Allowing Two Moves in Succession Increases the Game's Bias: a Theorem

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Allowing two moves in succession should not increase the game's bias, but it does: a seeming paradox. Some games are *biased* in the sense that whoever starts the game has an advantage. For example, in chess, it is widely believed that a player who plays the whites (and who, therefore, has the first move) have an advantage over the player who plays the blacks.

In such games, in some sense, each move of the 1st player "adds" some "bias" to the "value" of the position. With this interpretation in mind, if we consider, e.g., the sequence of four moves, the resulting bias should be the same whether we allow the players to play as usual: 1st, 2nd, 1st, 2nd, or whether we allow each player to make two moves at the same time: in both cases, the first player makes exactly two moves, so, informally, in both cases, the result is biased "twice".

However, in practice (e.g., in chess), if we allow each player to make two moves in succession, the bias noticeably increases. This empirical fact seems to be in contradiction with the above understanding of bias.

In this paper, we explain this seeming paradox, and prove that for four-move games, allowing players to make two moves in succession does increase the game's bias.

Motivations for the following definitions: formalizing bias. In order to formulate the problem in precise terms, let us describe what bias means.

We are considering zero-sum games like chess, in which one person's win is another person's loss (see, e.g., [1, 2]). So, to describe the payoffs, it is sufficient to describe the payoffs of the first player; the playoffs for the second player would be exactly opposite. The first player wants to maximize his payoff, the second player, accordingly, wants to maximize his playoffs, i.e., equivalently, minimize the first player's payoffs.

In this paper, we will follow traditional game theory and assume that both players can come up with the optimal strategies (for a possible formalism that enables us to take into account the fact that players are not perfect optimizers, see, e.g., [3]).

Our ultimate goal is to consider the first four moves of the game. Before we do that, let us first handle the first two moves, m_1 and m_2. Normally, the first player makes the move m_1, and then the second player makes the move m_2. For each sequence of moves (m_1, m_2), let $P_2(m_1, m_2)$ denote the expected payoff to the first player. After the first player makes a move m_1, the second player selects a move that would maximize his reward, i.e., equivalently, minimize the first player's playoff $P_2(m_1, m_2)$. So, after each move m_1, the expected playoff of the first player is equal to

$$\min_{m_2} P(m_1, m_2).$$

The first player wants to maximize his playoff, so he chooses a move m_1 for which this quantity takes the largest possible value. His resulting playoff is

$$\max_{m_1} \min_{m_2} P_2(m_1, m_2). \tag{1}$$

If the players switch turns, i.e., if the second player started and the first player made the next move, the resulting playoff for the first player would be, similarly, equal to

$$\min_{m_1} \max_{m_2} P_2(m_1, m_2).$$

For such two-move sequences, the fact that the game is favorably biased towards the first player means that the first player gets a better payoff if he starts first, i.e.,

$$\max_{m_1} \min_{m_2} P_2(m_1, m_2) > \min_{m_1} \max_{m_2} P_2(m_1, m_2). \tag{2}$$

Let us now consider four moves. Let $P_4(m_1, m_2, m_3, m_4)$ denote the expected payoff of the first player after the four moves m_i. The moves are, normally, done in turn, so that the 1st player makes the move m_1, the 2nd player makes the move m_2, the 1st player makes the move m_3, and then the 2nd player makes the move m_4. We want to define the "bias" by saying that in any position, if the first player starts a two-move sequence, he gets a better payoff. In particular, we can consider the positions after a first move m_1. In this case, after a two-move sequence (m_2, m_3), it is the second player's turn, so the expected payoff after the first three moves is

$$P_3(m_1, m_2, m_3) = \min_{m_4} P_4(m_1, m_2, m_3, m_4). \tag{3}$$

In our case, the move m_1 is fixed, so we can use the idea of the formula (2), with

$$\tilde{P}_2(m_2, m_3) = P_3(m_1, m_2, m_3), \tag{4}$$

to describe the fact that the game is favorably biased towards the first player:

$$\max_{m_2} \min_{m_4} \tilde{P}_2(m_2, m_3) > \min_{m_2} \max_{m_3} \tilde{P}_2(m_2, m_3). \tag{5}$$

Motivations for the following definitions: describing the payoff for the normal game and for the game which allows two moves in succession. Similarly to the way we derived the formula (1), we can conclude that for the normal game, the playoff for the first player is equal to

$$\max_{m_1} \min_{m_2} \max_{m_3} \min_{m_4} P_4(m_1, m_2, m_3, m_4). \tag{6}$$

Using the same idea, we can conclude that in the game in which the first player makes the first two moves and the second player makes the next two moves, the expected payoff of the first player is equal to

$$\max_{m_1} \max_{m_2} \min_{m_3} \min_{m_4} P_4(m_1, m_2, m_3, m_4). \tag{7}$$

Now, we are ready for the formal result.

Definition 1. *By a four-move game (or simply game, for short), we mean a triple $G = \langle M, \mathcal{M}, P_4 \rangle$, where:*

- *M be a finite set; its elements are called moves;*
- *$\mathcal{M} \subseteq M^4$ be a set of four-move sequences;*
- *$P : \mathcal{M} \to R$ is a function from the set $\mathcal{M}$ to the set of all real numbers.*

Definition 2. *For every game G:*

- *by an expected playoff for the first player, we mean a number*

$$E(G) = \max_{m_1} \min_{m_2} \max_{m_3} \min_{m_4} P_4(m_1, m_2, m_3, m_4); \tag{6a}$$

- *by an expected playoff for the first player when we allow two moves in succession, we mean a number*

$$E_2(G) = \max_{m_1} \max_{m_2} \min_{m_3} \min_{m_4} P_4(m_1, m_2, m_3, m_4). \tag{7a}$$

Definition 3. *We say that a game is favorably biased towards the first player (or simply biased, for short), if for every* m_1, *we have*

$$\max_{m_2} \min_{m_3} \tilde{P}_2(m_2, m_3) > \min_{m_2} \max_{m_3} \tilde{P}_2(m_2, m_3), \tag{5}$$

where $\tilde{P}_2(m_2, m_3)$ *is defined by the formula*

$$\tilde{P}_2(m_2, m_3) = P_3(m_1, m_2, m_3), \tag{4}$$

and P_3 *is defined by the formula*

$$P_3(m_1, m_2, m_3) = \min_{m_4} P_4(m_1, m_2, m_3, m_4). \tag{3}$$

Proposition. *If a game is favorably biased towards the first player, then the expected payoff for the first player increases if we allow two moves in succession (i.e.,* $E_2(G) > E(G)$*).*

Comment. This result is in good accordance with common sense, according to which, when fighting against a mightier opponent, it is better to respond after each attack than to wait. This is often true in sports, this is often true in debate, this is often true in international policy.

Proof. If we substitute the definition (4) of $\tilde{P}_2$ into the formula (5), we conclude that the fact that the game is biased towards the first player means that for every move m_1, we have

$$\max_{m_2} \min_{m_3} P_3(m_1, m_2, m_3) > \min_{m_2} \max_{m_3} P_3(m_1, m_2, m_3).$$

Substituting the expression (3) for P_3 into this formula, we conclude that

$$\max_{m_2} \min_{m_3} \min_{m_4} P_4(m_1, m_2, m_3, m_4) > \min_{m_2} \max_{m_3} \min_{m_4} P_4(m_1, m_2, m_3, m_4). \tag{8}$$

This inequality is true for each of finitely many possible first moves, so, the largest possible value of the left-hand sides must be larger than the larger possible value of the right-hand sides. According to Definition 2, the largest possible value of the left-hand side is $E_2(G)$, while the largest possible value of the right-hand sides is $E(G)$. Thus, $E_2(G) > E(G)$. The proposition is proven.

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