0$, and $\phi(i+1)=\phi(i)+1$ for all $i<j$. From this, we can conclude (using mathematical induction), that $\phi(i)=i$ for all $i<j$. By definition of ϕ this means that $f(a_i,a_j)=a_{\phi(i)}=a_i$, i.e., that $f(a,b)=\min(a,b)$.

If $i>j$, then the desired equality follows from the fact that f is commutative ($f(a_i,a_j)=f(a_j,a_i)$), and so this case is reduced to the previous one. The proposition is proven.

Comment. The ideas of this proof are similar to the proofs from [1, 13].

11.2 Proof of Proposition 2

The proof of Proposition 2 is similar, with the only difference that we must use F instead of T, $>$ instead of $<$, and $\ll$ instead of $\gg$.

11.3 Proof of Proposition 3

For every $a_i\in L$, $f(a_i)=a_k$ for some k. Let us denote this k by $\psi(i)$. In these terms, $f(a_i)=a_{\psi(i)}$. The definition of a negation operation means that $\psi(0)=n$, and $\psi(n)=0$. Continuity means that for each i, since $a_i\ll a_{i+1}$, there cannot be anything in between $a_{\psi(i)}=f(a_i)$ and $a_{\psi(i+1)}=f(a_{i+1})$. In other words, there cannot be anything in between $\psi(i)$ and $\psi(i+1)$. So, $\psi(i)$ and $\psi(i+1)$ must either coincide, or be neighbors: $|\psi(i+1)-\psi(i)|\le 1$. In particular, $\psi(i+1)-\psi(i)\ge -1$.

Now, the difference $\psi(n)-\psi(0)=0-n=-n$ can be represented as

$$-n=\psi(n)-\psi(0)=(\psi(n)-\psi(n-1))+\ldots+(\psi(2)-\psi(1))+(\psi(1)-\psi(0)).$$

So, $-n$ is represented as the sum of n terms each of which is ≥ -1. If one of them was greater than -1, then the entire sum would have been greater than $-n$. Since this sum is equal to $-n$, we can conclude that all the terms in this sum are exactly equal to -1: $\psi(i+1)-\psi(i)=-1$. Therefore, $\psi(0)=n$, and $\psi(i+1)=\psi(i)-1$ for all i. From these two conditions, one can easily conclude that $\psi(i)=n-i$. Hence, $f(a_i)=a_{\psi(i)}=a_{n-i}$. The proposition is proven.

11.4 Proof of Proposition 4

For every i and j, the value $f(a_i,a_j)$ belongs to L and is, therefore, equal to a_k for some k. Let us denote this k by $h(i,j)$, so that $f(a_i,a_j)=a_{h(i,j)}$.

We will consider two cases: $i\le j$, and $i>j$.

Let us first assume that $i \le j$. According to the definition of an $\to$-operation, $f(a_j, a_j) = T = a_n$, and $f(F, a_j) = f(a_0, a_j) = T = a_n$. In terms of h, it means that $h(j,j) = n$, and $h(0,j) = n$. From the fifth property of an $\to$-operation, we can conclude that $h(0,j) \ge h(1,j) \ge \ldots \ge h(j-1,j) \ge h(j,j)$. Since $h(0,j) = h(j,j) = n$, we can conclude that all the terms in this inequality are equal to n, i.e., $h(i,j) = n$ if $i \le j$.

Let us now consider the case, when $i > j$. According to the definition of an $\to$-operation, for every j, we have $f(T, a_j) = a_j$, and $f(a_j, a_j) = 1$. In terms of h, this turns into $h(n,j) = j$ and $h(j,j) = n$. Since f is continuous, we can conclude (just like we did in the proofs of Theorems 1 and 3) that $|h(i+1,j) - h(i,j)| \le 1$. So, the difference between $h(n,j)$ and $h(j,j)$ that is equal to $j - n = -(n-j)$, can be represented as the sum of $n - j$ differences $h(i+1,j) - h(i,j)$ $(j \le i < n)$, each of which is ≥ -1. If one of these differences was > -1, then the entire sum would be $> -(n-j)$. Therefore, all these difference are equal to -1. So, $h(j,j) = n$, and for $i \ge j$, $h(i+1,j) = h(i,j) - 1$. Therefore, for $i \ge j$, we have $h(i,j) = n - (i-j) = n + j - i$.

Combining the cases $i \le j$ and $i > j$, we get the desired formula. The proposition is proven.

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