# Chu Spaces: Towards New Justification for Fuzzy Heuristics

**Nhu Nguyen and Hung T. Nguyen**
Department of Mathematical Sciences
New Mexico State University
Las Cruces, NM 88003, USA
{nnguyen@nmsu.edu, hunguyen@nmsu.edu}

**Vladik Kreinovich**
Department of Computer Science
University of Texas at El Paso
El Paso, TX 79968, USA
vladik@cs.utep.edu

**Abstract**

We show that Chu spaces, a new formalism used to describe parallelism and information flow, provide uniform explanations for different choices of fuzzy methodology, such as choices of fuzzy logical operations, of membership functions, of defuzzification, etc.

## What Are Chu Spaces?

### World According to Classical Physics

It is well known that measurements can change the measured object: e.g., most methods of chemical analysis destroy a part of the analyzed substance; testing a car often means damaging it, etc. However, in classical (pre-quantum) physics it was assumed that in principle, we can make this adverse influence as small as possible.

Therefore, ideally, each measurement can be described as a function $r(x)$ from the set of all objects $X$ to the set $K$ of all measurement results. These measurements lead to a *complete knowledge* in the sense that an object $x$ can be uniquely reconstructed from the results $r(x)$ of all such measurements.

### Non-Determinism in Modern Physics: Enter Chu Spaces

In modern physics, starting from quantum mechanics, it was realized that ideal non-influencing measurements are impossible: the more accurately we measure, the more we change the object of measurement. As a result, it is not possible to uniquely reconstruct an object from measurement results. In other words, each measurement is a function $r(x,y)$ of *two* variables: an object $x$ and a (not completely known) measuring device $y$. Such a function describes a so-called *Chu space* (see, e.g., (Bar79; GP93; Gup94; Pra95; Pra95a; Pra95b; VP95; Bar96; Pra97; Pra97a; Pra98)).

### Precise Definition of a Chu Space

To be more precise, to define a Chu space, we must fix a set $K$ (of possible values). Then, *a $K$-Chu space is defined as a triple $(X,r,Y)$, where $X$ and $Y$ are sets, and $r: X\times Y\to K$ is a function which maps every pair $(x,y)$ of elements $x\in X$ and $y\in Y$ into an element $r(x,y)\in K$.*



### Back to Measurements: Enter Automorphisms of Chu Spaces

The fact that $x$ cannot be uniquely reconstructed from such measurements means that the same measurement results can be explained if we take slightly different objects ($f(x)$ instead of $x$) and, correspondingly, slightly different measuring instruments ($g(y)$ instead of $y$): $r(x,y)=r(f(x),g(y))$. This formula takes a more symmetric form if we consider, instead of $g(y)$, an inverse function $y=h(z)=g^{-1}(z)$:

$$r(x,h(z))=r(f(x),z). \tag{1}$$

A pair of functions $(f,h)$ which satisfies the property (1) for all $x\in X$ and $z\in Y$ is called an *automorphism* of a Chu space.

### From Physics to General Problem Solving

A *general problem* is: given $x$, find $y$ for which a known (easy to compute) function $r(x,y)$ takes the desired value $d$ (e.g., 0). A problem $r$ is *reduced* to a problem $r'$ if it is possible, for each instance $x$ of the first problem, to find the corresponding instance $f(x)$ of the second problem, so that from each solution $z$ of the second problem, we can compute a solution $h(z)$ to the original problem, i.e.,

$$r(x,h(z))=r'(f(x),z). \tag{2}$$

(This notion is central in computational complexity theory, in the definitions of NP-hardness etc., see, e.g., (GJ79; Pap94).) Such a pair $(f,h)$ is called a *morphism* of Chu spaces.

### Morphism of Chu Spaces: Precise Definition

In general: *If we have two Chu spaces $A=(X,r,Y)$ and $B=(X',r',Y')$, then a pair of functions*

$$(f:X\to X', h:Y'\to Y)$$

*is called a morphism of Chu spaces if it satisfies the property (2) for all $x\in X$ and for all $z\in Y'$.*

## Applications to Parallelism and to Information Flow

The notion of Chu spaces was actively used by V. Pratt (Stanford) for describing parallel problem-solving algorithms (see, e.g., (GP93; Gup94; Pra95; Pra95a; Pra95b; VP95; Pra97; Pra97a; Pra98)), and by J. Barwise (Indiana) to describe information flow in general (see, e.g., (BS97)).

# Fuzzy as a Natural Particular Case of Chu Spaces

Before we describe how Chu spaces can be used to justify fuzzy heuristics, let us show that fuzzy methodology can indeed be reformulated in Chu-space terms.

Indeed, the main idea of fuzzy methodology is as follows: We want to describe the experts' knowledge about objects from a certain set $O$. To describe these objects, experts use different *properties*; let us denote the set of such properties by $P$. For each object $o \in O$ and for each property $p \in P$, an expert decides whether the object $o$ has the property $p$.

- In some cases, the expert is absolutely sure that the object $o$ satisfies the property $p$.
- In some other cases, the expert is absolutely sure that the object $o$ does not satisfy the property $p$.
- In many other cases, however, the expert is not absolutely sure that the object $o$ satisfies the property $p$.

To describe this uncertain knowledge, we must therefore describe, for each $o \in O$ and for each $p \in P$, a degree $d(o,p)$ which characterizes the expert's certainty that the object $o$ satisfies the property $p$. Usually, this degree is described by a number from the interval $[0,1]$, so that 1 means that the expert is absolutely sure that $o$ satisfies the property $p$, 0 means that the expert is absolutely sure that $o$ does not satisfy the property $p$, and intermediate values represent uncertainty.

Thus, we get a *Chu space*, in which $X$ is the set of all objects, $Y$ is the set of all properties, and $r(x,y)$ is the degree to which the object $x$ satisfies the property $y$.

From this viewpoint, to describe a property $p$, we need to describe, for each object $o$, the number from the interval $[0,1]$ which characterizes our certainty that this object has a given property. In other words, a property can be described as a function from the set of all objects $O$ to the interval $[0,1]$. Such a function is called a *fuzzy set*. Thus, properties are described by fuzzy sets. For this description, the value $r(x,y)$ is the result of applying the function $y$ to the object $x$.

It is natural to consider, for each set of objects $O$, the set of all possible properties $[0,1]^O$. For the corresponding Chu space $(O,r,[0,1]^O)$, the function $r$ takes the form $r(x,y)=y(x)$. This Chu space is denoted by $FUZZ(O)$.

# Chu Spaces as a Uniform Justification for Fuzzy Techniques

## Fuzzy is a Particular Case of Chu Spaces

We have already mentioned that fuzzy knowledge can be naturally described as a Chu space $(X,r,Y)$, where $X$ is the set of all objects, $Y$ is the set of all linguistic properties, and $r(x,y)$ is a degree to which $x$ has a property $y$ (see, e.g., (Pap99)).

This relation was originally done in two steps:

- fuzzy logic can be interpreted as a particular case of so-called *linear logic* (see, e.g., (Gir95; KNW96; NK96; Pap99)), and
- linear logic is naturally interpreted in terms of Chu spaces.

## What We Are Planning to Do

We show that Chu description leads to a uniform justification of numerous choices of fuzzy membership functions, fuzzy logic operations, defuzzification procedures, etc. This justification is in line with a general group-theoretic approach described in our 1997 Kluwer book (NK97) (see also (Kre92; BKLN96; KLN99; NKL99; NKW99; NWK99)).

# The Main Technical Idea Behind Using Chu Spaces as a Foundation for Fuzzy Theory: A Simplified (Non-Fuzzy) Illustration

## Example: A Simple Physical Problem

To better present our main technical idea, we will first illustrate it on a simplified (crisp) example. Let us analyze how the period $t$ of a pendulum depends on its length $l$.

From the purely *mathematical* viewpoint, this dependency can be described by a function of one variable $t=F(l)$, i.e., as a function from real numbers to real numbers. However, from the *physical* viewpoint, such a mathematical description is somewhat unnatural, for the following reason:

- we really want a dependence between *physical quantities* $t$ and $l$;
- in order to describe this dependence as a dependence between real numbers, we must fix some units for measuring both length $l$ and time $t$; thus, the resulting function depends on the specific choice of these units;
- however, the choice of the units is a matter of convention (e.g., to describe length, we can use meters or feet without changing any physical meaning).

It is therefore desirable to have a mathematical description of the dependency of $t$ on $l$ which would reflect the physical dependency without adding any arbitrariness.

### A More Adequate Mathematical Description of the Physical Problem

Such a description can be obtained if we explicitly add the two measuring units $u_l$ (for length) and $u_t$ (for time) to the description of this function, i.e., if we consider the function of the type $t = F(l, u_l, u_t)$, where $l$ is a numerical value of the pendulum's length, $t$ is a numerical value of its period, and $u_l$ and $u_t$ are the measuring units used to describe the corresponding numerical values (described in terms of some standard measuring units).

If we know the dependence $t_0 = F(l_0)$ in standard units, then we can easily describe the new function: Indeed, if we use the length $u_l$ as a unit of length, then in these units, the numerical value $l$ of length means $l_0 = l \cdot u_l$ in the original units, so in the standard units, the pendulum's period is equal to $t_0 = F(l_0) = F(l \cdot u_l)$. Hence, if we use the unit $u_t$ for measuring time intervals, then in this unit, the numerical value of the time period is equal to $t = t_0/u_t = F(l \cdot u_l)/u_t$. In other words, we get $F(l, u_l, u_t) = F(l \cdot u_l)/u_t$.

### Mathematical Model Naturally Reformulated as a Chu Space

The above physically appropriate dependence can be naturally described as a Chu space, with $X$ being the set of all possible units of length, $Y$ the set of all possible time units, $K$ the set of all possible functions of one real variable, and the function $r(u_l, u_t)$ defined as $(r(u_l, u_t))(l) = F(l, u_l, u_t)$. From the mathematical viewpoint, the sets $X$ and $Y$ coincide with the set $R^+$ of all positive real numbers.

### Unit-Invariance Formulated in Precise Terms

Let us now formalize the requirement that this dependence be independent on the choice of the units for measuring length $l$ and time $t$. If we simply change a measuring unit for length or a measuring unit for time, then we get a different numerical dependence. However, for every change of the length unit, there is an appropriate change of a time unit after which the resulting numerical dependence stays the same. This requirement can be formulated as follows.

Suppose that we use a different standard unit for measuring length. Let $\lambda > 0$ be the value of the old standard unit in terms of the new one; then, 1 old standard unit = $\lambda$ new standard units, so $u_l$ old standard units = $u_l \cdot \lambda$ new standard units, i.e., the measuring unit for length whose value was $u_l$ in old standard units has a new value $u'_l = \lambda \cdot u_l$ in new standard units. Similarly, the choice of a new standard unit for time means that we replace the original value $u_t$ by a new value $u'_t = g(u_t)$, where $g(y) = \mu \cdot y$ and $\mu$ is the value of the old standard unit in terms of the new standard unit for time.

In these terms, the above requirement means that for every function $f : X \to X$ of the type $f(u_l) = \lambda \cdot u_l$, there exists a function $g(y)$ of the type $g(u_t) = \mu \cdot u_t$ for which, for every $x \in X$ and $y \in Y$, we have $r(x, y) = r(f(x), g(y))$.

### Unit-Invariance Reformulated in Terms of Chu Spaces

We have already mentioned that this equality describes an *automorphism* of the Chu space. Thus, the above requirement means that for every function $f : X \to X$ from a certain transformation class can be extended to an automorphism $(f, h)$ of the corresponding Chu space.

### The Chu-Space Requirement Describes the desired function

One can show that this condition is satisfied only by functions of the type $t = A \cdot l^\alpha$, with $A$ and $\alpha$ arbitrary constants; the actual pendulum corresponds to $\alpha = -0.5$.

Thus, the Chu space requirement leads to a description of a very narrow class of functions which contain the desired one.

## Application of Our Main Idea to Fuzzy Techniques: Illustration and Other Results

### General Idea

There exist several methods of eliciting fuzzy values from experts, and, in general, different elicitation methods lead to different results. In other words, different methods may result in values corresponding to different *scales* of uncertainty, just like measuring the length in feet or in meters leads to different scales in which the order is preserved but numerical values are different. Similarly to the above illustrative example, it is therefore reasonable to require that the operations with fuzzy values be independent on this choice of a scale.

### Re-scaling in fuzzy theory: an example

One of the most natural methods to ascribe the degree of truth $d(A)$ to a statement $A$ is *polling*: we take several ($N$) experts, and ask each of them whether she believes that $A$ is true. If $N(A)$ of them answer "yes", we take $d(A) = N(A)/N$. Knowledge engineers want the system to include the knowledge of the entire scientific community, so they ask as many experts as possible. But asking too many experts leads to the following negative phenomenon: when the opinion of the most respected professors, Nobel-prize winners, etc., is known, some less self-confident experts will not be brave enough to express their own opinions, so they will either say nothing or follow the opinion of the majority.

How does their presence influence the resulting uncertainty value? After we add $M$ experts who do not answer anything when asked about $A$, the number of experts who believe in $A$ is still $N(A)$, but the total number of experts is bigger $(M + N)$. So the new value of $d(A)$ is $d'(A) = N(A)/(N + M) = c \cdot d(A)$, where we denoted $c = N/(M+N)$. From mathematical viewpoint, this transformation is exactly the same as when we use a different measuring unit in physical measurements.

### Selecting a Hedge: An Example of Using Chu Spaces

How can we describe a *hedge*, i.e., an operation which transforms a degree of truth in a statement $A$ into a degree of truth for a statement "very $A$" or "slightly $A$"? From the purely mathematical viewpoint, we can describe this transformation as a function which transforms a numerical value $d$ of the original degree into the numerical value of the hedged degree $d' = H(d)$. However, the exact numerical type of this function would depend on the scales used to represent both degrees. It is therefore desirable to get a representation which is independent on the choice of these scales.

Similarly to the above illustrative example, we can achieve this representation independence if we describe the hedge function as a mapping $r(x, y)$, where $x$ is a parameter which describes the scale of original degrees, $y$ is a parameter which described the scale of the hedged degrees, and $r(x, y)$ is the description of a hedge function in the scales $x$ and $y$.

Similarly to the above example, the requirement that the hedge transformation be independent on the choice of scales means that every function $f : X \to X$ from the appropriate transformation class (of linear transformations) can be extended to an automorphism of the corresponding Chu space. As a result, we deduce that all such functions have the form $d' = A \cdot d^\alpha$ for some real numbers $A$ and $\alpha$. Indeed, the original Zadeh's hedges use $\alpha = 2$ for "very" and $\alpha = 0.5$ for "slightly". Thus, the Chu space requirement leads to a description of a very narrow class of functions which contain the desired one.

### General Results

In general, we can also have non-linear re-scalings (see, e.g., (Kre92; BKLN96; NK97)). It turns out that the use of these re-scalings enables us to justify all major choices of fuzzy techniques: the existing choices of membership functions, of "and" and "or" operations, of defuzzification, etc.

The mathematics is, in essence, already here: in (Kre92; BKLN96; NK97), we have shown that these choices can be explained by the natural symmetry requirements, and similar to the above examples, these symmetry requirements can be naturally reformulated in terms of Chu spaces.

## Chu Spaces Can Also Describe the General Dependence Between Different Quantities

### Formulation of the Problem

In the previous section, we have mentioned that the Chu space approach helps to justify the existing heuristic techniques of fuzzy methodology. These techniques include the description of possible values of different physical quantities, and the if-then rules describing the relation between these quantities.

The relation between different quantities is not always described by if-then rules; we may have more complicated constraints relating the values of different quantities. In this section, we will show that Chu spaces can describe not only the if-then rules, but also the most general relations between different quantities. In our description, we will use ideas first presented in (KSKI99).

### Crisp Case: The General Description of Possible Dependence Between Two Quantities

Let us start with the *crisp* case, in which, for each value of each physical quantity, we know for sure whether this value is possible or not. In this case, for each quantity, we have a (crisp) *set* of possible values. So, if we have two physical quantities $a$ and $b$, then we have two sets $A$ and $B$ of possible values.

In order to describe possible *dependencies* between two physical quantities $a$ and $b$, let us first describe what it means for $a$ and $b$ to be independent. Intuitively, it means that the set of possible values of the quantity $a$ should not depend on the value of the other quantity $b$, and, vice versa, the set of possible values of the quantity $b$ should not depend on the value of the quantity $a$. Therefore, the pair $(a, b)$ is possible if and only if $a$ is possible *and* $b$ is possible. As a result, the set $S$ of all possible pairs $(a, b)$ coincides with the Cartesian product $A \times B$ of the sets $A$ and $B$.

In general, if a pair $(a, b)$ is possible (i.e., if $(a, b) \in S$), then, of course, both $a$ and $b$ are possible, i.e., $a \in A$ and $b \in B$. Thus, in general, $S \subseteq A \times B$. Since *independence* corresponds to the case when $S = A \times B$, *dependence* corresponds to the situation when $S$ is a proper subset of the Cartesian product $A \times B$. In this case, this set $S$ describes the dependence: e.g., if $a$ is a function of $b$, then the set $S$ is a graph of this function. etc.

### Crisp Case: The General Description of Dependence Can Be Naturally Reformulated in Chu-Space Terms

One way to describe the set $S$ is to describe, for each possible value $a \in A$ of the first quantity, the corresponding set of all possible values of the second quantity $\{b \mid (a, b) \in S\}$. We will denote this set by $f(a)$.

Alternatively, we can describe the same set $S$ by describing, for each possible value $b \in B$ of the second quantity, the corresponding set of all possible values of the first quantity $\{a \mid (a, b) \in S\}$. We will denote this set by $h(b)$.

What is the relation between these two alternative descriptions of the same set $S$ (i.e., of the same dependence between the quantities $a$ and $b$)? To describe the set $S$, we must describe, for each pair $(a,b)\in A\times B$, whether this pair belongs to the set $S$ or not.

- If we use the first description, then the condition $(a,b)\in S$ can be described as $b\in f(a)$.
- If we use the second description, then the same condition $(a,b)\in S$ can be described as $a\in h(b)$.

The fact that the two descriptions describes the same set $S$ means that for every $a\in A$ and for every $b\in B$, the conditions $b\in f(a)$ and $a\in h(b)$ must have the same truth value. If we denote, by $t_\in(b,B)$, the truth value of the statement $b\in B$, then the above equivalence can be reformulated as the following equality:

$$t_\in(a,h(b))=t_\in(b,f(a)).$$

One can easily see that this condition is a particular case of the formula (2) which describes a morphism between two Chu spaces: Namely, here the first Chu space $(X,r,Y)$ is as follows:

- $X$ is the set of all possible values of the first quantity $a$, i.e., $X=A$.
- $Y$ is the set of all possible values of $h(b)$; since $h(b)$ is defined as a set of possible values of $a$, we can conclude that $h(b)$ is a subset of the set $A$. Therefore, $Y$ is the set of all subsets of $A$, i.e., $Y=2^A$.
- The set $K$ of possible values coincides with the binary set $\{0,1\}$ (=$\{$false,true$\}$).
- Finally, the mapping $r:X\times Y\to K$ results in $r(x,y)=1$ or $r(x,y)=0$ depending on whether $x\in y$ or not (i.e., $r(x,y)$ is the truth value of the statement $x\in y$).

Similarly, the second Chu space $(X',r',Y')$ has the form $(2^B,r',B)$, where $K$ is the same, and $r'(x',y')=1$ or $r'(x',y')=0$ depending on whether $y'\in x'$ or not.

## The Chu-Space General Description of Dependence Can Be Naturally Extended to the Fuzzy Case

In the previous section, we have analyzed the case of a *crisp* dependence between two physical quantities $a$ and $b$, i.e., the dependence in which for each pair $(a,b)$, we know for sure whether this pair is possible or not.

In real life, at least for some pairs $(a,b)$, we are often not 100% sure whether this pair $(a,b)$ is possible or not. To describe such an uncertain knowledge about the dependence between $a$ and $b$, we must describe, for each $a\in A$ and $b\in B$, the expert's degree of certainty $d(a,b)\in[0,1]$ that the pair $(a,b)$ is a possible pair of values of the two given physical quantities. In mathematical terms, this uncertainty is therefore characterized by a *function* $d:A\times B\to[0,1]$, i.e., by a fuzzy subset $d$ of the Cartesian product $A\times B$.

We can easily generalize the above Chu-space reformulation so that it applies to such fuzzy sets.

One way to describe the fuzzy set $d$ is to describe, for each possible value $a\in A$ of the first quantity, the corresponding fuzzy set $f(a)\in B$ of all possible values of the second quantity. By definition, a fuzzy subset $f(a)\subseteq B$ of a crisp set $B$ is a function from this crisp set $B$ to the interval $[0,1]$. For each $a$ and $b$, the degree to which the pair $(a,b)$ is possible is equal to $d(a,b)$; therefore, the function $f(a)$ can be defined by the following formula: for every $b$, the result $(f(a))(b)$ of applying this function $f(a)$ to the element $b$ is equal to $d(a,b)$.

Alternatively, we can describe the same fuzzy set $d$ by describing, for each possible value $b\in B$ of the second quantity, the corresponding fuzzy set $h(b)$ of all possible values of the first quantity; here, $(h(b))(a)=d(a,b)$.

These two descriptions of the same fuzzy set $d$ are related by the fact that for each pair $(a,b)$, both descriptions must lead to the same degree $d(a,b)$ of belief that $(a,b)$ is a possible pair. Therefore, the following equality must hold for every pair $(a,b)$:

$$(h(b))(a)=(f(a))(b).$$

This condition is also a particular case of the formula (2) which describes a morphism between two Chu spaces: Namely, here the first Chu space $(X,r,Y)$ is as follows:

- $X$ is the set of all possible values of the first quantity $a$, i.e., $X=A$.
- $Y$ is the set of all possible values of $h(b)$; since $h(b)$ is defined as a fuzzy set of possible values of $a$, we can conclude that $h(b)$ is a fuzzy subset of the set $A$. Therefore, $Y$ is the set of all fuzzy subsets of $A$, i.e., all functions from $A$ to $[0,1]$: $Y=[0,1]^A$.
- The set $K$ of possible values coincides with the interval $[0,1]$.
- Finally, the mapping $r:X\times Y\to K$ results, for each $x$ and $y$, in the degree with which an element $x$ belongs to the fuzzy set $y$; this degree is equal to $y(x)$.

This is a standard Chu-space description $FUZZ(A)$ of all fuzzy subsets of crisp set $A$.

Similarly, the second Chu space $(X',r',Y')$ has the form $([0,1]^B,r',B)$, where $K=[0,1]$ is the same, and $r'(x',y')=x'(y')$. This Chu space differs from the standard fuzzy-logic Chu space $FUZZ(B)$ only in that $X'$ and $Y'$ are swapped; such a Chu space is called a *dual* of the original Chu space $FUZZ(B)$ and it is usually denoted by $FUZZ(B)^\circ$.

So, we can say that a general (fuzzy) dependence of the two physical quantities $a$ and $b$ can be naturally described in Chu-space terms: namely, as a Chu morphism between two Chu spaces $FUZZ(A)$ and $FUZZ(B)^\circ$.

## Acknowledgments

This work was supported in part by NASA under cooperative agreement NCC5-209, by NSF grants No. DUE-9750858 and CDA-9522207, by United Space Alliance, grant No. NAS 9-20000 (PWO C0C67713A6), by the Future Aerospace Science and Technology Program (FAST) Center for Structural Integrity of Aerospace

Systems, effort sponsored by the Air Force Office of Scientific Research, Air Force Materiel Command, USAF, under grant number F49620-95-1-0518, and by the National Security Agency under Grant No. MDA904-98-1-0561.

The authors are thankful to the anonymous referees for useful remarks.

## References

M. Barr, **-Autonomous Categories*, Springer Lecture Notes in Mathematics, Vol. 752, Springer-Verlag, Berlin, 1979.

M. Barr, The Chu construction, *Theory and Applications of Categories*, 2:17–35, 1996.

J. Barwise and J. Seligman, *The logic of distributed systems*, Cambridge University Press, Cambridge, N.Y., 1997.

B. Bouchon-Meunier, V. Kreinovich, A. Lokshin, and H. T. Nguyen, On the formulation of optimization under elastic constraints (with control in mind), *Fuzzy Sets and Systems,* 81(1):5–29, 1996.

M. R. Garey and D. S. Johnson, *Computers and intractability: a guide to the theory of NP-completeness*, Freeman, San Francisco, 1979.

J.-Y. Girard, Linear logic: its syntax and semantics, In: J.-Y. Girard, Y. Lafont, and L. Regnier, eds., *Advances in Linear Logic*, Cambridge University Press, Cambridge, 1995, pp. 1–42.

V. Gupta, *Chu Spaces: A Model of Concurrency*, Ph.D. Dissertation, Stanford University, Sept. 1994.

V. Gupta and V. R. Pratt, Gates Accept Concurrent Behavior, *Proc. 34th Ann. IEEE Symp. on Foundations of Comp. Sci.*, 1993, pp. 62–71.

O. Katai, K. Sato, H. Kawakami, and S. Iwai, "Self-organizing decision making via fuzzy logic-based coding of constraints and objectives", *Proceedings of The Eighth International Fuzzy Systems Association World Congress IFSA'99*, Taipei, Taiwan, August 17–20, 1999, Vol. 2, pp. 556–560.

V. Kreinovich *et al.*, What non-linearity to choose? Mathematical foundations of fuzzy control, *Proceedings of the 1992 International Conference on Fuzzy Systems and Intelligent Control*, Louisville, KY, 1992, pp. 349–412.

V. Kreinovich, H. Nguyen, and P. Wojciechowski, Fuzzy Logic as Applied Linear Logic, *BUlletin for Studies and Exchanges on Fuzziness and its AppLications (BUSEFAL)*, 67:4–13, July 1996.

V. Kreinovich, G. Liu, and H. T. Nguyen, "Chu Spaces – A New Approach to Describing Uncertainty in Systems", *Proceedings of the 1999 IEEE Midwest Symposium on Circuits and Systems*, Las Cruces, New Mexico, August 8–11, 1999 (to appear).

H. T. Nguyen and V. Kreinovich, Fuzzy Logic, Logic Programming, and Linear Logic: Towards a New Understanding of Common Sense, *Proc. NAFIPS'96*, Berkeley, CA, June 20-22, 1996, pp. 546–550.

H. T. Nguyen and V. Kreinovich, *Applications of continuous mathematics to computer science*, Kluwer, Dordrecht, 1997.

H. T. Nguyen, V. Kreinovich, and G. Liu, "Chu Spaces: Towards New Foundations for Fuzzy Logic and Fuzzy Control, with Applications to Information Flow on the World Wide Web", *Proceedings of the 18th International Conference of the North American Fuzzy Information Society NAFIPS'99*, New York City, June 10–12, 1999, pp. 766–770.

H. T. Nguyen, V. Kreinovich, and B. Wu, "Chu Spaces - A New Approach to Diagnostic Information Fusion", *Proceedings of the 2nd International Conference on Information Fusion FUSION'99*, Sunnyvale, CA, July 6–8, 1999, Vol. 1, pp. 323–330.

H. T. Nguyen, B. Wu, and V. Kreinovich, "A New Look at Fuzzy Theory Via Chu Spaces", *Proceedings of The Eighth International Fuzzy Systems Association World Congress IFSA'99*, Taipei, Taiwan, August 17–20, 1999, pp. 237–240.

C. H. Papadimitriou, *Computational Complexity*, Addison Wesley, San Diego, 1994.

B. K. Papadopoulos and A. Syropoulos, Fuzzy sets and fuzzy relational structures as Chu spaces, In: B. K. Papadopoulos and A. Syropoulos (eds.), *Current Trends and Developments in Fuzzy Logic, Proceedings of the First International Workshop, Thessaloniki, Greece, October 16–20, 1998*, pp. 237–246.

V. R. Pratt, Chu Spaces and their Interpretation as Concurrent Objects, In: J. van Leeuwen (ed.), *Computer Science Today: Recent Trends and Developments*, Springer Lecture Notes in Computer Science, 1995, Vol. 1000, pp. 392–405.

V. R. Pratt, Rational Mechanics and Natural Mathematics, *Proceedings of TAPSOFT'95, Aarhus, Denmark, 1995*, Springer Lecture Notes in Computer Science, 1995, Vol. 915, pp. 108–122.

V. R. Pratt, The Stone Gamut: A Coordinatization of Mathematics, In: *Proceedings of Logic in Computer Science Conference LICS'95*, IEEE Computer Society, June 1995, pp. 444–454.

V. R. Pratt, Chu spaces from the representational viewpoint, *Parikh Festschrift*, 1997.

V. R. Pratt, Types as Processes, via Chu spaces, *EXRESS'97 Proceedings*, 1997.

V. R. Pratt, Chu spaces as a semantic bridge between linear logic and mathematics, *Theoretical Computer Science*, 1998.

R. Van Glabbeek and G. Plotkin, Configuration Structures, In: *Proceedings of Logic in Computer Science Conference LICS'95*, IEEE Computer Society, June 1995, pp. 199–209.