On Granularity in Fuzzy Logic: Minimum and Maximum are the Only Absolutely Granular t-Norm and t-Conorm

Vladik Kreinovich[1] and Bernadette Bouchon-Meunier[2]

[1]Department of Computer Science
University of Texas at El Paso
El Paso, TX 79968, USA
email vladik@cs.utep.edu

[2]LIP6, Pole IA
UPMC, Case 169
4 place Jussieu
75252 Paris Cédex 05, France
email bouchon@poleia.lip6.fr

1 The idea of granularity

In 1996, L. Zadeh described the following idea (see, e.g., [3]):

- We humans usually think in words. When we want to describe the size of an object we use words like "tiny", "small", "medium", "large", "huge". For each quantity (like size, velocity, distance, etc.), there are usually few possible words, and we can describe the value by indicating which of these words best corresponds to the current situation.

  In principle, to represent one of, say, 5 possible values, we need only 3 bits.

- However, traditional methods of representing the uncertain ("fuzzy") expert knowledge use *real numbers* to describe uncertainty. A real number usually requires at least 4 bytes (32 bits). Thus, we use at least 10 times more memory than we should, and when we process these numbers, then, to process all these bits, we use at least 10 times more time than we should.

For the computers to reach the ability of a human brain, we thus need to develop *new* methods of representing and processing uncertainty, methods in which the representation is *granular*, in which the computer stores the words and operates directly with words.

2 Granularity and fuzzy logic

In the traditional fuzzy logic, different degrees of certainty are described by real numbers from the interval $[0,1]$. In principle, there are *infinitely many* real numbers on this interval. Even if we take into consideration that in the computer, we can only represent finitely many different numbers, we still get a *huge amount* of different possible degree of certainty. For example, if a computer can represent numbers with an accuracy 10^{-6}, this means that we can have 10^6 different degrees of certainty.

An expert cannot distinguish between that many degrees of certainty. At best, he can have 10 or maybe 100 different degrees of certainty. For example, hardly anyone can distinguish between a degree of certainty 0.6 and a degree of certainty, say, 0.60001. As a result, different but almost equal numbers from the interval $[0,1]$, usually, describe the *same* degree of certainty.

To describe this "granularity" of fuzzy logic, let us explicitly combine the numbers from the interval $[0,1]$ that correspond to each degree of certainty into a separate set. It is reasonable to assume that if numbers $p < q$ actually describe the same degree of certainty (i.e., are, in this sense, indistinguishable by an expert), then all numbers that are intermediate between p and q also correspond to the same degree of certainty. Under this assumption, all the numbers that correspond to the same degree of certainty form an *interval*. The entire interval $[0,1]$ is thus divided into finitely many intervals of this type.

For example, an expert can have only 11 possible degrees of certainty that correspond to numbers 0, 0.1, 0.2, ..., 0.9, 1.0, and any other number corresponds to the same degree of certainty as the closest number from this finite list. In this case, the interval $[0,1]$ is granulated into the following 11 subintervals that correspond to different degrees of certainty: $[0,0.05]$, $[0.05,0.15]$, $[0.15,0.25]$, ..., $[0.85,0.95]$, $[0.95,1.0]$.

In general, we arrive at the following definition:

Definition 1.

- *By a granulation of fuzzy logic (or simply granulation, for short), we mean a finite sequence G of real numbers $c_1 = 0 < c_2 < c_3 < \ldots < c_{n-1} < c_n = 1$.*
- *Intervals $[c_1, c_2]$, $[c_2, c_3]$, ..., $[c_{n-1}, c_n]$, will be called granulation intervals.*
- *We say that two numbers a and a' are indistinguishable relative to granulation G (and denote it by $a \sim_G a'$) if they belong to the same granulation interval.*

3 Binary operations consistent with granularity

Since we interpret numbers from the interval $[0, 1]$ as representatives of degrees of certainty, operations with these numbers can also be interpreted as operations with degrees of certainty. In other words, we can view the corresponding operation not as an operation from numbers to numbers, but as an operation from degrees of certainty to degrees of certainty.

For example, in the above granulation $[0, 0.05]$, $[0.05, 0.15]$, etc., if the operation is $\max(a, b)$, and the only information that we know about a and b is that $a \in [0, 0.05]$ and $b \in [0.05, 0.15]$, then we do not know the exact value of $\max(a, b)$. This value can be anything from $\max(0, 0.05) = 0.05$ to $\max(0.05, 0.15) = 0.15$. In other words, $\max(a, b)$ can take any value from the interval $[0.05, 0.15]$. However, although we do not know the exact value of $\max(a, b)$, but all these value belong to the same granulation interval $[0.05, 0.15]$. Thus, if we know the degrees of certainty (i.e., granulation intervals) that correspond to a and b, we can thus uniquely determine the degree of certainty that corresponds to $\max(a, b)$.

Such *consistency* between an operation and a granulation is not always happening: For example, the same granulation is *not* consistent with the product $a \cdot b$; if we take $a = b = 1$ and $a' = b' = 0.96$, then all four elements a, b, a', b' belong to the same granulation interval $[0.95, 1]$, but the values $a \cdot b = 1 \in [0.95, 1]$ and $a' \cdot b' = 0.9216 \in [0.85, 0.95]$ belong to *different* granulation intervals.

This notion of consistency can be expressed in formal terms:

Definition 2.

- *By a binary operation of fuzzy logic (or simply operation, for short), we mean a function* $f : [0.1] \times [0, 1] \to [0, 1]$.
- *We say that an operation f is consistent with a granulation G if the following condition holds:*

  If $a \sim_G a'$ *and* $b \sim_G b'$, *then* $f(a, b) \sim_G f(a', b')$.

4 Absolute granularity

An operation may be consistent with one granulation and not consistent with another one: for example, every operation is consistent with the trivial granulation in which $n = 2$ and in which, therefore, there is only one granulation interval of width 1.

In general, if an operation is consistent with some granulation, it may not be consistent with finer granulations (with narrower granulation intervals).

An important question is *which operations are absolutely granular*, i.e., consistent with granulations of arbitrary small "fineness". Let us formulate this question formally:

Definition 2. *By a fineness $F(G)$ of a granulation $G = \{c_1 < c_2 < \ldots < c_n\}$, we mean the largest of the numbers $c_{i+1} - c_i$ (i.e., the width of the widest granulation interval).*

Definition 3. *We say that an operation f is absolutely granular if for every $\varepsilon > 0$, there exists a granulation G with fineness $F(G) < \varepsilon$ that is consistent with f.*

5 Main problem analyzed in this paper: Which t-norms and t-conorms are absolutely granular?

In this paper, we will analyze which t-norms and which t-conorms are absolutely granular. For completeness, let us give the definitions of a t-norm and a t-conorm:

Definition 4. (see, e.g., [1, 2]) *A function $T : [0,1] \times [0,1] \to [0,1]$ is called a t-norm if it satisfies the following four conditions:*

(i) $T(1, a) = a$ *for all* a;

(ii) $T(a, b) = T(b, a)$ *for all* a *and* b;

(iii) $T(a, T(b, c)) = T(T(a, b), c))$ *for all* a, b, *and* c;

(iv) *if* $a \le a'$ and $b \le b'$, *then* $T(a, b) \le T(a', b')$.

Definition 5. (see, e.g., [1, 2]) *A function $S : [0,1] \times [0,1] \to [0,1]$ is called a t-conorm if it satisfies the following four conditions:*

(i) $S(0, a) = a$ *for all* a;

(ii) $S(a, b) = S(b, a)$ *for all* a *and* b;

(iii) $S(a, S(b, c)) = S(S(a, b), c))$ *for all* a, b, *and* c;

(iv) *if* $a \le a'$ and $b \le b'$, *then* $S(a, b) \le S(a', b')$.

Comment. It is also usually required that a t-norm and a t-conorm are *continuous* functions.

6 Formulation of the main result: minimum and maximum are the only absolutely granular t-norm and t-conorm

THEOREM.

- $T(a, b) = \min(a, b)$ *is the only absolutely granular continuous t-norm.*
- $S(a, b) = \max(a, b)$ *is the only absolutely granular continuous t-conorm.*

7 Proof

1. We will first prove the result about a t-norm (the result about the t-conorm is proven in a similar way). Namely, we will prove that every absolutely granular continuous t-norm $T(a, b)$ is idempotent (i.e., $T(a, a) = a$ for all a). Our result will then follow from the known fact that $T(a, b) = \min(a, b)$ is the only idempotent t-norm [1, 2].

1.1. To prove idempotency, let us first prove that if a t-norm $T(a, b)$ is consistent with a granulation $G = \{c_1 < c_2 < \ldots < c_n\}$, then $T(c_i, c_i) = c_i$ for all i.

Indeed, from the definition of a t-norm (condition (i) with $a = 1$), it follows that $T(1, 1) = 1$, i.e., $T(c_n, c_n) = c_n$.

Similarly, from condition (i) and monotonicity (condition (iv)), it follows that $T(a, a) \le T(1, a) \le a$. For $a = 0 = c_1$, we get $T(0, 0) \le 0$ and thus, $T(0, 0) = 0$, i.e., $T(c_1, c_1) = c_1$.

Let us prove, by reduction to a contradiction, that $T(c_i, c_i) = c_i$ for all $i = 2, \ldots, n-1$. Indeed, let us assume that for some i, this condition is violated. Since $T(c_i, c_i) \le c_i$, the only possibility for violating this condition is when $T(c_i, c_i) < c_i$.

The value c_i belongs to ith granulation interval $[c_i, c_{i+1}]$. All the real numbers c between c_i and $c_n = 1$ belong to $n - 1 - i$ granulation intervals $[c_i, c_{i+1}], \ldots, [c_{n-1}, c_n]$. Since the operation T is consistent with the granulation, whenever two numbers c and c' belong to the same granulation interval ($c \sim_G c'$), the corresponding wto values $T(c, c)$ and $T(c', c')$ also belong to the same granulation interval ($T(c, c) \sim_G T(c', c')$). Thus, when we consider inputs c from $n - 1 - i$ different granulation intervals, we get the values $T(c, c)$ that belong to at most $n - 1 - i$ different granulation intervals.

On the other hand, the function $c \to T(c, c)$ is continuous, and thus, when c goes from c_i to c_n, it must take all the values from $T(c_i, c_i) < c_i$ to $c_n = 1$. In other words, every value t from the interval $[T(c_i, c_i), c_n]$ can be represented as $T(c, c)$ for some $c \in [c_i, c_n]$. But since $T(c_i, c_i) < c_i$, the values from the interval $[T(c_i, c_i), c_n]$ occupy at least $n - i$ different granulation intervals: namely, in addition to $n - i - 1$ granulation intervals $[c_i, c_{i+1}], \ldots, [c_{n-1}, c_n]$ that cover all the values from c_i to c_n, we need at least one extra granulation interval to cover values that are smaller than c_i.

Thus, we get a contradiction:

- On one hand, the values $T(c, c)$, where $c \in [c_i, c_n]$, occupy at most $n-i-1$ different granulation intervals.
- On the other hand, these same values occupy at least $n - i$ granulation intervals.

This contradiction shows that our initial assumption (that $T(c_i, c_i) < c_i$ for some i) is inconsistent, and therefore, $T(c_i, c_i) = c_i$ for all i.

1.2. Let us now show that $T(a, a) = a$ for all a.

Indeed, since T is absolutely granular, for every $\varepsilon > 0$, there exists a granulation of fineness $< \varepsilon$ that is consistent with T. Fineness $< \varepsilon$ means that each granulation interval is narrower than ε. In particular, the interval $[c_i, c_{i+1}]$ that contains a is narrower than ε, and hence, $|a - c_i| \leq \varepsilon$. For this c_i, we have already proven that $T(c_i, c_i) = c_i$.

Thus, if we take the values $c_i(\varepsilon)$ that correspond to $\varepsilon \to 0$, we get a sequence $c_i(\varepsilon)$ for which $|c_i(\varepsilon) - a| \leq \varepsilon \to 0$ and hence, $c_i(\varepsilon \to a$, and for which $T(c_i(\varepsilon), c_i(\varepsilon)) = c_i(\varepsilon)$. Since T is continuous, we conclude that $T(a, a) = a$.

The result about a t-norm is proven.

2. Let us now prove the result about a t-conorm. Namely, we will prove that every absolutely granular continuous t-conorm $S(a, b)$ is idempotent (i.e., $S(a, a) = a$ for all a). Our result will then follow from the known fact that $S(a, b) = \max(a, b)$ is the only idempotent t-conorm [1, 2].

2.1. To prove idempotency, let us first prove that if a t-conorm $S(a, b)$ is consistent with a granulation $G = \{c_1 < c_2 < \ldots < \ldots < c_n\}$, then $S(c_i, c_i) = c_i$ for all i.

Indeed, from the definition of a t-norm (condition (i) with $a = 0$), it follows that $S(0, 0) = 0$, i.e., $S(c_1, c_1) = c_1$.

Similarly, from condition (i) and monotonicity (condition (iv)), it follows that $a \leq a = S(0, a) \leq S(a, a)$. For $a = 1 = c_n$, we get $1 \leq S(1, 1)$, hence, $S(1, 1) = 1$, i.e., $S(c_n, c_n) = c_n$.

Let us prove, by reduction to a contradiction, that $S(c_i, c_i) = c_i$ for all $i = 2, \ldots, n-1$. Indeed, let us assume that for some i, this condition is violated. Since $c_i \leq S(c_i, c_i)$, the only possibility for violating this condition is when $c_i < S(c_i, c_i)$.

The value c_i belongs to the $(i-1)$–st granulation interval $[c_{i-1}, c_i]$. All the real numbers c between $c_1 = 0$ and c_i belong to $i - 1$ granulation intervals $[c_1, c_2], \ldots, [c_{i-1}, c_i]$. Since the operation S is consistent with the granulation, whenever two numbers c and c' belong to the same granulation interval ($c \sim_G c'$), the corresponding two values $S(c, c)$ and $S(c', c')$ also belong to the same granulation interval ($T(c,c) \sim_G T(c',c')$). Thus, when we consider inputs c from $i-1$ different granulation intervals, we get the values $S(c,c)$ that belong to at most $i-1$ different granulation intervals.

On the other hand, the function $c \to S(c,c)$ is continuous, and thus, when c goes from c_1 to c_i, it must take all the values from $S(c_1,c_1)=c_1$ to $S(c_i,c_i)>c_i$. In other words, every value s from the interval $[c_1, S(c_i,c_i)]$ can be represented as $S(c,c)$ for some $c \in [c_1,c_i]$. But since $S(c_i,c_i) > c_i$, the values from the interval $[c_1,S(c_i,c_i)]$ occupy at least i different granulation intervals: namely, in addition to $i-1$ granulation intervals $[c_1,c_2],\ldots,[c_{i-1},c_i]$ that cover all the values from c_1 to c_i, we need at least one extra granulation interval to cover values that are greater than c_i.

Thus, we get a contradiction:

- On one hand, the values $S(c,c)$, where $c \in [c_1,c_i]$, occupy at most $i-1$ different granulation intervals.
- On the other hand, these same values occupy at least i granulation intervals.

This contradiction shows that our initial assumption (that $S(c_i,c_i) > c_i$ for some i) is inconsistent, and therefore, $S(c_i,c_i)=c_i$ for all i.

2.2. Similar to part 1.2, we can show that $S(a,a)=a$ for all a.

The theorem is proven.

Acknowledgments. This work was supported in part by NASA under cooperative agreement NCC5-209, by NSF grants No. DUE-9750858 and CDA-9522207, by United Space Alliance, grant No. NAS 9-20000 (PWO C0C67713A6), by the Future Aerospace Science and Technology Program (FAST) Center for Structural Integrity of Aerospace Systems, effort sponsored by the Air Force Office of Scientific Research, Air Force Materiel Command, USAF, under grant number F49620-95-1-0518, and by the National Security Agency under Grant No. MDA904-98-1-0561.

References

[1] G. Klir and B. Yuan, *Fuzzy sets and fuzzy logic: theory and applications*, Prentice Hall, Upper Saddle River, NJ, 1995.

[2] H. T. Nguyen and E. A. Walker, *A first course in fuzzy logic*, CRC Press, Boca Raton, Florida, 1999.

[3] L. A. Zadeh, "Toward a theory of fuzzy information granulation and its centrality in human reasoning and fuzzy logic", *Fuzzy Sets Syst.*, 1997, Vol. 90, No. 2, pp. 111–127.